Arab J. Math. Sc.
Volume 5, Number 2, December 1999
pp. 51-66

DIFFRACTION OF TRANSIENT SH-WAVES IN A HALF SPACE

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ABSTRACT. We consider the diffraction of a transient SH-wave obliquely incident on a plane semi-infinite crack lying in a homogeneous and isotropic half space. The Wiener-Hopf technique together with the Cagniard-de Hoop method has been used to obtain the solution in the two regions separated by the crack.

1. INTRODUCTION

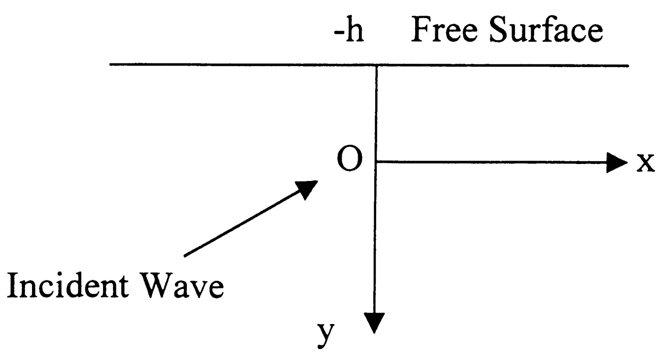
The diffraction of elastic waves is of considerable importance in material and earth sciences. Some interesting problems of this nature have been discussed by Kazi [1], Achenbach [2] and Asghar and Zaman [3]. These authors have employed the Wiener-Hopf technique to solve the boundary value problems resulting from plane discontinuities lying in the medium of propagation. The problem discussed by Achenbach [2] involves an obliquely incident transient SH-wave by a semi-infinite crack of vanishing thickness lying in an unbounded elastic medium. The results have been subsequently applied to problems arising in seismology by Achenbach and Harris [4]. it seems that the diffraction of such waves in a half space would be of interest due to the fact that if the discontinuity lies near the surface of the earth, the medium of propagation can no longer be considered to be unbounded.

Mathematics Subject Classification. 35A22, 73D25.

In this paper, we set up and solve the problem of diffraction of obliquely incident transient SH-wave on a semi infinite plane crack lying in a half space. The modified Wiener-Hopf technique (Jones [5]), together with the Cagniard-de Hoop method has been used to obtain the solution of the resulting boundary value problem. It is found that the transmitted wave in the region below the crack agrees with Achenbach's solution obtained in the unbounded medium. This agreement is attributed to the similarity of the geometry of this region. The transmitted wave in the upper layer formed by the free surface above and the crack below satisfies the dispersion relation of SH-wave travelling in a layer of uniform thickness with free upper and lower surfaces.

2. FORMULATION OF THE PROBLEM

We consider the diffraction of a horizontally polarized shear wave by a plane semi-infinite crack lying at a distance h parallel to the free surface of the elastic half space. The half space is filled with homogeneous and isotropic elastic solid with rigidity μ and shear velocity β. We choose the coordinate system in such a way that the crack coincides with the xz-plane, the y-axis is directed into the half space and the free surface is $y = -h$; $-\infty < x < \infty$. The geometry of the plane is shown in the figure below.



The incident wave has transient displacement

$$w^{inc} = G(t - s_T x \cos\alpha - s_T y \sin\alpha), \tag{1}$$

where

$$G(t) = H(t)\int_0^t g(s)ds. \tag{2}$$

Therefore

$$w^{inc} = H(t - s_T \cos\alpha - s_T \sin\alpha)\int_0^{t - s_T \cos\alpha - y \sin\alpha} g(s)ds. \tag{3}$$

In equation (3), $H(x)$ denotes the Heaviside step function, α is the angle between the normal to the wave front and the x-axis, and $s_T = \frac{1}{\beta}$ is the slowness of the shear wave. We restrict the angle α to the range $0 \leq \alpha \leq \pi/2$.

In the presence of the crack at $y = 0\,; x \geq 0$, the stress component τ_{yz} of the wave motion must satisfy the following condition at $y = 0$

$$\tau_{yz} = \mu s_T \sin\alpha H(t - s_T x \cos\alpha) g(t - s_T x \cos\alpha). \tag{4}$$

Let the total displacement field due to the presence of crack be decomposed as follows:

$$w^{tot} = \begin{matrix} w^{inc} + w_1 \;, \; -h \leq y \leq 0, \; -\infty < x < \infty \\ w^{inc} + w_2 \;, \; y \geq 0, \qquad -\infty < x < \infty. \end{matrix} \tag{5}$$

The geometry of the problem leads to the following boundary conditions:

(a) At

$$\left.\begin{aligned} x \geq 0,\ y = 0^-,\ \frac{\partial w_1}{\partial y} \\ x \geq 0,\ y = 0^+,\ \frac{\partial w_2}{\partial y} \end{aligned}\right\} = -\frac{\partial w^{inc}}{\partial y}$$

$$= s_T \sin\alpha H(t - s_T\cos\alpha) g(t - s_T\cos\alpha).$$

At $x \leq 0,\ y = 0$

$$\begin{aligned} w_1 &= w_2 \\ \frac{\partial w_1}{\partial y} &= \frac{\partial w_2}{\partial y}, \end{aligned} \tag{6}$$

(b) At $y = -h,\ -\infty < x < \infty$

$$\frac{\partial w_1}{\partial y} = 0. \tag{7}$$

The displacement satisfies the differential equation

$$\frac{\partial^2 w^{inc}(x,y,t)}{\partial x^2} + \frac{\partial^2 w^{inc}(x,y,t)}{\partial y^2} = s_T^2 \frac{\partial^2 w^{inc}(x,y,t)}{\partial t^2}, \tag{8}$$

while the initial conditions are

$$w^{inc}(x,y,0) = \frac{\partial w^{inc}(x,y,0)}{\partial t} = 0. \tag{9}$$

We now find the transmitted solutions w_1 and w_2 , where w_1 and w_2 satisfy the differential equation

$$\frac{\partial^2 w_i}{\partial x^2} + \frac{\partial^2 w_i}{\partial y^2} = s_T^2 \frac{\partial^2 w_i}{\partial t^2}, \quad i = 1, 2. \tag{10}$$

The differential equation (10) together with the boundary conditions (6,7) constitute the boundary value problem.

3. THE WIENER-HOPF EQUATIONS

We use the one-sided Laplace transform in time and the two-sided Laplace transform in x to derive the Wiener-Hopf equations. The one-sided Laplace transform in time of a function $f(x,y,t)$ is given by

$$\overline{f}(x,y,p) = \int_0^\infty f(x,y,t)e^{-pt}dt. \tag{11}$$

It may be noted that if $f(x,y,t)$ as a function of t is of order e^{at} as $t \to \infty$, then $\overline{f}(x,y,p)$ defined by equation (11) is an analytic function of p if $\mathrm{Re}(p) > a$.

The two-sided transform in x is defined as

$$f^*(\xi,y,t) = \int_{-\infty}^\infty f(x,y,t)e^{-\xi x}dx. \tag{12}$$

If $f(x,y,t)$ is of order $e^{-a|x|}$ as $|x| \to \infty$, then it follows that the two-sided transform defined in (12) will be an analytic function of ξ if $-a < \mathrm{Re}(\xi) < a$ (Achenbach[2]).

Using (11) and (12) we can write (10) as

$$\frac{d^2\overline{w_i}^*}{dy^2} - \sigma^2\overline{w_i}^* = 0, \tag{13}$$

where

$$\sigma^2 = s_T^2p^2 - \xi^2. \tag{14}$$

The solution of equation (13) in the layer $-h \leq y \leq 0$, $-\infty < x < \infty$, is given by

$$\overline{w_1}^*(\xi,y,p) = A(p,\xi)e^{-\sigma y} + B(p,\xi)e^{\sigma y}. \tag{15}$$

The solution of equation (13) in the half space $y \geq 0$ is

$$\overline{w_2}^*(\xi, y, p) = C(p, \xi)e^{-\sigma y}. \tag{16}$$

Using boundary conditions (6), equation (15) can be written as

$$\overline{w}_1^*(\xi, y, p) = E(p, \xi) \cosh \sigma(y + h). \tag{17}$$

We now attempt to eliminate the constants $C(p, \xi)$ and $E(p, \xi)$ from the equations (16) and (17). We differentiate these equations and then put $y = 0$, to obtain

$$\overline{w}_1^{*'}(\xi, 0, p) = E(p, \xi)\sigma \sinh \sigma h \tag{18}$$

and

$$\overline{w}_2^{*'}(\xi, 0, p) = -\sigma C(p, \xi). \tag{19}$$

Using equations (18) and (19), equations (16) and (17) can be written as

$$\overline{w}_1^*(\xi, y, p) = \frac{\cosh \sigma(y + h)}{\sigma \sinh \sigma h} \overline{w}_1^{*'}(\xi, 0, p) \tag{20}$$

and

$$\overline{w}_2^*(\xi, y, p) = -\frac{1}{\sigma} e^{-\sigma y} \overline{w}_2^{*'}(\xi, 0, p). \tag{21}$$

Substituting $y = 0$ into equations (20) and (21), we get

$$\overline{w}_1^*(\xi, 0, p) = \frac{\coth \sigma h}{\sigma} \overline{w}_1^{*'}(\xi, 0, p) \tag{22}$$

and

$$\overline{w}_2^*(\xi, 0, p) = -\frac{1}{\sigma} \overline{w}_2^{*'}(\xi, 0, p). \tag{23}$$

Following a notation similar to that of Achenbach [2], we define

$$\text{(24)} \qquad \begin{aligned} \overline{w}^*(\xi, y, p) &= \int_{-\infty}^{\infty} e^{-\xi x}\overline{w}(x, y, p)dx \\ &= \overline{w}^*_+(\xi, y, p) + \overline{w}^*_-(\xi, y, p) \end{aligned}$$

where

$$\overline{w}^*_+(\xi, y, p) = \int_0^{\infty} e^{-\xi x}\overline{w}(x, y, p)dx$$

and

$$\overline{w}^*_-(\xi, y, p) = \int_{-\infty}^{0} e^{-\xi x}\overline{w}(x, y, p)dx.$$

Here

$$\xi = \alpha + i\tau$$

$$|w| \leq A\exp(\alpha_- x) \text{ as } x \to \infty$$

and

$$|w| \leq B\exp(\alpha_+ x) \text{ as } x \to -\infty$$

where we use $\alpha_+ = \mathrm{Re}(\xi)$ and $\alpha_- = -\mathrm{Re}(\xi)$, then $\overline{w}^*_+(\xi, y, p)$ is analytic for $\alpha > \alpha_-$ and $\overline{w}^*_-(\xi, y, p)$ is analytic for $\alpha < \alpha_+$. Thus $\overline{w}^*(\xi, y, p)$ is analytic in the strip $\alpha_- < \alpha < \alpha_+$ (Noble[6]).

Following (24), equations (22) and (23) can be written as

$$\text{(25)} \quad \overline{w}^*_{1+}(\xi, 0, p) + \overline{w}^*_{1-}(\xi, 0, p) = \frac{\coth \sigma h}{\sigma}\left[\overline{w}^{*'}_{1+}(\xi, 0, p) + \overline{w}^{*'}_{1-}(\xi, 0, p)\right]$$

and

$$\text{(26)} \quad \overline{w}^*_{2+}(\xi, 0, p) + \overline{w}^*_{2-}(\xi, 0, p) = -\frac{1}{\sigma}\left[\overline{w}^{*'}_{2+}(\xi, 0, p) + \overline{w}^{*'}_{2-}(\xi, 0, p)\right].$$

Transforming the boundary conditions (6), we get for $y = 0$, $x > 0$

$$
\begin{aligned}
\overline{w}^{*'}_{1+}(\xi,0,p) &= \overline{w}^{*'}_{2+}(\xi,0,p) \\
&= -\overline{w}^{*'}(\xi,0,p) \\
&= \frac{s_T \sin\alpha \overline{g}(p)}{\xi + s_T p\cos\alpha}
\end{aligned} \tag{27}
$$

and for $y = 0$, $x < 0$

$$
\begin{aligned}
\overline{w}^{*}_{1-} &= \overline{w}^{*}_{2-} \\
\overline{w}^{*'}_{1-} &= \overline{w}^{*'}_{2-}
\end{aligned} \tag{28}
$$

Equations (25) and (26) together with equations (27) and (28) give

$$
\begin{aligned}
\overline{w}^{*}_{2+}(\xi,0,p) &- \overline{w}^{*}_{1+}(\xi,0,p) + \overline{w}^{*}_{1+}(\xi,0,p) - \overline{w}^{*}_{1-}(\xi,0,p) \\
&= \frac{1}{\sigma}\left[\overline{w}^{*'}_{1+}(\xi,0,p) + \overline{w}^{*'}_{1-}(\xi,0,p)\right]
\end{aligned} \tag{29}
$$

and

$$
\begin{aligned}
\overline{w}^{*}_{2+}(\xi,0,p) &+ \overline{w}^{*}_{2-}(\xi,0,p) + \overline{w}^{*}_{1+}(\xi,0,p) - \overline{w}^{*}_{2+}(\xi,0,p) \\
&= \frac{\coth\sigma h}{\sigma}\left[\overline{w}^{*'}_{2+}(\xi,0,p) + \overline{w}^{*'}_{2-}(\xi,0,p)\right].
\end{aligned} \tag{30}
$$

Using equations (25) and (26) in equations (29) and (30), and rearranging, we have

$$
\begin{aligned}
\overline{w}^{*}_{2+}(\xi,0,p) &- \overline{w}^{*}_{1+}(\xi,0,p) = \\
&- \frac{1}{\sigma}\frac{e^{\sigma h}}{\sinh\sigma h}\left[\overline{w}^{*'}_{1+}(\xi,0,p) + \overline{w}^{*'}_{1-}(\xi,0,p)\right]
\end{aligned} \tag{31}
$$

and

$$
\begin{aligned}
\overline{w}^{*}_{2+}(\xi,0,p) &- \overline{w}^{*}_{1+}(\xi,0,p) = \\
&- \frac{1}{\sigma}\frac{e^{\sigma h}}{\sinh\sigma h}\left[\overline{w}^{*'}_{2+}(\xi,0,p) + \overline{w}^{*'}_{2-}(\xi,0,p)\right]
\end{aligned} \tag{32}
$$

Multiplying and dividing the right hand side of equations (31) and (32) by σh and rearranging, we have

$$\begin{aligned}\overline{w}^{*'}_{1+}(\xi,0,p) \;&+\; \overline{w}^{*'}_{1-}(\xi,0,p) = \\ &-\; \sigma^2 h\left[\frac{e^{-\sigma h}\sinh\sigma h}{\sigma h}\right]\left[\overline{w}^{*}_{2+}(\xi,0,p)-\overline{w}^{*}_{1+}(\xi,0,p)\right]\end{aligned} \tag{33}$$

and

$$\begin{aligned}\overline{w}^{*'}_{2+}(\xi,0,p) \;&+\; \overline{w}^{*'}_{2-}(\xi,0,p) = \\ &-\sigma^2 h\left[\frac{e^{-\sigma h}\sinh\sigma h}{\sigma h}\right]\left[\overline{w}^{*}_{2+}(\xi,0,p)-\overline{w}^{*}_{1+}(\xi,0,p)\right].\end{aligned} \tag{34}$$

The equations (33) and (34) are the required Wiener-Hopf equations and can be solved by the usual Wiener-Hopf procedure. We point out that the two Wiener-Hopf equations are identical, as would be expected.

4. DETERMINATION OF THE WIENER-HOPF SOLUTION

R. Mittra and S. W. Lee [7] have fully described the factorization of $\frac{e^{-\sigma h}\sinh\sigma h}{\sigma h}$. Using these results, we write

$$\frac{e^{-\sigma h}\sinh\sigma h}{\sigma h} = G_+(\xi)G_-(\xi). \tag{35}$$

where

$$\begin{aligned}G_\pm(\xi) \;&=\; \sqrt{(\sin\kappa b)/\kappa b}\,\exp\left[\pm\frac{ib\xi}{\pi}(1-c+\ln(\frac{2\pi}{\kappa b})+\frac{i\pi}{2}\right]\times \\ &\exp\left[\frac{ib\gamma}{\pi}\ln(\frac{\pm\xi-\gamma}{\pi})\prod_{n=1}^{\infty}(1\pm\frac{\xi}{i\gamma_{nb}})\right]e^{\pm\frac{i\xi b}{n\pi}}\end{aligned}$$

and $G_\pm(\xi)\sim\xi^{-\frac{1}{2}}$ as $|\xi|\to\infty$ in the upper half plane.

By making use of equation (35), we can write (33) as

$$\frac{1}{(\xi - s_T p)G_-(\xi)}\left[\overline{w}^{*'}_{1+}(\xi,0,p) + \overline{w}^{*'}_{1-}(\xi,0,p)\right] = (\xi + s_T p)G_+(\xi)h\left[\overline{w}^{*}_{2+}(\xi,0,p) - \overline{w}^{*}_{1+}(\xi,0,p)\right]. \tag{36}$$

Now we use equation (27) to achieve the following decomposition of equation (36)

$$\frac{\overline{w}^{*'}_{1-}(\xi,0,p)}{(\xi - s_T p)G_-(\xi)} + \frac{s_T \sin\alpha \overline{g}(p)}{G_-(\xi)(\xi - s_T p)(\xi + s_T p\cos\alpha)} = (\xi + s_T p)G_+(\xi)h\left[\overline{w}^{*}_{2+}(\xi,0,p) - \overline{w}^{*}_{1+}(\xi,0,p)\right]. \tag{37}$$

The right side of this equation is analytic in the right half plane while the left side is so in the left half plane except that the second term has a pole at $\xi = -s_T p\cos\alpha$. To remove this pole we subtract (from both sides) the contribution due to this pole and write the equation (37) as

$$\begin{aligned}\frac{\overline{w}^{*'}_{1-}(\xi,0,p)}{(\xi - s_T p)G_-(\xi)} &+ \frac{s_T \sin\alpha \overline{g}(p)}{\xi + s_T p\cos\alpha}\left\{\frac{1}{G_-(\xi)(\xi - s_T p)} + \frac{1}{s_T p G_-(-s_T p\cos\alpha)(1+\cos\alpha)}\right\} \\ &= (\xi + s_T p)G_+(\xi)h\left[\overline{w}^{*}_{2+}(\xi,0,p) - \overline{w}^{*}_{1+}(\xi,0,p)\right] + \frac{s_T \sin\alpha \overline{g}(p)}{s_T p(\xi + s_T p\cos\alpha)G_-(-s_T p\cos\alpha)(1+\cos\alpha)}.\end{aligned} \tag{38}$$

We now note that the left side of the equation (38) is analytic in the left half plane $\mathrm{Re}(\xi - s_T p) < 0$ and the right side is analytic in the right half plane $\mathrm{Re}(\xi + s_T p\cos\alpha) > 0$. Therefore, both the expressions define an entire function in the strip common to both the half planes. If both sides tend to zero as $|\xi| \to \infty$ in the appropriate half planes, then the entire function can be shown to be zero using Liouville's theorem. This can be done as in Achenbach [2]. Thus we can obtain the following from (38) with the help of (27)

$$\overline{w}_1^{*\prime}(\xi, 0, p) = \frac{s_T \sin\alpha \overline{g}(p) G_-(\xi)(s_T p - \xi)}{s_T p G_-(-s_T p\cos\alpha)(1+\cos\alpha)(\xi + s_T p \cos\alpha)}. \tag{39}$$

In order to obtain $\overline{w}_1^*(\xi, y, p)$, we make use of equation (39) in equation (20). This gives

$$\overline{w}_1^*(\xi, y, p) = \frac{\cosh\sigma(y+h) s_T \sin\alpha \overline{g}(p) G_-(\xi)(s_T p - \xi)}{s_T p \sigma \sinh\sigma h G_-(-s_T p\cos\alpha)(1+\cos\alpha)(\xi + s_T p\cos\alpha)}. \tag{40}$$

With

$$M = \frac{\sin\alpha \overline{g}(p)}{s_T p G_-(-s_T p \cos\alpha)(1+\cos\alpha)} \tag{41}$$

which is independent of ξ, equation (40) becomes

$$\overline{w}_1^*(\xi, y, p) = \frac{M\cosh\sigma(y+h)(s_T p - \xi)^{1/2} G_-(\xi)}{(\xi + s_T p\cos\alpha)\sinh\sigma h (s_T p + \xi)^{1/2}}. \tag{42}$$

In order to obtain $\overline{w}_2^*(\xi, y, p)$, we return to (21) and use equations (27) and (28) to obtain

$$\overline{w}_2^*(\xi, y, p) = \frac{-MG_-(\xi)(s_T p - \xi)^{1/2} e^{-\sigma y}}{(s_T p + \xi)^{1/2}(\xi + s_T p \cos\alpha)}. \tag{43}$$

The equations (42) and (43) give the transmitted waves in the two regions of interest in the transformed ξ-plane.

5. THE TRANSMITTED WAVES

We determine the transmitted waves in the two regions that consist of a semi-infinite layer that lies above the crack and a half space lying below the crack. This can be done by taking the inverse transforms of the expressions in (42) and (43). We do this for each region separately.

(a) The region $-h \le y \le 0$, $x > 0$.

The two sided Laplace inversion formula gives

$$\overline{w}_1(x, y, p) = \frac{1}{2\pi i} \int_{\xi_1 - i\infty}^{\xi_1 + i\infty} e^{\xi x} \overline{w}_1^*(\xi, y, p) d\xi, \tag{44}$$

where $\overline{w}_1^*(\xi, y, p)$ is given by equation (42). The path of integration is restricted to the strip $-s_T p \cos\alpha < \xi_1 < s_T p$. For $-h \le y \le 0$, $x > 0$, we can close the contour in the upper half plane. Using equation (40) in (44), we write

$$\overline{w}_1(x, y, p) = \frac{1}{2\pi i} \int_{\xi_1 - i\infty}^{\xi_1 + i\infty} F_1 d\xi, \tag{45}$$

where

$$F_1 = \frac{s_T \sin\alpha \overline{g}(p) \cosh\sigma(y + h) G_-(\xi)(s_T - \xi)^{1/2} e^{\xi x}}{s_T p(1 + \cos\alpha)(s_T p + \xi)^{1/2} \sinh\sigma h(\xi + s_T p \cos\alpha) G_-(-s_T p \cos\alpha)}.$$

To invert the one-sided Laplace transform (45), we introduce the substitution $\xi = p\eta$ to obtain

$$\overline{w}_1(x, y, p) = \frac{\overline{g}(p) \sin\alpha}{p(1 + \cos\alpha) 2\pi i} \int_{\eta_1 - i\infty}^{\eta_1 + i\infty} F_2 d\eta, \tag{46}$$

where

$$F_2 = \frac{\cosh\left[p(s_T^2 - \eta^2)^{1/2}(y + h)\right] G_-(p\eta)(s_T - \eta)^{1/2} e^{p\eta x}}{G_-(-s_T p \cos\alpha)(s_T + \eta)^{1/2}(\eta + s_T \cos\alpha) \sinh\left[p(s_T^2 - \eta^2)^{1/2} h\right]},$$

and $-s_T < \eta < s_T$, $\eta = \eta_1 + i\eta_2$.

The poles of the integrand in equation (46) consist of a simple pole at $\eta = -s_T \cos\alpha$ and the zeros of $\sinh \sigma h$. The contribution due to the pole at $\eta = -s_T \cos\alpha$ is given by

$$(47) \qquad \overline{w}_{1,1}(x, y, p) = \frac{\overline{g}(p) \cosh\left[(s_T p \sin\alpha)(y+h)\right] e^{-s_T p \cos\alpha x}}{p \sinh(s_T p h \sin\alpha)}.$$

For the contribution at the poles arising from the zeros of $\sinh \sigma h$, we note that

$$(48) \qquad \begin{aligned} \sinh \sigma h &= 0 \Rightarrow \sigma = \frac{in\pi}{h}, \\ \xi^2 &= s_T^2 p^2 + \frac{n^2\pi^2}{h^2} = p_n^2 \\ \xi &= \pm p_n. \end{aligned}$$

Thus, the contribution at $\xi = -p_n$ to equation (45) is

$$(49) \qquad \overline{w}_{1,2}(x, y, p) = \sum_{n=1}^{\infty} \frac{\sin\alpha \overline{g}(p) G_-(-p_n) \cosh\left[\frac{n\pi}{h}(y+h)\right] (s_T p + p_n)^{1/2} e^{-p_n x}}{p(-p_n + s_T p \cos\alpha) G_-(-s_T p \cos\alpha)(1+\cos\alpha)(s_T p - p_n)^{1/2}}.$$

Therefore

$$(50) \qquad \overline{w}_1(x, y, p) = \overline{w}_{1,1}(x, y, p) + \overline{w}_{1,2}(x, y, p).$$

Again, the inverse Laplace transform of the corresponding displacement wave is

$$(51) \qquad \begin{aligned} w_1(x, y, t) &= 2 \sum_{n=1}^{\infty} g\left[\frac{in\pi}{h s_T \sin\alpha}\right] \times \\ &\quad \cos\left[\frac{n\pi}{h}(y+h)\right] e^{\frac{in\pi}{h s_T \sin\alpha}(t - s_T x \cos\alpha)} \end{aligned}$$

It may be noted that this solution corresponds to $\sinh \sigma h = 0$, which is the dispersion relation of the SH-wave travelling in a uniform layer of thickness h with free upper and lower surfaces.

(b) The region $y \geq 0$, $x > 0$.

The transmitted wave in this region is determined by applying the inversion formula to equation (43). The two-sided Laplace inversion formula gives

$$\overline{w}_2(x,y,p)=\frac{1}{2\pi i}\int_{\xi_1-i\infty}^{\xi_1+i\infty}e^{\xi x}\overline{w}_2^*(\xi,y,p)d\xi. \tag{52}$$

For $y\geq 0$, $x>0$,The contour is closed in the lower half plane. Following the Cagniard-de Hoop method, we use the substitution $\xi=\eta p$. This gives

$$\begin{aligned} w_2(x,y,p) &= -\frac{\overline{g}(p)\sin\alpha}{2\pi i p G_-(-s_T p\cos\alpha)(1+\cos\alpha)}\times \\ &\int_{\eta_1-i\infty}^{\eta_1+i\infty}\frac{G_-(p\eta)(s_T-\eta)^{1/2}e^{-p\{(s_T^2-\eta^2)^{1/2}y-\eta x\}}d\eta}{(s_T+\eta)^{1/2}(\eta+s_T\cos\alpha)}. \end{aligned} \tag{53}$$

The contribution from the pole $\eta=-s_T\cos\alpha$ is

$$\overline{w}_2(x,y,p)=-\frac{\overline{g}(p)}{p}e^{-ps_T(x\cos\alpha+y\sin\alpha)}. \tag{54}$$

Again, by the use of the Laplace inversion formula

$$\overline{w}(x,y,t)=-H(t-s_T x\cos\alpha-s_T\sin\alpha)\int_0^{t-s_T x\cos\alpha-s_T y\sin\alpha}g(s)ds. \tag{55}$$

On the other hand, we deform the path of integration for the integrand defined by equation (53) from $R(\eta)=\eta_1$ to a path along which the integral can be recognized as a one-sided Laplace transform. The appropriate path is given by substituting $(s_T^2-\eta^2)^{1/2}y-\eta x=t$ and solving for η to obtain

$$\eta_{T\pm}(r,\theta,t)=-\frac{t}{r}\cos\theta\pm i\left[\frac{t^2}{r^2}-s_T^2\right]^{1/2}\sin\theta, \tag{56}$$

where $r^2=x^2+y^2$ and $\tan\theta=\frac{y}{x}$, respectively. When $s_T r<t<\infty$, equation (55) represents a hyperbola whose points of intersection with the real axis is located in between the branch points $\eta=\pm s_T$. The contribution of the pole $\eta=-s_T\cos\alpha$ has to be taken into account for values of θ in the region $0\leq\theta\leq\alpha$.

The equation (55) is a plane wave of the same form as the incident wave in the zone $0 \leq \theta \leq \alpha$. The integrand along the path defined by equation (53) represents a cylindrical wave. The solution in this region, therefore, agrees with that given by Achenbach [2] in his problem of diffraction in an unbounded medium.

6. CONCLUSION

The transmitted wave has been shown to consist of a guided or channel SH-wave travelling in the layer formed by the free surface $y = -h$ and the crack $y = 0$ in the region $x \geq 0$. As has been pointed out, this wave satisfies the dispersion relation of the SH-wave travelling in a layer of uniform thickness with both the upper and lower surfaces free. From the applications point of view, one is interested in looking at the contribution of this disturbance at the surface of the earth. To obtain this, we put $y = -h$ in the equation (51) to get

$$w_1 = 2 \sum_{n=1}^{\infty} \overline{g} \left(\frac{in\pi}{hs_T \sin\alpha} \right) e^{\frac{in\pi}{hs_T \sin\alpha}(t - s_T x \cos\alpha)} \tag{57}$$

The transmitted wave in the quarter space $x > 0$, $y > 0$, is a cylindrical wave emitting from the edge $x = 0$ of the crack which acts as a source placed at that point. We notice from equation (51) that the transmitted wave in the layer $-h < y < 0$, $x > 0$, will not be possible if h vanishes. This is so because SH-waves are not propagated as surface waves in a half space model with a free surface. All these observations are in agreement with the physical situation of our model.

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Date received September 24, 1998.