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A METHOD OF TESTING INTERACTION IN A SINGLE REPLICATE TWO-WAY CLASSIFICATION EXPERIMENT

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ABSTRACT. In this paper, we present a new statistical procedure to detect interaction in single replicate two-way cross-classification experiments without assuming any functional form of the interaction term of the model. This procedure enables us to identify individual nonzero interaction effects.

1. INTRODUCTION

Two-way cross-classification models are important in statistical applications. In order to use these models in experimental and survey investigations, it is necessary to make a judgement about the presence of interaction in the model and to find an estimate of the experimental error variance. In the case of two-way cross-classification models with only one observation per treatment combination, the conventional linear model theory cannot be applied to test for the presence of the interaction nor can it be used to estimate the error variance simultaneously. Therefore, alternative approaches are necessary to test for the presence of interaction, and to estimate the error variance for the single replicate two-way classification models when the additivity assumption is not assumed.

In the literature there are several methods dealing with this situation which are reviewed in Section 2. In this paper, we propose a new statistical procedure to deal with this situation. The method proposed in this paper is based on the likelihood principle.

Key Words: interaction; additivity; contrast matrices; likelihood; outliers.

Let $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n$ be the the best linear unbiased estimators (BLUEs) of a specific set of interaction contrasts of the cell means of an unreplicated two-way cross-classification experiment where these contrasts are denoted by $\theta_1, \theta_2, \ldots, \theta_n$. It will be shown that these contrasts can be chosen so that $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n$ are independently normally distributed random variables with a common variance σ^2 and means $\theta_1, \theta_2, \ldots, \theta_n$, respectively. The proposed test is based on only the $\hat{\theta}_i$'s and unlike other known tests is model independent.

2. REVIEW OF SOME PREVIOUS WORK

In this section, we give a brief review of some known methods found in literature concerning the situations of single replicate two-way cross-classification experiments.

The general interaction model of two-way cross-classification designs with no replications is generally defined by

$$y_{ij} = \mu + \tau_i + \beta_j + \eta_{ij} + \epsilon_{ij}; \qquad i = 1, 2, \ldots, t, \; j = 1, 2, \ldots, b, \tag{2.1}$$

where τ_i is the i-th row effect, β_j is the j-th column effect, η_{ij} is the (i, j) cell's interaction effect, and the ϵ_{ij}'s are independently normally distributed random variables with mean zero and variance σ^2. The standard restrictions that usually imposed on this model are

$$\begin{aligned} \sum_{i=1}^{t} \tau_i &= \sum_{j=1}^{b} \beta_j = 0, \\ \sum_{i=1}^{t} \eta_{ij} &= 0; \qquad \text{for all } j = 1, 2, \ldots, b, \\ \sum_{j=1}^{b} \eta_{ij} &= 0; \qquad \text{for all } i = 1, 2, \ldots, t. \end{aligned} \tag{2.2}$$

Most known methods proposed for testing interaction involve replacing the interaction term of the model by some function of the row and column effects τ_i and β_j, a function of additional parameters α_i and γ_j, or a function of some combination of these parameters. These multiplicative interaction models have been used as approximations to the general interaction model. They represent the many attempts for testing interaction without replication.

Tukey (1949) was the first one to propose a test for interaction in the general model. In the development of his test, Tukey did not assume any particular type of interaction model. However, Scheffé (1959), and Graybill (1961) showed that Tukey's test can be derived from the multiplicative interaction model:

$$(2.3)\quad y_{ij} = \mu + \tau_i + \beta_j + \lambda\tau_i\beta_j + \epsilon_{ij}; \qquad i = 1, 2, \ldots, t,\ j = 1, 2, \ldots, b.$$

It has relative good power for this special model which essentially is testing linear by linear interaction. If the additivity hypothesis is not rejected, it is concluded that interaction of the form $\lambda\tau_i\beta j$ is not significant but still the general additivity of the data can not be concluded if model (2.3) is not the correct model for the data.

Milliken and Graybill (1970) generalized Tukey's original idea to the general linear model. They showed that any known function of the row and column main effects' parameters can be used to model the interaction in the general model (2.1).

A second model was proposed by Williams (1952) and is given by

$$y_{ij} = \mu + \tau_i + \alpha_i\beta_j + \epsilon_{ij}; \qquad i = 1, 2, \ldots, t,\ j = 1, 2, \ldots, b.$$

Mandel (1959) generalized Tukey's method by assuming the model

$$y_{ij} = \mu + \tau_i + \beta_j + \lambda\alpha_i\beta_j + \epsilon_{ij}; \qquad i = 1, 2, \ldots, t,\ j = 1, 2, \ldots, b.$$

Mandel (1961) also generalized William's model when he considered what he called a bundle of straight lines model that is given by

$$y_{ij} = \mu + \tau_i + \beta_j + \alpha_i\beta_j + \epsilon_{ij}; \qquad i = 1, 2, \ldots, t,\ j = 1, 2, \ldots, b.$$

Gollob (1968) and Mandel (1969) have proposed another more general multiplicative interaction model. This model assumes that the interaction term has the functional form $\eta_{ij} = \lambda\alpha_i\gamma_j$, and the model is

$$(2.4)\quad y_{ij} = \mu + \tau_i + \beta_j + \lambda\alpha_i\gamma_j + \epsilon_{ij}; \qquad i = 1, 2, \ldots, t,\ j = 1, 2, \ldots, b,$$

where

$$\sum_{i=1}^{t}\tau_i = \sum_{j=1}^{b}\beta_j = \sum_{i=1}^{t}\alpha_i = \sum_{j=1}^{b}\gamma_j = 0 \quad \text{and} \quad \sum_{i=1}^{t}\alpha_i^2 = \sum_{j=1}^{b}\gamma_j^2 = 1.$$

Assuming that the ϵ_{ij}'s are independently normally distributed random variables with mean zero and variance σ^2, Johnson and Graybill (1972) obtained maximum likelihood estimates for the parameters of the model (2.4). They also derived a likelihood ratio test of no interaction hypothesis (i.e., $H_0 : \lambda = 0$). Marasinghe and Johnson (1981) gave a procedure which enables one to find subareas of the data in which the data may be additive. They also derived an improved estimator of σ^2.

Milliken and Rasmuson (1977) proposed a test for interaction in model (2.1). This test does not assume any form for the interaction. However, there are some forms of interaction the method cannot detect (see Milliken and Johnson, 1989). In their procedure, the problem of testing presence of an interaction is reduced to a problem of testing homogeneity of variance.

All models mentioned above are, in some manner, special cases of the general multiplicative interaction model (2.4). The validity of these procedures depends on the correctness of the assumed multiplicative interaction. Of course, if the assumption of multiplicative interaction is not correct, non-multiplicative interaction may not be detected by these procedures and erroneous conclusions may be made. Therefore, alternative approaches are needed to handle the general interaction model. An alternative approach is our procedure presented in this paper.

3. NOTATION AND SET-UP

In this section, we present the notations, the model, and the assumptions used for the model. We consider a two-way cross classification experiment. There are two factors of interest, say A and B. The factor A is applied with t levels and B is applied with b levels and there are tb treatment combinations.

Suppose there is only one observation per treatment combination. No functional form for the interaction is assumed, and we will only work with the mean model:

$$y_{ij} = \mu_{ij} + \epsilon_{ij}; \qquad i = 1, 2, \ldots, t, \; j = 1, 2, \ldots, b, \tag{3.1}$$

where y_{ij} and μ_{ij} are respectively the observed response and the mean response of the (i,j)-th treatment combination. The random errors ϵ_{ij}'s are assumed to be independently normally distributed with mean zero and unknown variance σ^2.

The vector of the observed responses of the i-th level of the factor A is given by $\mathbf{y}_{i.} = (y_{i1}, y_{i2}, ..., y_{ib})^t, i = 1, 2, \ldots, t$. Similarly, we define the vector of the mean responses of the i-th level of the factor A by $\boldsymbol{\mu}_{i.}$, and the vector of the errors of the i-th level of the factor A by $\boldsymbol{\epsilon}_i$. Now, model (3.1) can be written in matrix notation as follows

$$\mathbf{y} = \boldsymbol{\mu} + \boldsymbol{\epsilon} \tag{3.2}$$

where $\mathbf{y}^t = (\mathbf{y}_{1.}^t, \mathbf{y}_{2.}^t, \ldots, \mathbf{y}_{t.}^t)$, $\boldsymbol{\mu}^t = (\boldsymbol{\mu}_{1.}^t, \boldsymbol{\mu}_{2.}^t, \ldots, \boldsymbol{\mu}_{t.}^t)$, $\boldsymbol{\epsilon}^t = (\boldsymbol{\epsilon}_{1.}^t, \boldsymbol{\epsilon}_{2.}^t, \ldots, \boldsymbol{\epsilon}_{t.}^t)$, and $\boldsymbol{\epsilon} \sim \mathbf{N}_{tb}(\mathbf{0}, \sigma^2\mathbf{I})$.

In the case of single replication of the two-way cross-classifications, there are $n = (t-1)(b-1)$ degrees of freedom for the interaction and zero degrees of freedom for the error term. For the purpose of testing the additivity of model (3.2), the interaction effect can be partitioned into $(t-1)(b-1)$ orthogonal contrasts. This means that the interaction's sum of squares can be partitioned into $(t-1)(b-1)$ single degree of freedom sum of squares.

When two factors in a two-way classification model do not behave independently, they are said to interact. Interaction is the failure of the differences in response to change in levels of one factor to be the same at all levels of another factor. When two factors interact, the response to changes in one factor is conditioned by the level of the other.

The general definition that the two factors do not interact is that $\mu_{ij} - \mu_{i'j} - \mu_{ij'} + \mu_{i'j'} = 0$ for all possible values of i, i', j, and j'. Testing for the presence and absence of interaction in model (3.2) is equivalent to testing the following null and alternative hypotheses:

$$\begin{aligned} H_0 : \mu_{ij} - \mu_{i'j} - \mu_{ij'} + \mu_{i'j'} &= 0, \qquad \text{for all } \ i, i', j, \text{ and } j', \\ H_1 : \mu_{ij} - \mu_{i'j} - \mu_{ij'} + \mu_{i'j'} &\neq 0, \qquad \text{for some } \ i, i', j, \text{ and } j'. \end{aligned}$$

Equivalently, we can test the following null and alternative hypotheses:

$$
\begin{aligned}
H_0 : (\mathbf{B}_b \otimes \mathbf{A}_t)\boldsymbol{\mu} &= \mathbf{0} \\
H_1 : (\mathbf{B}_b \otimes \mathbf{A}_t)\boldsymbol{\mu} &\neq \mathbf{0},
\end{aligned} \tag{3.3}
$$

where $\mathbf{A}_t$ and $\mathbf{B}_b$ are any $(t-1)\times t$ and $(b-1)\times b$ contrast matrices of ranks $(t-1)$ and $(b-1)$, respectively, and $\otimes$ is the direct or Kronecker product. A matrix $\mathbf{H}$ is defined as a contrast matrix if $\mathbf{Hj} = \mathbf{0}$, where $\mathbf{j}$ is a vector of ones. To ensure that the estimates of the interaction contrasts of the μ_{ij}'s are independent with a common variance σ^2, the rows of $\mathbf{B}_b \otimes \mathbf{A}_t$ need to be orthonormal. If the rows of $\mathbf{A}_t$ and the rows of $\mathbf{B}_b$ are orthonormal, then the rows of $\mathbf{X}_{tb} = \mathbf{A}_t \otimes \mathbf{B}_b$ will also be orthonormal. For example, let the i-th row of $\mathbf{A}_t$ be $\mathbf{a}_i^t = (c_{i1}, c_{i2}, \ldots, c_{it})/\sqrt{i(i+1)}$, $(i = 1, 2, \ldots, t-1)$, where

$$
c_{ij} = \begin{cases} 1; & \text{if } j < i+1 \\ -i; & \text{if } j = i+1, \quad \text{for } j = 1, 2, \ldots, t. \\ 0; & \text{if } j > i+1, \end{cases}
$$

Then it is clear that the rows of $\mathbf{A}_t$ are orthonormal. The rows of $\mathbf{B}_b$ can be similarly defined yielding a matrix $\mathbf{X}_{tb}$ with orthonormal rows (i.e., $\mathbf{X}_{tb}\mathbf{X}_{tb}' = \mathbf{I}$).

If meaningful orthonormal contrasts can be constructed, they should be used as the rows of $\mathbf{A}_t$ and $\mathbf{B}_b$. For example, if the levels of the factors are quantitative, then the coefficients of orthonormal polynomials can be used as the coefficients of the contrasts.

Let $\mathbf{a}_i^t$ and $\mathbf{b}_j^t$ be respectively the i-th row of $\mathbf{A}_t$ and j-th row of $\mathbf{B}_b$, and define $\mathbf{z}_{ij}^t = \mathbf{b}_j^t \otimes \mathbf{a}_i^t, i = 1, 2, \ldots, t-1$, and $j = 1, 2, \ldots, b-1$. Then the matrix of coefficients of the orthonormal interaction contrasts, $\mathbf{X}_{tb} = \mathbf{A}_t \otimes \mathbf{B}_b$, which is a $(t-1)(b-1) \times tb$ matrix, can be written as

$$
\mathbf{X}_{tb}^t = [\mathbf{z}_{11},\ \mathbf{z}_{12},\ \ldots, \mathbf{z}_{1,b-1},\ \ldots,\ \mathbf{z}_{t-1,1},\ \mathbf{z}_{t-1,2},\ \ldots,\ \mathbf{z}_{t-1,b-1}].
$$

We define the vector of the orthonormal interaction contrasts of the mean responses as

$$
\boldsymbol{\theta} = \mathbf{X}_{tb}\boldsymbol{\mu}, \tag{3.4}
$$

where $\boldsymbol{\theta}^t = (\theta_1, \theta_2, \ldots, \theta_n)$, and $n = (t-1)(b-1)$. Now, (3.3) can be rewritten as

$$H_0 : \boldsymbol{\theta} = \mathbf{0}, \quad H_1 : \boldsymbol{\theta} \neq \mathbf{0}. \tag{3.5}$$

In the following section, we introduce the proposed testing procedure.

4. THE PROPOSED TEST FOR SIGNIFICANT INTERACTION EFFECTS

In this section, we present a method for testing significant interaction effects in a single replicate two-way classification model. This method is a modified version of the method proposed by Al-Shiha and Yang (1999) for detecting significant effects in single replicate factorial experiments.

Based on only one observed value of the response for each treatment combination in the model (3.2), the BLUE of $\boldsymbol{\mu}$ is $\hat{\boldsymbol{\mu}} = \mathbf{y}$. Therefore, the BLUE of $\boldsymbol{\theta}$ is given by

$$\hat{\boldsymbol{\theta}} = \mathbf{X}_{tb}\mathbf{y}. \tag{4.1}$$

Since $\mathbf{X}_{tb}\mathbf{X}_{tb}^t = \mathbf{I}$, the estimates of the orthonormal interaction contrasts $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n$ are independently normally distributed with means $\theta_1, \theta_2, \ldots, \theta_n$, respectively and an unknown common variance σ^2 which is the variance of the experimental error. Hence $\hat{\theta}_i \sim N(0, \sigma^2)$ if the i-th orthonormal interaction contrast is not significant; and $\hat{\theta}_i \sim N(\theta_i, \sigma^2)$ with $\theta_i \neq 0$ if the i-th orthonormal interaction contrast is significant. Therefore, the estimates of non-significant orthonormal interaction contrasts will be considered as a random sample from $N(0, \sigma^2)$ while the estimates of the significant orthonormal interaction contrasts will be considered contaminants.

The particular statistical inference problem concerning the parameters θ_1, θ_2, ..., θ_n, to be considered in this section may be stated formally as follows. Let $\hat{\theta}_1$, $\hat{\theta}_2$,..., $\hat{\theta}_n$, be n independent normal random variables with means

$\theta_1, \theta_2, \ldots, \theta_n$, respectively, and a common variance σ^2. It is desired to decide upon, based on the given sample $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n$, which, if any, of the means are significantly different from zero.

Let r be the unknown actual number of nonzero means among $\theta_1, \theta_2, \ldots, \theta_n$. We treat the problem as the problem of detecting outliers with the outliers being the estimates of significant interaction effects. Of course, here the inliers are normal random variables with zero means and the outliers are normal random variables with nonzero means. Let $\hat{\kappa}_i = \hat{\theta}_i^2, i = 1, 2, \ldots, n$, and denote the i-th order statistic of $\hat{\kappa}_1, \hat{\kappa}_2, \ldots, \hat{\kappa}_n$ by $\hat{\kappa}_{(i)}$. Let $(j_1, j_2, \ldots, j_n)$ be the permutation of $(1, 2, \ldots, n)$ such that

$$\hat{\kappa}_{j_i} = \hat{\kappa}_{(i)}, \qquad i = 1, 2, \ldots, n.$$

Let

$$\theta_{[i]} = \theta_{j_i}, \qquad i = 1, 2, \ldots, n.$$

A likelihood ratio test statistic for testing $H_0 : \boldsymbol{\theta} = \mathbf{0}$ against H_1 : exactly k of the θ_i's are nonzero, denoted by $L_{n,k}$, was developed by Al-Shiha and Yang (1999) to detect significance effects for single replicate factorial experiments. The proposed test statistic $L_{n,k}$ is defined as follows

$$L_{n,k} = \frac{\sum_{i=n-k+1}^{n} \hat{\kappa}_{(i)}/k}{\sum_{i=1}^{n-k} \hat{\kappa}_{(i)}/(n-k)}.$$

Some critical values of this test statistic are given in Table 4.1. Extensive table of critical values of $L_{n,k}$ can be found in Al-Shiha and Yang (2000).

Table 4.1: Some $(1-\alpha)$-Quantiles of $L_{n,k}$

		α			
n	k	0.01	0.025	0.05	0.10
12	6	72.397	51.277	37.309	27.701
11	5	62.725	43.657	32.402	23.356
10	4	52.615	36.747	27.585	19.873
9	3	42.321	30.064	22.791	16.736
8	2	35.408	25.055	18.894	14.226
7	1	31.770	21.423	16.322	11.720

The proposed procedure using $L_{n,k}$ statistics must select a k such that $k \geq r$ which must be determined prior to the use of the $L_{n,k}$ procedure. Ideally, k should be equal to r. However, in practice most likely a k that is slightly larger than r is used. As shown by Al-Shiha and Yang (1999), the reduction in the power from using $L_{n,k}$ statistic is small when k is chosen to be slightly larger than the actual number of significant effects, r. By using a slightly overestimate value k of r, the possible effect of the masking problem is minimized as shown by Al-Shiha and Yang (1999).

For selecting k, we will use the half-normal probability plot proposed by Daniel (1959). First, we identify gaps in the half-normal plots which are potential breaks between significant and nonsignificant interaction effects. Then, interaction effects corresponding to the absolute values of the estimated interaction effects which are to the right of the gaps are tentatively declared significant.

If $L_{n,k}$ is greater than some critical value, then we may conclude that interaction exists between the two factors A and B. Next, we propose a multistep procedure to determine the set of all nonzero interaction effects. This multistep procedure involves testing the following nested sequence of hypotheses. First, the null hypothesis $H_0^{(1)}$: there is no significant orthonormal contrasts against the alternative hypothesis $H_1^{(1)}$: there are at most k significant orthonormal contrasts is tested using $L_{n,k}$. If the observed value of $L_{n,k}$ is greater than the critical value, $H_0^{(1)}$ is rejected and $\theta_{[n]}$ is declared non-zero. Then, $\hat{\theta}^2_{(n)}$ is removed from the calculation of $L_{n-1,k-1}$ which is used to test $H_0^{(2)}$: there is no significant orthonormal contrasts, against $H_1^{(2)}$: there are at most $(k-1)$ significant orthonormal contrasts. If $H_0^{(2)}$ is rejected, $\theta_{[n-1]}$ is declared nonzero and $\hat{\theta}^2_{(n-1)}$ is removed. This process is continued until the null hypothesis $H_0^{(m+1)}$ (say) is failed to be rejected. Then, m is the number of significant orthonormal contrasts determined by this sequential procedure. If $\theta_{[n]}, \theta_{[n-1]}, \ldots, \theta_{[n-m+1]}$ are all non-zero, in which case the probability of Type-I error rate at each stage is the level specified for the tests, then an estimate of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{n-m-1} \sum_{i=1}^{n-m} (\hat{\theta}_{[i]} - \hat{\theta}_o)^2,$$

where

$$\hat{\theta}_o = \frac{1}{n-m} \sum_{i=1}^{n-m} \hat{\theta}_{[i]}.$$

In the following section, we present an example to demonstrate the application of the $L_{n,k}$ procedure for detecting significant interaction contrasts in a single replicate two-way classification experiment.

5. AN EXAMPLE

This example is from Davies (1954). It is a 5×4 design in which the factor A is quantitative and the factor B is qualitative. The data is given in Table 5.1. The vector of the expected responses of the orthonormal interaction contrasts is

$$\begin{aligned} \boldsymbol{\theta} &= (\mathbf{B} \otimes \mathbf{A})\boldsymbol{\mu}, \\ &= [(b_1 \otimes a_1)\boldsymbol{\mu}, (b_2 \otimes a_1)\boldsymbol{\mu}, (b_3 \otimes a_1)\boldsymbol{\mu}, \ldots, (b_1 \otimes a_4)\boldsymbol{\mu}, (b_2 \otimes a_4)\boldsymbol{\mu}, (b_3 \otimes a_4)\boldsymbol{\mu}]^t, \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 & 0 \\ 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & -3/\sqrt{12} & 0 \\ 1/\sqrt{20} & 1/\sqrt{20} & 1/\sqrt{20} & 1/\sqrt{20} & -4/\sqrt{20} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^t \\ \mathbf{a}_2^t \\ \mathbf{a}_3^t \\ \mathbf{a}_4^t \end{bmatrix},$$

and

$$\mathbf{B} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & -3/\sqrt{12} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1^t \\ \mathbf{b}_2^t \\ \mathbf{b}_3^t \end{bmatrix}.$$

The absolute values of the estimates of the orthonormal interaction contrasts are given in Table 5.2 and the half-normal probability plot of this example is given in Figure 5.1.

Figure 5.1 suggests that $k = 5$. Table 5.2 also gives the values of $L_{n,k}$ for this example. From Table 5.2 and Table 4.1, we conclude that $\theta_{12}, \theta_8, \theta_{11}, \theta_{10}$, and θ_9 are significantly different from zero at 0.05 significance level. Also, we notice that interaction effects involving a_1 and a_2 are not significant and that a_1 and a_2 are the orthogonal contrasts involving only the first three levels of factor A. This suggests that the first three levels of factor A are free of interaction with the levels of factor B. These findings agree with those of Marasinghe and Johnson (1981), in which they tested the null hypothesis $H_0 : \alpha_1 = \alpha_2 = \alpha_3$ in the multiplicative model (2.4). Their test failed to reject H_0. Our estimate of σ based on the nonsignificant $\hat{\theta}_i$'s is 1.618. Using the same example, Marasinghe and Johnson (1982) estimated σ by 2.441. We speculate that the reason that their estimate is greater than ours is that their interaction model does not include all interaction effects.

6. COMMENTS

There are many choices of the orthogonal interaction contrasts as discussed in Section 3. If meaningful contrasts can be constructed, they should be used as the rows of the matrices $\mathbf{A}_t$ and $\mathbf{B}_b$. However, the choice of the orthogonal contrasts is not unique, consequently, many types of interaction contrasts can be constructed. In view of this, a matter that needs to be investigated further is the choice of the contrasts and the effect of using different sets of contrasts.

Table 5.1: A 5×4 Design

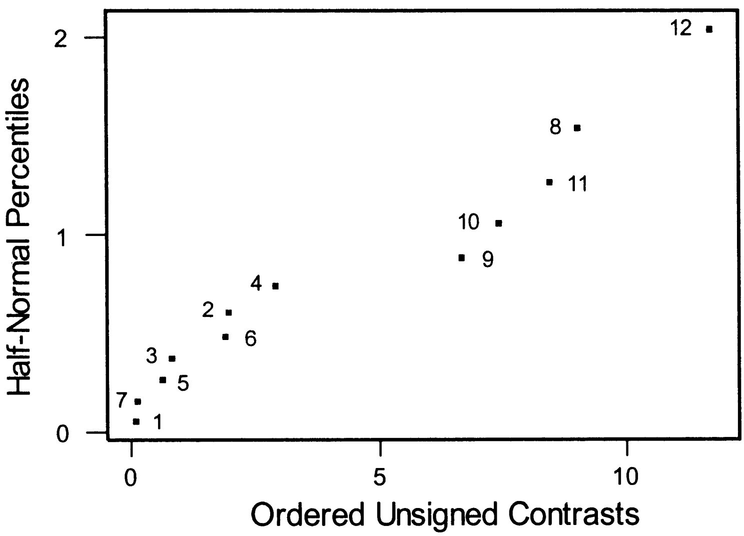
Levels of Factor A	Levels of Factor B 1	2	3	4
1	28. 2	23.5	17.4	10.1
2	29. 3	24.8	15.2	11.5
3	33. 7	24.1	17.8	15.6
4	41. 2	34.7	14.7	9.9
5	50. 9	32.8	16.6	4.7

Table 5.2: Values of $|\hat{\theta}|_{(i)}$ and $L_{n,k}$

					Percentiles of $L_{n,k}$ α		
θ_i	$\lvert\hat{\theta}\rvert_{(i)}$	n	k	$L_{n,k}$	0.01	0.05	0.1
$\theta_1 = (b_1 \otimes a_1)\boldsymbol{\mu}$	0.1000	12	5	32.490	47.88	26.82	19.88
$\theta_7 = (b_1 \otimes a_3)\boldsymbol{\mu}$	0.1429	11	4	26.348	41.36	22.97	17.27
$\theta_5 = (b_2 \otimes a_2)\boldsymbol{\mu}$	0.6333	10	3	23.810	34.36	19.61	14.74
$\theta_3 = (b_3 \otimes a_1)\boldsymbol{\mu}$	0.8165	9	2	20.833	30.06	16.87	12.80
$\theta_6 = (b_3 \otimes a_2)\boldsymbol{\mu}$	1.8856	8	1	18.613	27.36	14.79	11.07
$\theta_2 = (b_2 \otimes a_1)\boldsymbol{\mu}$	1.9630						
$\theta_4 = (b_1 \otimes a_2)\boldsymbol{\mu}$	2.8868						
$\theta_9 = (b_3 \otimes a_3)\boldsymbol{\mu}$	6.6917*						
$\theta_{10} = (b_1 \otimes a_4)\boldsymbol{\mu}$	7.4472*						
$\theta_{11} = (b_2 \otimes a_4)\boldsymbol{\mu}$	8.4623*						
$\theta_8 = (b_2 \otimes a_3)\boldsymbol{\mu}$	9.0392*						
$\theta_{12} = (b_3 \otimes a_4)\boldsymbol{\mu}$	11.7158*						

* Significant at $\alpha = 0.05$

Figure 5.1: The Half-Normal Probability Plot



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