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IMPROVING ESTIMATES IN REGULAR GROUP DIVISIBLE DESIGNS

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ABSTRACT. The general approach of improving estimates in PBIB designs established in Kanjo (1991b) is applied to the case of group divisible designs, and a general method of improving treatment estimates in this class of designs is made available.

1. INTRODUCTION

The problem of utilization of between block information for improving the estimation of treatment effects was first tackled by Yates (1940) for special designs and then by Rao (1947) for general incomplete block designs. During the past three decades the properties of estimates obtained by these procedures have been investigated by several authors. A review of many of these procedures is given by Shah (1975) who provided, again, an update review in (1992). In this paper, a general method of improving treatment estimates in a special class of PBIB designs called "Regular Group Divisible Designs" is suggested. It enjoys two properties (i) it is applicable in regular GD designs with 7 treatments or more. (ii) No method of improving treatments estimates in PBIB designs gives an answer to the question of how much we have improved? Here a lower bound of the recovery achieved is always given.

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2. DEFINITIONS AND SOME USEFUL RELATIONS

Consider a P.B.I.B. experiment with b blocks of k units each, and $v(>k)$ treatments each of which occurs in r blocks, and each treatment has exactly n_i i-th associates $(i=1,2)$. It is clear that

$$vr = bk,\ n_1+n_2 = v-1,\ n_1\lambda_1+n_2\lambda_2 = r(k-1), \tag{2.1}$$

Bose, et al. (1954), defines the constants Δ, H, c_1, c_2 as

$$k^2\Delta = (a+\lambda_1)(a+\lambda_2)+(\lambda_1-\lambda_2)[a(f-g)+f\lambda_2-g\lambda_1], \tag{2.2}$$

$$kH = (2a+\lambda_1+\lambda_2)+(f-g)(\lambda_1-\lambda_2), \tag{2.3}$$

$$k\Delta c_1 = \lambda_1(a+\lambda_2)+(\lambda_1-\lambda_2)(f\lambda_2-g\lambda_1), \tag{2.4}$$

$$k\Delta c_2 = \lambda_2(a+\lambda_1)+(\lambda_1-\lambda_2)(f\lambda_2-g\lambda_1), \tag{2.5}$$

where

$$a = r(k-1),\ f = p^1_{12},\ g = p^2_{12}. \tag{2.6}$$

Given any pair of treatments which are i-th associate, p^i_{jk} represents the number of treatments which are j-th associate to one treatment of the pair and k-th associate to the other.

In the intra-block analysis, the best linear estimate $\hat{t}_i$ of the treatment effect t_i is given by, (see Bose et al. (1954))

$$\hat{t}_i = [(k-c_2)/a]Q_i + [(c_1-c_2)/a]S_1(Q_i), \tag{2.7}$$

In the inter-block analysis Zelen (1957) gives the best linear estimates $\hat{t}'_i$ of the treatment effect t_i as:

$$\hat{t}'_i = [(k - c'_2)/r]Q'_i + [(c'_1 - c'_2)/r]S_1(Q'_i), \tag{2.8}$$

where

$$Q'_i = T_i - Q_i - \frac{rG}{N} \text{ and } c'_j = \frac{c_j\Delta - r\lambda_j}{\Delta - rH + r^2}; \ (j = 1, 2). \tag{2.9}$$

where Q_i denotes the adjusted yield for the i-th treatment; $S_1(Q_i)$ denotes the sum of the adjusted yields for all the first associates of the i-th treatment, and G denotes the grand total.

Consider the restriction $\sum_{i=1}^{v} \hat{t}_i = 0$, and let $V_1 = var(\hat{t}_i)$, $i = 1, 2, ..., v$; $C_1 = cov(\hat{t}_i, \hat{t}_{i1}); C_2 = cov(\hat{t}_i, \hat{t}_{i2}); V'_1, C'_1, C'_2$ the inter-analysis counter part, where t_{ij} denotes a j-th associate of t_i. It can be shown that:

(2.10a)
$$\begin{aligned} V_1 &= [k(v-1) - n_1c_1 - n_2c_2]\sigma^2/av; \\ V'_1 &= [k(v-1) - n_1c'_1 - n_2c'_2]\sigma'^2/vr, \end{aligned}$$

(2.10b)
$$\begin{aligned} C_1 &= [c_1(n_2+1) - n_2c_2 - k]\sigma^2/av; \\ C'_1 &= [c'_1(n_2+1) - n_2c'_2 - k]\sigma'^2/vr, \end{aligned}$$

(2.10c)
$$\begin{aligned} C_2 &= [c_2(n_1+1) - n_1c_1 - k]\sigma^2/av; \\ C'_2 &= [(n_1+1)c'_2 - n_1c'_1 - k]\sigma'^2/vr, \end{aligned}$$

$\sigma'^2 = \sigma^2 + k\sigma_b^2$, where σ^2 is the error variance in the intra-model and σ_b^2 is the variance of the block effect in the inter-model. Let,

$$V = V_1 + V'_1, \ C = C_1 + C'_1, \ C = C_2 + C'_2 \tag{2.11}$$

It can be shown that,

$$C_1 - C_2 = \frac{\lambda_1 - \lambda_2}{k\Delta}\sigma^2; \ C'_1 - C'_2 = \frac{\Delta(c_1 - c_2) - r(\lambda_1 - \lambda_2)}{r(\Delta - rH + r^2)}\sigma'^2 \tag{2.12}$$

$$(2.13)\ C\ -C' = (\lambda_1 - \lambda_2)\left[\frac{-r(h-r)}{k\Delta(\Delta - rH + r^2)}\sigma 2 - \frac{1}{\Delta - rH + r^2}\sigma_b^2\right].$$

Now, if $\Delta > 0$, $\Delta - rH + r^2 > 0$, and $H > r$, then:

$$C - C' > 0, \text{ if } \lambda_1 < \lambda_2,$$
$$C - C' < 0, \text{ if } \lambda 1 > \lambda_2,$$

It is to be noticed that the above conditions are satisfied in every design listed in Bose et al. (1954).

Consider now t independent parameters $\tau_1, \ldots, \tau_t$ and suppose that for each τ_i there exists two independent unbiased estimates U_i and X_i, where U_i is distributed as $N(\tau_i, V_i' = \theta_i\sigma^2)$ and X_i is distributed as $N(\tau_i, V_i > V_i)$. Suppose also that independently of the X_i's and U_i's there exists an unbiased estimator S^2 for σ^2 where S^2 is distributed as $(\sigma^2/e)\chi^2(e)$. It was shown in Kanjo (1991b) that there exists a constant B such that the unbiased estimator:

$$\hat{\tau}_i = U_i + \frac{\theta_i B S^2}{\sum\limits_{j=1, j\neq i}^{t} (X_j - U_j)^2}(X_i - U_i).$$

has a variance less than $var(U_i)$ whenever $t > 5$.

This result was applied to P.B.I.B designs. A specific relations giving optimum value of B and a lower bound of improvement achieved, were determined. In section 3 formulation of the problem in terms of group divisible class of designs is given and some useful relations are derived. In section 4 results for regular group divisible designs have been established.

3. A FORMULATION OF THE PROBLEM IN TERMS OF REGULAR GROUP DIVISIBLE DESIGNS

3.1. Reparametrization and some relations.

In group divisible designs we have $v = mn$, and the treatments can be divided into m groups of n treatments each, such that any two treatments of the same group are first associates, while two treatments of different groups are second associates. We have also, $n_1 = n - 1; n_2 = n(m - 1)$; $f = 0$, $g = n - 1$. Clearly:

$$(3.1) \qquad \begin{aligned} &(n-1)\lambda_1 + n(m-1)\lambda_2 = r(k-1), \text{ or} \\ &rk - \lambda_2 v = r - \lambda_1 + n(\lambda_1 - \lambda_2). \end{aligned}$$

In regular group divisible (GD) designs:

$$(3.2) \qquad r > \lambda_1, \qquad rk - \lambda_2 v > 0.$$

Consider now the $mn \times mn$ matrix M_1 defined as follows:

$$
M_i = \left[\begin{array}{r|ccccc|ccccc|ccccc|c|ccccc|ccccc|}
 & \multicolumn{5}{c}{1} & \multicolumn{5}{c}{2} & \multicolumn{5}{c}{3} & & \multicolumn{5}{c}{m-1} & \multicolumn{5}{c}{m} \\
 & 1 & 2 & 3 & \cdots & n & 1 & 2 & 3 & \cdots & n & 1 & 2 & 3 & \cdots & n & \cdots & 1 & 2 & 3 & \cdots & n & 1 & 2 & 3 & \cdots & n \\
\hline
1 & 1 & -1 & & & & & & & & & & & & & & & & & & & & & & & & \\
2 & 1 & 1 & -2 & & & & & & & & & & & & & & & & & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & & & & & & & & & & & & & & & & & & & & & & \\
n-1 & 1 & 1 & 1 & \cdots & (-n+1) & & & & & & & & & & & & & & & & & & & & & \\
\hline
n & & & & & & 1 & -1 & & & & & & & & & & & & & & & & & & & \\
\vdots & & & & & & 1 & 1 & -2 & & & & & & & & & & & & & & & & & & \\
\vdots & & & & & & \vdots & \vdots & \vdots & \ddots & & & & & & & & & & & & & & & & & \\
2n-2 & & & & & & 1 & 1 & 1 & \cdots & (-n+1) & & & & & & & & & & & & & & & & \\
\hline
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \\
\hline
\vdots & & & & & & & & & & & & & & & & & 1 & -1 & & & & & & & & \\
\vdots & & & & & & & & & & & & & & & & & 1 & 1 & -2 & & & & & & & \\
\vdots & & & & & & & & & & & & & & & & & \vdots & \vdots & \vdots & \ddots & & & & & & \\
\vdots & & & & & & & & & & & & & & & & & 1 & 1 & 1 & \cdots & (-n+1) & & & & & \\
\hline
\vdots & & & & & & & & & & & & & & & & & & & & & & 1 & -1 & & & \\
\vdots & & & & & & & & & & & & & & & & & & & & & & 1 & 1 & -2 & & \\
\vdots & & & & & & & & & & & & & & & & & & & & & & \vdots & \vdots & \vdots & \ddots & \\
m(n-1) & & & & & & & & & & & & & & & & & & & & & & 1 & 1 & 1 & \cdots & (-n+1) \\
\hline
m(n-1)+1 & 1 & 1 & 1 & \cdots & 1 & -1 & -1 & -1 & \cdots & -1 & & & & & & & & & & & & & & & & \\
\vdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & -2 & -2 & -2 & \cdots & -2 & & & & & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & & & \vdots & \cdots & & \vdots & \vdots & \vdots & \cdots & \\
mn-2 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & \cdots & (-m+2) & (-m+2) & \cdot & \cdots & (-m+2) & & & & & \\
mn-1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & (-m+1) & \cdot & \cdot & \cdots & (-m+1) \\
mn & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & \cdot & \cdot & \cdots & 1
\end{array}\right]
$$

Consider the following contrasts between the mn intra-block estimates of the treatments $\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_v$:

$$\underline{U} = (U_1, U_2, \ldots, U_{mn-1}, 0)' = M(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{nm})' = M\hat{\underline{t}}.$$

where M is the same as M_1 after normalizing its rows. Define similarly the contrasts between the inter-block estimates of the treatments $\hat{t}'_1, \hat{t}'_2, \ldots, \hat{t}'_v$, i.e. $\underline{X} = M\hat{\underline{t}}'$.

The problem now is to combine $\underline{U}$ and $\underline{X}$ to get new estimates $\hat{\tau} = (\hat{\tau}_1, \ldots, \hat{\tau}_{nm})'$. It should be noted that $U_{nm} = X_{nm} = 0$, and the corresponding combined estimate is assumed to be zero, i.e. $\hat{\tau}_{nm} = 0$.

One notices that U_i is uncorrelated with U_j for $i \neq j$, also X_i is uncorrelated with X_j for $i \neq j$, and the U's are independent of the X's.

It should also be noted that in the vector $\hat{\underline{t}}$ or $\hat{\underline{t}}'$, the treatments are grouped according to group divisible association plan, moreover, the contrasts included in the matrix M are of two types, within group contrasts and among group contrasts. It can be shown that the variance of any within group contrast is $V_1 - C_1$, and the variance of any among group contrast is $V_1 - C_1 + n(C_1 - C_2)$.

Similarly one can see that the variance of X_i is $V'_1 - C'_1$ or $V'_1 - C'_1 + n(C'_1 - C'_2)$ according as X_i represents a within contrast or among contrast respectively. Consider now the vector:

$$\underline{Z} = (Z_1, Z_2, \ldots, Z_{mn-1})' = \underline{X} - \underline{U}, \tag{3.3}$$

Z_i and Z_j are independent for $i \neq j$. For within group Z's, one has,

$$V(Z_i) = V(X_i) + V(U_i) = (V_1 + V'_1) - (C_1 + C'_1) = V - C. \tag{3.4}$$

For among group Z's, one has,

$$\begin{aligned} V(Z_i) &= V_1 - C_1 + n(C_1 - C_2) + V'_1 - C'_1 + n(C'_1 - C'_2) \\ &= V - C + n(C - C') \end{aligned} \tag{3.5}$$

where V, C, C' are defined in (2.10a) through (2.11). From (2.14) it is seen that,

$$V - C > (V - C) + n(C - C'), \text{ if } \lambda_1 > \lambda_2, \tag{3.6a}$$

and

(3.6b) $$V - C < (V - C) + n(C - C'), \text{ if } \lambda_1 < \lambda_2,$$

This will give rise to two divisions of the problem according to whether $\lambda_1 > \lambda_2$ or $\lambda_1 < \lambda_2$.

3.2. Study of the ratio $(V - C)/[V - C + n(C - C')]$.

From (2.10a) through (2.10c) one has:

(3.7a) $$V - C = (k - c_1)\sigma^2/a > +(k - c_1')\sigma'^2/r,$$

(3.7b) $$V - C + n(C - C') = \left[\frac{k - c_1}{a} + \frac{n(\lambda_1 - \lambda_2)}{k\Delta}\right]\sigma^2 + \left[\frac{k - c_1'}{r} - \frac{n(\lambda_1 - \lambda_2)}{k(\Delta - rH + r^2)}\right]\sigma'^2$$

putting $f = 0,\ g = n - 1$ in (2.2) through (2.5) one gets:

(3.8a) $$k^2\Delta = \lambda_2 v(a + \lambda_1); \quad kH = a + \lambda_1 + \lambda_2 v; k\Delta c_1 = \lambda_1\lambda_2 v,$$

(3.8b) $$c_1 - c_2 = [ak(\lambda_1 - \lambda_2)]/[\lambda_2 v(a + \lambda_1)]; \quad (k - c_1)/a = k/(a + \lambda_1).$$

Let $A = \Delta - rH + r^2$, then $kA = k\Delta - rkH + r^2k$, using (3.8a) and (2.6) one obtains:

(3.9) $$kA = (r - \lambda_1)(rk - \lambda_2 v)/k.$$

Since $r > \lambda_1,\ rk > \lambda_2 v$ in regular GD., one can say that $A = \Delta - rH + r^2 > 0$ in every regular GD design. Also from (2.9) using (3.8a) and (3.9) one obtains:

(3.10) $$(k - c_1')/r = k/(r - \lambda_1).$$

Substituting from (3.8b) and (3.10) into (3.7a), one can write,

(3.11) $$V - C = k\sigma^2/(a + \lambda_1) + k\sigma'^2/(r - \lambda_1).$$

From (3.1), using (3.8a), one obtains:

$$n(\lambda_1 - \lambda_2)/k\Delta = k/\lambda_2 v - k/(a + \lambda_1) \tag{3.12}$$

and using (3.9), one obtains:

$$n(\lambda_1 - \lambda_2)/kA = k/(r - \lambda_1) - k/(rk - \lambda_2 v). \tag{3.13}$$

Substituting from (3.8b), (3.10), (3.12) and (3.13) into (3.7b), one can write:

$$V - C + n(C - C') = k\sigma^2/\lambda_2 v + k\sigma'^2/(rk - \lambda_2 v). \tag{3.14}$$

Since $\sigma'^2 = \sigma^2 + k\sigma_b^2$, if one puts $\sigma_b^2/\sigma^2 = R > 1$, then:

$$\frac{V - C}{V - C + n(C - C')} = \frac{\dfrac{r}{(r - \lambda_1)(a + \lambda_1)} + \dfrac{1}{r - \lambda_1}R}{\dfrac{r}{\lambda_2 v(rk - \lambda_2 v)} + \dfrac{1}{rk - \lambda_2 v}R} = \frac{\alpha_1 + \beta_1 R}{\alpha_2 + \beta_2 R} = F(R). \tag{3.15}$$

Since $r - \lambda_1 > 0$, $rk - \lambda_2 v > 0$ in regular GD, one can say that,

$$\frac{dF(R)}{dR} = \begin{cases} > 0 & \text{if} \quad \lambda_1 > \lambda_2, \\ < 0 & \text{if} \quad \lambda_1 < \lambda_2. \end{cases} \tag{3.16}$$

so that,

$$F(1) < F(R) < F(\infty), \quad \text{when} \quad \lambda_1 > \lambda_2, \tag{3.17a}$$

$$F(\infty) < F(R) < F(1), \quad \text{when} \quad \lambda_1 < \lambda_2. \tag{3.17b}$$

where,

$$F(1) = \frac{\alpha_1 + \beta_1}{\alpha_2 + \beta_2} = \frac{\lambda_2 v(rk - \lambda_2 v)(rk + \lambda_1)}{(r - \lambda_1)(a + \lambda_1)(r + \lambda_2 v)}, \tag{3.18a}$$

$$F(\infty) = \beta_1/\beta_2 = (rk - \lambda_2 v)/(r - \lambda_1). \tag{3.18b}$$

For the inverse ratio $G(R) = 1/F(R)$, one has:

$$G(\infty) < G(R) < G(1), \quad \text{when} \quad \lambda_1 > \lambda_2, \tag{3.19a}$$

(3.19b) $$G(1) < G(R) < G(\infty), \quad \text{when} \quad \lambda_1 < \lambda_2,$$

where $G(1) = 1/F(1)$ and $G(\infty) = 1/F(\infty)$.

4. APPLICATION TO REGULAR GROUP DIVISIBLE DESIGNS

The results obtained in Kanjo (1991b) are exactly what is needed here where $\phi_1 = V - C, \quad \phi_2 = V - C + n(C - C')$.

For within comparisons with variance $\phi_1 = V - C$, one has $\nu_1 = m(n-1) - 1 = v - m - 1, \quad \nu_2 = m - 1$, ($m$ should be ≥ 2) thus $L < \phi_2/\phi_1 < P$ if $\lambda_1 > \lambda_2$, and $P < \phi_2/\phi_1 < L$ if $\lambda_1 < \lambda_2$, where

(4.1) $$L = \frac{r - \lambda_1}{rk - \lambda_2 v'}, \quad P = \frac{(r - \lambda_1)(a + \lambda_1)(r + \lambda_2 v)}{\lambda_2 v(rk - \lambda_2 v)(rk + \lambda_1)}.$$

For among comparisons with variance $\phi_2 = V - C + n(C - C')$, one has $\nu_1 = m(n-1) = v - m, \nu_2 = m - 2$, ($m$ should be ≥ 3), thus $1/P < \phi_2/\phi_1 < 1/L$ if $\lambda_1 > \lambda_2$ and $1/L < \phi_2/\phi_1 < 1/P$ if $\lambda_1 < \lambda_2$.

The combined estimate is given in (2.15) where θ_i is the coefficient of σ^2 in $V(U_i) = V_1 - C_1$. Using (2.10a), (2.10b), and (3.8b) one gets $\theta_i = k/(a + \lambda_1)$, and the combined estimate for within comparisons is:

(4.2) $$\hat{\tau}_i = U_i + \frac{kBS^2}{(a + \lambda_1)\sum\limits_{\substack{j=1 \\ j \neq i}}^{v-1}(X_j - U_j)^2}(X_i - U_i),$$

For among comparisons, one has $V(U_i) = V_1 - C_1 + n(C_1 - C_2)$, using (2.10a), (2.10b), (2.12), (3.8b) and (3.12):

(4.3) $$V(U_i) = \left(\frac{k - c_1}{a} + \frac{n(\lambda_1 - \lambda_2)}{k\Delta}\right)\sigma^2 = \frac{k}{\lambda_2 v}\sigma^2$$

hence $\theta_i = k/\lambda_2 v$ and the combined estimate is:

(4.4) $$\hat{\tau}_i = U_i + \frac{kBS^2}{\lambda_2 v \sum\limits_{\substack{j=1 \\ j \neq i}}^{v-1}(X_j - U_j)^2}(X_i - U_i),$$

The combining constant B_w and a conservative lower bound of the improvement ratio D_w, for within comparisons, and B_A and D_A for among comparisons have been computed for 62 regular GD designs that appear in Bose et al. (1954). The results are listed in table 1.

5. GENERAL PROCEDURE FOR IMPROVING ESTIMATES IN REGULAR GROUP DIVISIBLE DESIGNS

Compute:

1. $\hat{t}'_i - \hat{t}_i, i = 1, 2, \ldots, v$. where $\hat{t}'_i$ and $\hat{t}_i$ are the inter and intra-estimate of t_i respectively.

2. $\underline{X} = \underline{U} = M(\hat{\underline{t}}' - \hat{\underline{t}})$; where M is the orthogonal matrix defined in section 3.1.

3. $\sum_{j \neq i}^{v-1} (X_j - U_j)^2 = \sum_{j=1}^{v-1} (X_j - U_j)^2 - (X_i - U_i)^2$.

4. $J = kB_w S^2/(a + \lambda_1) \sum_{j \neq i}^{v-1} (X_j - U_j)^2$, for within contrasts and
   $J = kB_A S^2/\lambda_2 v \sum_{j \neq i}^{v-1} (X_j - U_j)^2$ for among contrasts; B_W and B_A are constants to be taken from table 1 and S^2 is the error mean square in the ANOVA table.

5. The combined estimate of U_i and X_i,

$$\hat{\tau}_i = U_i + J(X_i - U_i), \quad i = 1, 2, \ldots, v - 1.$$

If the combined estimates of the treatment effects t_i's are desired, one should compute:

$$\underline{T} = (T_1, \ldots, T_v)' = M'(\hat{\tau}'_1, \hat{\tau}'_2, \ldots, \hat{\tau}'_{v-1}, 0)' = M'\hat{\underline{\tau}}$$

where M' is the transpose of M.

Table 1

*Design No.	R_5	R_6	R_8	R_9	R_{10}	R_{11}	R_{12}	R_{13}
v	8	8	8	9	9	9	9	9
B_W	2.57	2.99	1.63	2.23	2.02	3.35	2.54	3.79
D_W	0.34	0.40	0.46	0.52	0.50	0.53	0.56	0.52
B_A	0.75	0.87	11.78	4.24	5.31	1.50	3.74	1.07
D_A	0.35	0.40	0.40	0.46	0.40	0.51	0.53	0.50

*Design No.	R_{14}	R_{15}	R_{16}	R_{17}	R_{18}	R_{20}	R_{21}	R_{22}
v	10	12	12	12	12	12	12	12
B_W	7.57	3.95	6.87	4.58	5.08	7.82	7.14	6.44
D_W	0.55	0.66	0.65	0.68	0.69	0.64	0.68	0.70
B_A	1.65	8.17	4.05	6.86	8.20	2.34	4.21	4.10
D_A	0.56	0.65	0.65	0.66	0.60	0.62	0.68	0.68

*Design No.	R_{23}	R_{24}	R_{25}	R_{26}	R_{27}	R_{28}	R_{29}	R_{30}
v	12	14	14	14	15	15	15	15
B_W	6.71	10.40	8.95	10.81	13.52	10.35	7.66	14.02
D_W	0.71	0.70	0.72	0.73	0.68	0.73	0.77	0.71
B_A	3.08	5.03	5.93	5.23	3.05	4.93	10.91	3.17
D_A	0.70	0.71	0.72	0.74	0.68	0.70	0.72	0.71

*Design No.	R_{31}	R_{32}	R_{33}	R_{34}	R_{35}	R_{36}	R_{37}	R_{38}
v	15	15	15	15	16	16	16	16
B_W	10.89	8.05	7.08	10.22	12.06	8.43	7.98	8.82
D_W	0.75	0.78	0.75	0.79	0.73	0.79	0.78	0.79
B_A	2.73	12.04	12.62	6.15	3.64	16.04	21.19	12.67
D_A	0.73	0.66	0.65	0.78	0.68	0.72	0.66	0.71

*Design No.	R_{39}	R_{40}	R_{41}	R_{42}	R_{43}	R_{44}	R_{45}	R_{46}
v	16	18	18	20	20	21	24	24
B_W	11.38	13.04	10.24	15.08	12.05	16.31	28.73	21.04
D_W	0.79	0.80	0.81	0.83	0.84	0.83	0.80	0.85
B_A	5.35	9.82	13.04	11.79	27.58	10.67	5.28	11.77
D_A	0.77	0.80	0.80	0.83	0.70	0.81	0.80	0.84

*Design No.	R_{47}	R_{48}	R_{49}	R_{50}	R_{51}	R_{52}	R_{53}	R_{54}
v	24	24	24	24	25	25	26	27
B_W	20.74	19.57	16.82	29.18	16.86	16.22	22.42	23.93
D_W	0.83	0.85	0.88	0.81	0.88	0.87	0.87	0.88
B_A	6.30	11.34	29.88	5.37	32.42	44.30	16.82	14.73
D_A	0.76	0.81	0.77	0.81	0.82	0.77	0.88	0.86

*Design No.	R_{55}	R_{56}	R_{57}	R_{58}	R_{59}	R_{60}	R_{61}	R_{62}
v	28	28	30	33	35	39	40	45
B_W	25.65	19.90	21.20	25.36	26.98	29.69	39.32	44.96
D_W	0.88	0.89	0.90	0.92	0.92	0.92	0.91	0.93
B_A	12.42	28.77	26.41	76.26	47.97	40.38	21.45	21.98
D_A	0.85	0.86	0.90	0.84	0.86	0.91	0.91	0.91

*Design No.	R_{63}	R_{64}	R_{65}	R_{66}	R_{67}	R_{68}
v	48	49	49	63	64	80
B_W	71.06	39.85	38.85	97.99	54.37	128.51
D_W	0.88	0.95	0.96	0.91	0.96	1.00
B_A	9.54	77.83	110.29	11.60	106.69	13.68
D_A	0.89	0.91	0.88	0.91	0.94	0.92

* Design serial number as it appears in Bose et al (1954).

REFERENCES

1. R.C. Bose, W.H. Clatworthy and S.S. Shrikhande, *Technical Bulletin No. 107,* North Carolina Agricultural Experiment station, 1954.

2. A.I. Kanjo, *Recovery of Inter-block Information. Unpublished Ph.D. Dissertation,* Virginia Polytechnic Institute (Blacksburg, VA), 1965..

3. A.I. Kanjo, *Improving Estimates in B.I.B. Designs,* Statist.Prob. Lett., **12** (1991a), 167-174.

4. A.I. Kanjo, *Improving Estimates in P.B.I.B. Designs,* Statist. Prob. Lett. **12**(1991b), 175-181.

5. C.R. Rao, *General methods of analysis for incomplete block designs,* J. Amer. Statist. Assoc. **42**(1947), 541-561.

6. K.R. Shah, *Analysis of Block Designs: A Review Article,* Coujarot Statistical Review **2**(1975), 1-11.

7. K.R. Shah, *Recovery of interblock information: on update,* J. Statist. Plann. Inference **30**(1992), 163-172.

8. M. Zelen, *The analysis of incomplete block designs,* J. Amer. Statist. Assoc., **52**(1957), 204-217.

9. F. Yates, *The recovery of inter-block information in balanced incomplete block designs,* Ann. Eugenics **10**(1940), 317-325.

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