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# ON SOME CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. Let $P$ be the class of functions $p$ analytic in the unit disc $E$ with the property $Re\ p(z) > 0, z \in E$. The class $P_k$ is defined as follows. A function $p \in P_k$ if and only if there exist $p_1, p_2 \in P$ such that $p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z), z \in E$ and $k \geq 2$. Using this class $P_k$ some new classes of analytic functions are defined in $E$ and their properties are studied.

## 1. INTRODUCTION

Let $A$ denote the class of functions $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disc $E = \{z : |z| < 1\}$. By $S, K, S^*$ and $C$, we denote the subclasses of $A$ which are, respectively, univalent, close- to-convex, starlike, and convex in $E$. Let $P$ be the class of analytic functions $h : h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ such that $Re\ h(z) > 0$ for $z \in E$.

**Definition 1.1.** Let $b \neq 0$ be a complex number. Then $h \in P(b)$ if and only if, for $z \in E, h(z) = bp(z) + (1-b), p \in P$.

**Definition 1.2.** A function $H : H(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$ is said to belong to the class $P_k(b)$ if and only if there exist $h_1, h_2 \in P(b)$ such that

$$H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z), \quad k \geq 2 \quad \text{and } z \in E$$

We note that $P_2(b) = P(b)$ and $P_2(1) = P$. The class, $P_k(1) = P_k$ has been studied in [6] by Pinchuk.

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**Definition 1.3.** A function $f \in A$ is in $K_k(b), b \neq 0$ (complex), $k \geq 2$, if and only if there exists $g \in C$ such that $\frac{f'(z)}{g'(z)} \in P_k(b)$ for $z \in E$.

For $b = 1, k = 2$ we obtain the well-known class $K$ of close-to-convex functions, see [4]. The class $K_2(b) = K(b)$ of close-to-convex functions of complex order has been studied in [1] by Al-Amiri and Fernando. The convolution (or Hadamard product) of two functions $f, g \in A$ is $f * g = z + \sum_{n=2}^{\infty} a_n b_n z^n$, where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

## MAIN RESULTS

**Theorem 2.1.** *Let $f \in K_k(b)$ and $\phi \in C$. Then $f * \phi \in K_k(b)$ for $z \in E$.*

*Proof.* Since $f \in K_k(b)$, there exists $g \in C$ such that $\frac{f'(z)}{g'(z)} \in P_k(b)$ for $z \in E$.

$$\left[1 + \frac{1}{b}\left(\frac{f'(z)}{g'(z)} - 1\right)\right] \in P_k, \quad z \in E.$$

Since $g * \phi \in C$, see [7], it is sufficient to prove that

$$1 + \frac{1}{b}\left[\left(\frac{(f * \phi)'}{(g * \phi)'}\right) - 1\right] \in P_k, \quad z \in E.$$

Now

$$\begin{aligned} 1 + \frac{1}{b}\left[\frac{(f * \phi)'}{(g * \phi)'} - 1\right] &= \frac{1}{b}\left[\frac{\phi * f'}{\phi * g'}\right] + \left(1 - \frac{1}{b}\right) \\ &= \frac{1}{b}\left[\frac{\phi * zf'}{\phi * zg'}\right] + \left(1 - \frac{1}{b}\right) \\ &= \frac{1}{b}\left[\frac{\phi * \frac{zf'}{g'} g'}{\phi * zg'}\right] + \left(1 - \frac{1}{b}\right) \\ &= \frac{\phi * \left[\frac{1}{b}\left(\frac{f'}{g'} - 1\right) + 1\right] zg'}{\phi * zg'} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\phi * Fzg'}{\phi * zg'} \\
&= \frac{\phi * FG}{\phi * G}, \quad G = zg'
\end{aligned}
$$

where $F \in P_k, G \in S^*$ and $\phi \in C$.

Since $F \in P_k$ we can write $F(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z)$, $h_1, h_2 \in P$. So

$$
\begin{aligned}
\frac{\phi * FG}{\phi * G} &= \frac{\phi * \left[\left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z)\right] G}{\phi * G} \\
\frac{\phi * FG}{\phi * G} &= \frac{\left[\left(\frac{k}{4} + \frac{1}{2}\right) (\phi * h_1(z)) - \left(\frac{k}{4} - \frac{1}{2}\right) (\phi * h_2(z))\right] G}{\phi * G} \\
&= \frac{\left(\frac{k}{4} + \frac{1}{2}\right) (\phi * h_1(z))G - \left(\frac{k}{4} - \frac{1}{2}\right) (\phi * h_2(z))G}{\phi * G}
\end{aligned}
$$

Since $h_1, h_2 \in P, \phi \in C$ and $G \in S^*$ we use the result due to Ruscheweyh and Shiel Small [7], to have $\frac{(\phi * h_1(z))G}{\phi * G}, \frac{(\phi * h_2(z))G}{\phi * G} \in P, \ z \in E$.

This implies that

$$
\frac{\phi * FG}{\phi * G} \in P_k
$$

and so $1 + \frac{1}{b}\left[\frac{(f * \phi)'}{(g * \phi)'} - 1\right] \in P_k$. Hence $\frac{(f * \phi)'}{(g * \phi)'} \in P_k(b)$ and consequently $(\phi * f) \in K_k(b)$

**Corollary 2.1.** *Let $f \in K_k(b)$. Then the class $K_k(b)$ is invariant under the following integral operators;*

*i)* $f_1(z) = \int_0^z \frac{f(\xi)}{\xi} d\xi$

*ii)* $f_2(z) = \frac{2}{z} \int_0^z f(\xi) d\xi$ *(Libera's operator see [5])* *iii)* $f_3(z) = \int_0^z \frac{f(\xi) - f(x)}{\xi - x\xi} d\xi, \qquad |x| = 1, x \neq 1.$

*iv)* $f_4(z) = \frac{1+c}{z^c} \int_0^z \xi^{c-1} f(\xi) d\xi, \quad Re\ c > 0$

*Proof.* We may write, see [2],

$$f_i = f * \phi_i, \qquad i = 1, 2, 3, 4.$$

where $\phi_i$, $1 \leq i \leq 4$, are convex and

$$\begin{aligned} \phi_1(z) &= -\log(1-z) \\ \phi_2(z) &= \frac{-2[z + \log(1-z)]}{z} \\ \phi_3(z) &= \frac{1}{1-x} \log\left[\frac{1-xz}{1-z}\right], \quad |x| = 1, x \neq 1 \\ \phi_4(z) &= \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \quad Re\ c > 0 \end{aligned}$$

The result follows by applying Theorem (2.1).

**Corollary 2.2.** *Let* $f \in K_k(b)$. *Then* $\mu_1(f) = f * h_1(z) = zf'$ *is in* $K_k(b)$ *for* $|z| < 2 - \sqrt{3}$ *and*

$$\mu_2(f) = \frac{(zf)'}{2} = f * h_2(z) \in K_k(b) \quad for\ |z| < \tfrac{1}{2}.$$

*where* $h_1(z) = \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2}$, *and* $h_2(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n = \frac{z - \frac{z^2}{2}}{(1-z)^2}$.

The result follows immediately by applying Theorem (2.1) and taking note of the fact that $h_1 \in C$ for $|z| < 2 - \sqrt{3}$ and $h_2 \in C$ for $|z| < \frac{1}{2}$.

**Theorem 2.2.** *Let* $0 < b_1 < b_2$. *Then* $K_k(b_1) \subset K_k(b_2)$.

*Proof.* Let $f \in K_k(b_1)$. Then there exists $g \in C$ such that $\frac{f'}{g'} = b_1 h + (1 - b_1)$, $h \in P_k$.

Now

$$1+\frac{1}{b_2}\left(\frac{f'}{g'}-1\right)=\frac{b_1}{b_2}h+\left(1-\frac{b_1}{b_2}\right)$$

Since $0<b_1<b_2$, we have $0<\dfrac{b_1}{b_2}<1$. This means

$$0<\left(1-\frac{b_1}{b_2}\right)=\alpha_1<1.$$

It is known that the class $P_k$ is a convex set, so $1+\dfrac{1}{b_2}\left(\dfrac{f'}{g'}-1\right)\in P_k$.

Hence $f\in K_k(b_2)$

**Theorem 2.3.** *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n\in K_k(b)$.

*Then*

$$||a_{n+1}|-|a_n||<\left(\frac{3}{2}\right)ek|b|$$

To prove this we need the following lemma

**Lemma 2.1.** *Let* $p\in P_k(b)$ *in* $E$ *be given by* $p(z)=1+\sum_{n=1}^{\infty}c_nz^n$. *Then*

*i)* $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}|p(re^{i\theta})|^2d\theta\le\frac{1+\{k^2|b|^2-1\}r^2}{1-r^2}$

*ii)* $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}|p'(re^{i\theta})|d\theta\le\frac{k|b|}{1-r^2}$

*Proof.* (i) Using Parseval's identity, we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}|p(re^{i\theta})|^2d\theta &= \sum_{n=0}^{\infty}|c_n|^2r^{2n}\\ &= 1+k^2|b|^2\sum_{n=1}^{\infty}r^{2n},\quad \text{since } |c_n|\le k|b|\end{aligned}$$

$$= \frac{1 + \{k^2|b|^2 - 1\}r^2}{1 - r^2}$$

(ii) We want to show that $\frac{1}{2\pi}\int_0^{2\pi} |p'(re^{i\theta})|d\theta \leq \frac{k|b|}{1 - r^2}$.

We can write, for $h_1(z), h_2(z) \in P$,

$$\begin{aligned} p(z) &= \left(\frac{k}{4} + \frac{1}{2}\right)[(bh_1(z)) + (1 - b)] - \left(\frac{k}{4} - \frac{1}{2}\right)[(bh_2(z)) + (1 - b)] \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)(bh_1(z)) - \left(\frac{k}{4} - \frac{1}{2}\right)(bh_2(z)) + (1 - b) \\ &= b\left[\left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z)\right] + [(1 - b)] \end{aligned}$$

Therefore

(1) $$p'(z) = b\left(\frac{k}{4} + \frac{1}{2}\right)h'_1(z) - b\left(\frac{k}{4} - \frac{1}{2}\right)h'_2(z)$$

Now, for all $h_1(z) \in P$, $i = 1, 2$, it is well known that $h_i(z) = \frac{1 - w(z)}{1 + w(z)}$ and $h_i(z) = \frac{-2w'(z)}{[1 + w(z)]^2}$, where $w(z)$ is a Schwarz function. So

(2) $$\frac{1}{2\pi}\int_0^{2\pi} |h'_i(re^{i\theta})|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{2|w'(re^{i\theta})|}{|1 + w(re^{i\theta})|^2} d\theta \leq \frac{2}{1 - r^2}$$

Hence, from (1) and (2), we have the required result.

We proceed to prove Theorem 2.3. Since $f \in K_k(b)$, there exists $g \in C$ such that

$$f'(z) = g'(z)p(z), \quad p \in P_k(b)$$

Set

$$\begin{aligned} F(z) &= (zf'(z))' \\ &= (zg'(z)p(z))' \\ &= (zg'(z))'p(z) + zg'(z)p'(z) \end{aligned}$$

$$\begin{aligned}
&= g'(z)\left\{\frac{(zg'(z))'}{g'(z)}p(z) + zp'(z)\right\} \\
&= g'(z)[H(z)p(z) + zp'(z)] \\
&= g'(z)[H(z)p(z) + zbh'], \quad p(z) = bh(z) + (1-b), \quad h \in P_k,
\end{aligned}$$

where $H(z) = \dfrac{(zg'(z))'}{g'(z)} \in P$. $g \in C$ implies $zg'(z) = G \in S^*$ and it is univalent, So we can choose a $z_1 = z_1(r)$ with $|z_1| = r$ such that $Q(r) = \max|(z-z_1)G(z)| \leq \frac{2r^2}{1-r^2}$, see [3].

With $z = re^{i\theta}$, we therefore obtain

$$\begin{aligned}
\int_0^{2\pi} |(z-z_1)F(z)|d\theta &\leq r^{-1}Q(r)\left[\frac{1}{2\pi}\right]\int_0^{2\pi} |H(z)p(z)|d\theta \\
&\quad + \left[\frac{r}{2\pi}\right]\int_0^{2\pi} |bh'(z)|d\theta \\
&\leq \frac{2r}{1-r^2}\left[\left(\frac{1+\{k^2|b|^2-1\}_{r^2}}{1-r^2}r^2\right)^{\frac{1}{2}}\left(\frac{1+3r^2}{1-r^2}\right)^{\frac{1}{2}}\right. \\
&\quad \left. + \frac{k|b|}{1-r^2}\right]
\end{aligned}$$

where we have applied the Schwarz inequality and Lemma (2.1) for $P_k(b)$ and $P_2(1)$.

Thus we have

$$\begin{aligned}
&|n^2 a_n - (n+1)^2 a_{n+1} z_1| \\
&\leq \frac{1}{r^{n+1}}\frac{2r}{(1-r^2)^2}[(1+3r^2)^{\frac{1}{2}}\{1+(k^2|b|^2-1)^2r^2\}^{\frac{1}{2}} + k|b|] \\
&\leq \frac{1}{r^n}\frac{1}{2(1-r)^2}[(1+3)^{\frac{1}{2}}\{1+(k^2|b|^2-1)\}^{\frac{1}{2}} + k|b|] \\
&= \frac{1}{r^n}\frac{3k|b|}{2(1-r)^2}
\end{aligned}$$

Setting $r = 1 - \dfrac{1}{n}$, we have $n^2||a_n| - |a_{n+1}|| < \dfrac{3ek|b|}{2}n^2$

This gives us the required result.

For $b = 1, k = 2$ we have

$$||a_n| - |a_{n+1}|| < 3e$$

**Theorem 2.4.** *Let $f \in A$ and $\dfrac{f'(z)}{g'(z)} \in P_k(b)$ for $g \in C$, $z \in E$. Then $f$ is close-to-convex of complex order $b$ for $|z| < r_k = \frac{1}{2}[k - \sqrt{k^2 - 4}]$.*

*Proof.* We can write

$$\begin{aligned} \frac{f'(z)}{g'(z)} &= p(z) \in P_k(b), \quad g \in C \\ &= bh(z) + (1-b), \quad h \in P_k. \end{aligned}$$

We notice that $h \in P_k$ implies that $h \in P$ for $|z| < r_k$, see [6]. Using this fact our result follows immediately.

**Theorem 2.5.** *Let $f(z) = (1-\lambda)F(z) + \lambda zF'(z), g(z) = (1-\lambda)G(z) + \lambda zG'(z)$ and let $\dfrac{(zF')'}{G'} \in P_k(b)$, $G \in S^*$. Then $\dfrac{(zf')'}{g'} \in P_k(b)$ for $|z| < r_\lambda$, where*

$$r_\lambda = \frac{1}{3\lambda + \sqrt{9\lambda^2 - 2\lambda + 1}}$$

*Proof.* We can write $F(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-2} f(t)dt$, and so

$$\begin{aligned} zF'(z) &= \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \left[ \left(1 - \frac{1}{\lambda}\right) \int_0^z t^{\frac{1}{\lambda}-2} f(t)dt + z^{\frac{1}{\lambda}-1} f(z) \right] \\ &= \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \left( \int_0^z t^{\frac{1}{\lambda}-1} f'(t)dt \right) \end{aligned}$$

Thus $\dfrac{(zF'(z))'}{G'(z)} = \dfrac{[z^{\frac{1}{\lambda}} f'(z) - \left(\frac{1}{\lambda} - 1\right) \int_0^z t^{\frac{1}{\lambda}-1} f'(t)dt]}{\int_0^z t^{\frac{1}{\lambda}-1} g'(t)dt} = h(z), \quad h \in P_k(b).$

So $z^{\frac{1}{\lambda}} f'(z) - \left(\frac{1}{\lambda} - 1\right) \int_0^z t^{\frac{1}{\lambda}-1} f'(t)dt = h(z) \int_0^z t^{\frac{1}{\lambda}-1} g'(t)dt$. Differentiating both sides and simplifying, we obtain

$$(2.1) \qquad \frac{(zf'(z))'}{g'(z)} = h(z) + h'(z) \frac{\int_0^z t^{\frac{1}{\lambda}-1} g'(t)dt}{z^{\frac{1}{\lambda}-1} g'(z)}$$

For $h \in P_k(b)$, we can write

$$h(z) = bp(z) + (1-b), \quad p \in P_k.$$

Using this, (2.1) becomes

$$(2.2) \qquad \begin{aligned} 1 + \frac{1}{b}\left\{\frac{(zf'(z))'}{g'(z)} - 1\right\} &= p(z) + p'(z) \frac{\int_0^z t^{\frac{1}{\lambda}-1} g'(t)dt}{z^{\frac{1}{\lambda}-1} g'(z)} \\ &= p(z) + \frac{zp'(z)}{H(z)} \end{aligned}$$

Now

$$H(z) = \frac{z^{\frac{1}{\lambda}} g'(z)}{\int_0^z t^{\frac{1}{\lambda}-1} g'(t)dt} = \left(\frac{1}{\lambda} - 1\right) + \frac{(zG'(z))'}{G'(z)},$$

and it follows that

$$(2.3) \qquad \begin{aligned} \left|\frac{z^{\frac{1}{\lambda}} g'(z)}{\int_0^z t^{\frac{1}{\lambda}-1} g'(t)dt}\right| &\geq Re \frac{(zG'(z))'}{G'(z)} + \left(\frac{1}{\lambda} - 1\right) \\ &\geq \left(\frac{1}{\lambda} - 1\right) + \frac{1 - 4r + r^2}{1 - r^2} \end{aligned}$$

Since $G \in S^*$, see [5], then $H \in P$ for $|z| < r_\lambda$. Also, for $p \in P_k$, we can write

$$(2.4) \qquad \begin{aligned} p(z) + \frac{zp'(z)}{H(z)} &= \left(\frac{k}{4} + \frac{1}{2}\right)\left[p_1(z) + \frac{zp_1'(z)}{H(z)}\right] \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right)\left[p_2(z) + \frac{zp_2'(z)}{H(z)}\right] \end{aligned}$$

where $p_1, p_2 \in P$.

Now for $i = 1, 2$

$$Re\left[p_i(z) + \frac{zp_i'(z)}{H(z)}\right] \geq Re\, p_i(z)\left[1 - \frac{2r}{1-r^2}\frac{\lambda(1-r^2)}{1-4\lambda r+(2\lambda-1)r^2}\right]$$

$$= Re\, p_i(z)\left[\frac{1-4\lambda r+(2\lambda-1)r^2-2\lambda r}{1-4\lambda r+(2\lambda-1)r^2}\right] \tag{2.5}$$

The right-hand side of (2.5) is positive for

$$|z| < r = \frac{1}{3\lambda \pm \sqrt{9\lambda^2 - 2\lambda + 1}}$$

From (2.4) and (2.5) the result follows.

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