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ASYMPTOTIC TEST FOR MONOTONE VARIANCE RESIDUAL LIFE

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ABSTRACT. In this paper we give an outline of the main properties of decreasing (increasing) variance remaining life distributions, DVRL (IVRL). The connections of this class with DMRL and equilibrium distributions are displayed. An asymptotic test for exponentiality against DVRL (IVRL) class is developed. It can be used to test exponentiality against DMRL (IMRL) property of the associated class of renewal distributions.

1. INTRODUCTION

Let T, denote the life time of an equipment with distribution function $F(t) = P(T \leq t)$ and survival function $\bar{F}(t) = P(T > t)$, where $\bar{F}(O) = 1$. We assume that the density $f(t)$ can be obtained by differentiating $F(t)$. The failure rate $r(t) = f(t)/\bar{F}(t)$. The mean life $\mu = \int_0^\infty \bar{F}(t)dt$ and the variance life $\sigma^2 = \text{var } (T)$, will be assumed finite. It is well known that $\mu(t) = E(T - t|T > t) = \int_t^\infty \bar{F}(x)dx/\bar{F}(t)$, and $\sigma^2(t)$ will denote var $(T - t|T > t) = \text{var } (T|T > t)$.

A distribution function F is said to have decreasing (increasing) variance residual life DVRL, (IVRL) if $\sigma^2(t)$ is nonincreasing (nondecreasing) function of t on $(0,\infty)$. Launer [9]; Gupta [4]; Gupta, Kirmani and Launer (5); studied characterization of this class and used it to find better bounds on moments and survival functions. In section 2 below we outline some known results for the convenience of reader, although some of them are derived in a new different way. In section 3, some preliminary work is presented and in section 4 an asymptotic test for exponentiality against DVRL (IVRL) class is derived and an example is given.

2. CHARACTERIZATION OF DVRL(IVRL) AND SOME USEFUL RESULTS

(a) Consider $E(u^2|t) = -\int_0^\infty u^2 d\bar{F}(u|t)$. Integrating by parts we have,

$$\sigma^2(t) + \mu^2(t) = E(u^2|t) = 2\int_0^\infty u[\bar{F}(u+t)/\bar{F}(t)]du.$$

Let $u + t = x$, then,

$$\begin{aligned}
\sigma^2(t) + \mu^2(t) &= [2/\bar{F}(t)]\int_t^\infty (x-t)\bar{F}(x)dx \\
&= [2/\bar{F}(t)]\int_t^\infty \int_t^x \bar{F}(x)dydx \\
&= [2/\bar{F}(t)]\int_t^\infty \int_y^\infty \bar{F}(x)dxdy. \qquad (2.1)
\end{aligned}$$

(b) As in Hall and Wellner [6], let

$$\bar{F}^{(r)}(t) = \int_t^\infty \bar{F}^{(r-1)}(x)dx, \quad r = 1, 2, \ldots; \quad \bar{F}^{(0)} = \bar{F} = 1 - F. \qquad (2.2)$$

and let $\phi_r(t) = r!\bar{F}^{(r)}(t)/\bar{F}(t)$. Then from (2.1) and (2.2) we have

$$\begin{aligned}
\phi_1(t) = \mu(t); \phi_2(t) &= [2/\bar{F}(t)]\int_t^\infty \int_y^\infty \bar{F}(x)dxdy \\
&= \sigma^2(t) + \mu^2(t). \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
\sigma^2(t) &= [2/\bar{F}(t)]\int_t^\infty \int_y^\infty \bar{F}(x)dxdy - \mu^2(t) \\
&= \phi_2(t) - \mu^2(t) \qquad (2.4)
\end{aligned}$$

or, using the definition of $\mu(y)$:

$$\sigma^2(t) = [2/\bar{F}(t)]\int_t^\infty \bar{F}(y)\mu(y)dy - \mu^2(t) \qquad (2.5)$$

$$\begin{aligned}
\frac{d\sigma^2(t)}{dt} &= [2f(t)/\bar{F}^2(t)]\int_t^\infty \bar{F}(y)\mu(y)dy \\
&\quad -[2/\bar{F}(t)]\bar{F}(t)\mu(t) - 2\mu(t)\ \mu'(t) \\
&= r(t)[\sigma^2(t) + \mu^2(t)] - 2\mu(t)[1 + \mu'(t)] \\
&= r(t)[\sigma^2(t) - \mu^2(t)] \\
&= r(t)\mu^2(t)[\gamma^2(t) - 1], \quad \text{where} \gamma^2(t) = \sigma^2(t)/\mu^2(t). \qquad (2.6)
\end{aligned}$$

Hence F $\in$ DVRL (IVRL) iff

$$\gamma^2(t) \leq (\geq) 1. \tag{2.7}$$

(c) It can be easily shown that $\frac{d}{dt}[\phi_2(t)/\phi_1(t)] = \gamma^2(t) - 1$; so that

$$[\phi_2(t)/\phi_1(t)] \downarrow \Leftrightarrow F \in \text{ DVRL} \tag{2.8}$$

$$[\phi_2(t)/\phi_1(t)] \uparrow \Leftrightarrow F \in \text{ IVRL} \tag{2.9}$$

(d) To see the connection with the equilibrium (renewal) distribution, let $g(x) = \bar{F}(x)/\mu,\ x > 0$, be the density of the equilibrium (or renewal) distribution corresponding to F, then $\bar{G}(t) = [1/\mu]\int_t^\infty \bar{F}(x)dx$, where μ, σ^2, r without subscript are μ_F; σ_F^2, and r_F respectively.

Now,

$$\begin{aligned} \phi_2(t)/\phi_1(t) &= 2\int_t^\infty \int_y^\infty \bar{F}(x)dxdy / \int_t^\infty \bar{F}(y)dy \\ &= 2\int_t^\infty \mu\bar{G}(y)dy/\mu\bar{G}(t) = 2\mu_G(t). \end{aligned}$$

or, using (2.3),

$$\mu_G(t) = [\sigma_F^2(t) + \mu_F^2(t)]/2\mu_F(t) \tag{2.10}$$

From (2.8), (2.9) and (2.10) we have immediately:

$$\begin{aligned} F \in \text{ DVRL} &\Leftrightarrow G \in \text{ DMRL} \\ F \in \text{ IVRL} &\Leftrightarrow G \in \text{ IMRL.} \end{aligned} \tag{2.11}$$

(e) From (2.5) we have:

$$\begin{aligned} [2/\bar{F}(t)]\int_t^\infty \bar{F}(y)[\mu(y) - \mu(t)]dy &= \sigma^2(t) + \mu^2(t) - 2\mu^2(t) \\ &= \sigma^2(t) - \mu^2(t), \end{aligned} \tag{2.12}$$

thus, $\mu(y) - \mu(t) \leq 0\ \forall\ y > t \Rightarrow \gamma^2(t) \leq 1 \Leftrightarrow F \in$ DVRL; so that

$$F \in \text{DMRL} \Rightarrow F \in \text{ DVRL.} \tag{2.13}$$

i.e. the class DMRL of a subset of the class DVRL. From (2.12) and (2.13) we have:

$$F \in \text{NBUE} \Leftarrow F \in \text{DMRL} \quad \rightarrow \quad F \in \text{ DVRL} \Leftrightarrow G \in \text{ DMRL}$$

$$\rightarrow \quad G \in \text{ NBUE}. \tag{2.14}$$

(f) Solving (2.6) as a differential equation in $\sigma^2(t)$ we get:

$$\sigma^2(t) = \frac{1}{\bar{F}(t)}[\sigma^2(0) - \int_0^t r(x)\mu^2(x)\bar{F}(x)dx].$$

$$\frac{\sigma^2(t)}{\sigma^2(0)} = 1 + \frac{F(t)}{\bar{F}(t)} - \frac{1}{\sigma^2(0)\bar{F}(t)}\int_0^t \mu^2(x)f(x)dx \tag{2.15}$$

If $\mu(x)$ is $\downarrow$ we can write (2.15), using the mean value theorem for integrals as

$$\frac{\sigma^2(t)}{\sigma^2(0)} = 1 + \frac{F(t)}{\bar{F}(t)} - \frac{\mu^2(0)}{\sigma^2(0)\bar{F}(t)}\int_0^{\theta t} f(x)dx$$

$$= 1 + \frac{F(t)}{\bar{F}(t)} - \frac{F(\theta t)}{\gamma^2(0)\bar{F}(t)}, \quad 0 < \theta < 1 \tag{2.16}$$

but $F \in$ DMRL $\Rightarrow F \in$ DVRL $\Rightarrow \frac{\sigma^2(t)}{\sigma^2(0)} < 1, \quad \forall\, t > 0$, this leads in turn to,

$$\frac{F(t)}{\bar{F}(t)} - \frac{F(\theta t)}{\gamma^2(0)\bar{F}(t)} < 0, \quad \forall\, t > 0.$$

or $\gamma^2(0) < \frac{F(\theta t)}{F(t)} \leq 1$. Thus we have the result

$$F \in \text{ DMRL} \Rightarrow F \in \text{ DVRL} \Rightarrow \gamma^2(0) < 1. \tag{2.17}$$

(g) Further we have from (2.10),

$$\mu_G(t)/\mu_F(t) = \frac{1}{2}[\gamma_F^2(t) + 1]. \tag{2.18}$$

so that

$$\mu_G(t) < \mu_F(t) \quad \text{iff} \quad \gamma_F^2(t) < 1 \tag{2.19}$$

or

(2.20) $$\mu_G(t) < \mu_F(t) \Leftrightarrow F \in \text{ DVRL}.$$

From the definition of the equilibrium distribution $G(t)$ we can see easily that:

(2.21) $$F \in \text{ NBUE } \Leftrightarrow \bar{G}(t) < \bar{F}(t), \quad \forall\, t > 0.$$

But $\bar{G}(t) < \bar{F}(t) \Rightarrow \mu_G(0) < \mu_F(0)$. Using (2.18) again we have the result:

(2.22) $$F \in \text{ NBUE } \Rightarrow \gamma^2(0) < 1.$$

3. PRELIMINARIES

Let $x_1, x_2, \ldots, x_n$ be a random sample from

(3.1) $$f_1(x;\theta) = \theta e^{-\theta x}; \quad x > 0, \ \theta > 0.$$

Let $z_i = \frac{X_i}{T}$, $\quad i = 1, 2, \ldots, n$; where $T = \sum_{i=1}^n X_i$. It is well known (see [3]) that the joint distribution of any $r(< n)$ of these variables is:

$$f(z_1, \ldots, z_r) = \frac{(n-1)!}{(n-r-1)!}(1-z_1-z_2-\ldots-z_r)^{n-r-1}, \quad r = 1, 2, \ldots, n-1$$

This is a Dirichlet distribution $D(1, \ldots, 1, n-r)$, (see [10]) from which $E(z_i) = 1/n$; $E(z_i^2) = 2/n(n+1)$; $E(z_i^2 z_j^2) = 4/p(n)$; $E(z_i^4) = 24/p(n)$; where

(3.2) $$p(n) = n(n+1)(n+2)(n+3) \quad i, j = 1, \ldots, r.$$

We notice that $\sum_{i=1}^n z_i^2 - \frac{1}{n} = \frac{ns^2}{T^2}$ where $s^2 = \frac{1}{n}\sum_1^n (x_i - \bar{x})^2$ so that

(3.3) $$\hat{\gamma}^2 = \frac{s^2}{\bar{X}^2} = n\sum_1^n z_i^2 - 1$$

4. TEST FOR DVRL (IVRL) CLASS

We notice from (2.10) that

$$\mu_G(t) = \frac{\sigma_F^2(t) + \mu_F^2(t)}{2\mu_F(t)} \tag{4.1}$$

It can be shown that, (see [6])

$$F(t) = \frac{\mu}{\mu(t)} \exp[-\int_0^t \frac{du}{\mu(u)}] \tag{4.2}$$

Thus $\mu(t)$ determines uniquely, a continuous survival distribution with finite mean. From (4.1) it is clear that $\gamma^2(t) = 1 \Leftrightarrow \mu_G(t) = \mu_F(t)$ and this leads to $F(t) = G(t)$ leading to $F(t)$ is exponential. Thus $\gamma^2(t) = 1$ is a characteristic property of exponential distribution . Consider now the hypothesis:

$H_0 : F(x)$ is exponential; against $H_1 : F(x)$ is DVRL (IVRL). Since $\gamma^2(t) = 1$ characterizes exponentiality, while $\gamma^2(t) < 1$ (> 1) characterizes DVRL (IVRL) class of survival functions, it is reasonable to adopt as a measure of departure from H_0, the parameter

$$\int_0^\infty (\gamma^2(t) - 1))dF(t) = \int_0^\infty \gamma^2(t)dF(t) - 1.$$

In other words, small (large) values of

$$\Delta = \int_0^\infty \gamma^2(t)dF(t) \tag{4.3}$$

will indicate the rejection of H_0 leading to the claim that $F(t) \in$ DVRL (IVRL). In terms of Δ, the hypothesis can be formulated as:

$$H_0 : \Delta = 1 \quad \text{ag.} \quad H_1 : \Delta < 1 \ (> 1).$$

Consider now, $\hat{\gamma}_k^2 = s_K^2/\bar{x}_k^2, \quad k = 0, 1, \ldots, n_1 \leq n-2.$

Where $\bar{x}_k = \sum_{i=k+1}^n x_i/(n-k)$, and $s_k^2 = \sum_{i=k+1}^n (x_i - \bar{x}_k)^2/(n-k)$;

$$\bar{x}_0 = \bar{x} = \sum_{i=1}^n x_i/n; s_0^2 = s^2 = \sum_{i=1}^n (x_i - \bar{x}_n)^2/n. \tag{4.4}$$

Notice that $\hat{\gamma}_k^2$ is based on the last $(n-k)$ observations, of the sample, $k = 0, 1, \ldots, n_1$, (Remember that life time data is naturally ordered) . Consider the test statistic:

$$Q = \frac{1}{n_1} \sum_{k=0}^{n_1-1} \hat{\gamma}_k^2 \tag{4.5}$$

which is the sample analogue of Δ in (4.3).

Lemma 4.1. *Let $\bar{x}$ and s^2 be the mean amd variance of a random sample of size n from a distribution with mean μ and variance σ^2 then:*

$$\sqrt{n}\left[\frac{s^2}{\bar{x}^2} - \gamma^2\right] \xrightarrow{D} N[0, \gamma^2(\alpha_4 - 1)] \tag{4.6}$$

where γ^2 and α_4 are square coefficient of variation and the kurtosis μ_4/σ^4, respectively. In case of exponential

$$\sqrt{n}\left[\frac{s^2}{\bar{x}^2} - \gamma^2\right] \xrightarrow{D} N(0, 8) \tag{4.7}$$

Proof. It is well known that $\sqrt{n}(s^2 - \sigma^2) \xrightarrow{D} N(0, \mu_4 - \sigma^4)$; see [8]; from which

$$\sqrt{n}\left(\frac{s^2}{\bar{x}^2} - \frac{\sigma^2}{\mu^2}\right) \xrightarrow{D} N\left[0, \frac{\mu_4 - \sigma^4}{\mu^4}\right] \equiv N[0, \gamma^4(\alpha_4 - 1)],$$

using Slutsky's theorem. In exponential $\gamma^2 = 1$, $\alpha_4 = 9$, hence we have (4.8). □

Now, by lemma 4.1 we have, under H_0:

$$\sqrt{(n-k)}[\hat{\gamma}_k^2 - 1] \xrightarrow{D} N(0, 8) \tag{4.8}$$

Thus the test statistic $Q = \frac{1}{n_1} \sum_{k=0}^{n_1-1} \hat{\gamma}_k^2$ has a normal limiting distribution for every value of $n_1 \geq 1$. Since $\hat{\gamma}_k$'s are not independent for different values of k, it is not easy to compute the variance of Q. This difficulty will be overcomed by considering another statistic which is asymptotically equivalent, and whose variances and covariances can be computed more easily.

Lemma 4.2. $\hat{\gamma}_k^2+1$ *has the same limiting distribution as* $\dfrac{n^2}{n-k}\sum_{i=k+1}^{n} z_i^2$

Proof,

$$\hat{\gamma}_k^2+1 = (n-k)\frac{x_{k+1}^2+\cdots+x_n^2}{T_k^2}; \quad \text{where} \quad T_k=\sum_{i=k+1}^{n} x_i$$

$$(4.9) \qquad = (n-k)\frac{T^2}{T_k^2}\frac{x_{k+1}^2+\cdots+x_n^2}{T^2} \equiv \frac{(n-k)}{U_k^2}\sum_{i=k+1}^{n} z_i^2,$$

$$\text{where} U_k = T_k/T.$$

Thus $\dfrac{n^2}{n-k}\sum_{i=k+1}^{n} z_i^2 = \dfrac{n^2}{(n-k)^2}U_k^2(\hat{\gamma}_k^2+1)$. It is clear that $\dfrac{n^2}{(n-k)^2}U_k^2 = \dfrac{n^2}{(n-k)^2}\cdot\dfrac{T_k^2}{T^2} = \dfrac{\bar{x}_k^2}{\bar{x}^2} \xrightarrow{P} 1.$

Since both $\bar{x}_k^2$ & $\bar{x}^2$ converge in probability to the same μ^2. The required result follows by Slutsky's theorem. □

For n large enough, one can write,

$$(4.10) \qquad \text{cov}\,(\hat{\gamma}_k^2,\hat{\gamma}_j^2) \simeq \text{cov}\left(\frac{n^2}{n-k}y_k, \frac{n^2}{n-j}y_j\right)$$

where $y_k = \sum_{i=k+1}^{n} z_i^2$.

The product $y_j y_k$, $j>k$, contains $(n-j)(n-k-1)$ pairs like $z_r^2 z_t^2 (r\neq t)$, and $(n-j)$ pairs like z_t^4. From (3.2), we have therefore,

$$(4.11) \qquad \text{cov}\,(Y_k,Y_j) = \frac{4(n-j)(n^2+4nk-n+6k)}{n(n+1)p(n)}.$$

Asymptotically, one has,

$$\text{cov}\,(\hat{\gamma}_k^2,\hat{\gamma}_j^2) = \frac{n^4}{(n-j)(n-k)}\,\text{cov}\,(y_k,y_j) =$$

$$\frac{4n^4(n-j)(n^2+4nk-n+6k)}{(n-j)(n-k)n(n+1)p(n)} = \frac{4n^2(n^2+4nk-n+6k)}{(n-k)(n+1)^2(n+2)(n+3)} \tag{4.12}$$

which is of the order $\frac{4}{n-k}$ as $n \to \infty$, and k fixed.

$$\begin{aligned} V(Q) &= V\left[\frac{1}{n_1}\sum_{k=0}^{n_1-1}\hat{\gamma}_k^2\right] \\ &= \frac{1}{n_1^2}\left\{\sum_{k=0}^{n_1-1} V(\hat{\gamma}_k^2) + 2\sum_{k=0}^{n_1-1}\sum_{j=k+1}^{n_1-1} \operatorname{cov}(\hat{\gamma}_k^2, \hat{\gamma}_j^2)\right\} \\ &= \frac{1}{n_1^2}\left\{\sum_{k=0}^{n_1-1}\frac{8}{n-k} + 2\sum_{k=0}^{n_1-1}\sum_{j=k+1}^{n_1-1}\frac{4}{n-k}\right\} \\ &= \frac{8}{n_1^2}\sum_{k=0}^{n_1-1}\frac{n_1-k}{n-k}. \end{aligned} \tag{4.13}$$

For $n - n_1$ large enough, the statistic $Q = \frac{1}{n_1}\sum_{k=0}^{n_1-1}\hat{\gamma}_k^2$ could be used to test H_0 vs H_1, using the standardized normal table. Small (large) values of $D = (Q-1)/\sigma_Q$ indicate that $F(t)$ belongs to DVRL (IVRL). From (2.11) we can say that this test is at the same time a test of $H_0' : G(t)$ is exponential vs $H_1' : G(t)$ belongs to DMRL (IMRL), where $G(t)$ is the renewal distribution corresponding to $F(t)$.

Notice that Q is the average of n_1 asymptotically normal variables, so that it approaches normality for every $n_1 \geq 1$. For this test $n - n_1$ should exceed 30, and n_1 should be fairly large.

Example: Bryson & Siddiqui (1969) have analysed data which are survival times, in days from diagnosis of patients suffering from chronic granalocytic leukemia. The order statistics $X_1 < \ldots < X_{43}$ $(n = 43)$ are: 7, 47, 58, 74, 177, 232, 273, 285, 317, 429, 440, 445, 455, 468, 495, 497, 532, 571, 579, 581, 650, 702, 715, 779, 881, 900, 930, 968, 1077, 1109, 1314, 1334, 1367, 1534, 1712, 1784, 1877, 1886, 2045, 2056, 2260, 2429, 2509.

We shall test here whether this data is exponential against, the data belongs to DVRL (IVRL) class.

Using $n_1 = 13$, so that we leave $n - n_1 = 30$, (i.e. $n - k \geq 30$), allowing the use of asymptotic results stated above.

$$\begin{aligned} Q &= \frac{1}{n_1}\sum_{k=0}^{n_1-1} \hat{\gamma}_k^2 = 0.400255727; \quad \sigma_Q = 0.333506121 \\ D &= -1.798, \quad p\text{-value} = 0.036. \end{aligned}$$

A significant result indicating that $F(t) \in$ DVRL class and that the corresponding renewal distribution belongs to DMRL class.

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