Arab J. Math. Sc.
Volume 14, Number 2, December 2008
pp 25-41

CLOSED POLAR SETS IN A RIEMANNIAN MANIFOLD

Victor Anandam and Suad AlHemedan

Abstract. Closed polar sets e in a Riemannian manifold can be characterised by the possibility of extension as harmonic or superharmonic functions across e of bounded harmonic functions defined outside e.

1. Introduction

Like sets of measure zero in the theory of measures and integration, polar sets play the role of exceptional sets in potential theory which is the study of harmonic functions and potential functions on Euclidean domains (see Brelot[10], Aikawa-Essn [1] and Armitage-Gardiner[4]) and more generally on locally compact spaces like Riemann surfaces and Riemannian manifolds. In the case of function theory on the complex plane, it is known that a bounded analytic (respectively bounded harmonic) function defined outside a point z_0 can be extended analytically (respectively harmonically) to a domain including z_0. $\{z_0\}$ is the simplest example of a polar set in the study of harmonic functions on $\Re^n$, $n \geq 2$. The notion of locally polar sets in $\Re^n$ was introduced by Brelot in 1941 to replace the notion of sets of (classical) inner capacity zero. A set e is said to be locally polar in $\Re^n$ if for every point y in $\Re^n$ there exists a neighbourhood N of y and a potential p on N such that $p = \infty$ on $N \cap e$. Soon afterwards, Cartan [13] and [14] remarked that in a ball in $\Re^n$, the locally polar sets are the same as sets of outer capacity zero. If Ω is a Green domain (that is, the Green kernel can be defined on Ω), and if e is a locally polar set in Ω, then there exists a positive potential on Ω such that $p = \infty$ on e (see the proof in Brelot [11, p.47] in a general setting). Later in

2000 Mathematics Subject Classification. 31C12.
Key words. Polar sets, Harmonic extensions.

[2, Thorme 3.18] it was shown in particular that if ω is a domain in $\Re^n$ with or without positive potentials and if e is locally polar in ω , then there exists a superharmonic function s on ω such that $e \subset \{x : s(x) = \infty\}$; s may not be lower bounded in the general case, but if ω is a Green domain, one can take s as a potential. Consequently, we shall leave out the term locally polar and say that a set e in an open set ω is polar if and only if there exists a superharmonic function s on ω such that $e \subset \{x : s(x) = \infty\}$.

Some other descriptions of polar sets include the characterisation of polar F_σ sets in Euclidean spaces by means of hitting times (Port-Stone [20]); and the characterisation by using the fine topology on $\Re^n$ given in Doob [15, p.177]: A polar subset of $\Re^n$ has no fine limit point and conversely a subset of $\Re^n$ with no fine limit point is polar.

Among the various properties of polar sets, the property that is of interest to us here is that closed polar sets in $\Re^n$ are removable singularities for lower bounded superharmonic functions and for bounded harmonic functions (see Brelot [10]). Some of the results proved in this note are as follows.

Let R denote an orientable hyperbolic or parabolic $\mathbb{C}^\infty$ -Riemannian manifold of dimension $d \geq 2$ with a countable base or a Riemann surface. A set F in an open set $\omega \subset R$ is said to be polar if there exists a superharmonic function s on ω such that $F \subset \{x : s(x) = \infty\}$. It is known that if e is a closed polar set contained in an open set ω in R, then any bounded harmonic function on $\omega \setminus e$ extends harmonically on ω. Conversely suppose that e is a closed set in R and that every bounded harmonic function defined on $\omega \setminus e$ extends as a harmonic or a superharmonic function on ω. Then we prove in Section 3 that e is polar, provided some additional condition such as $R \in O_{HB}$ (that is, there is no non-constant bounded harmonic function on R) or $\overset{\circ}{e} = \emptyset$ or e is compact is satisfied.

As a consequence we show, in Section 4, that if R is parabolic and if Ω is a proper subdomain of R , then either there exists a harmonic function $u(x)$ on Ω tending to ∞ at every boundary point of Ω in R or there exists a positive harmonic function $h(x)$ on Ω such that $\limsup_{x \to y} h(x) = \infty$ for every $y \in \partial\Omega$ (which is equivalent to saying that there exists a bounded non-constant harmonic function on Ω). In the hyperbolic case, this dichotomy does not exist;

for in this case, we remark that given any proper open set ω, one can always construct a positive harmonic function $u(x)$ in ω such that $\limsup_{x\to y} u(x) = \infty$ for every $y \in \partial\omega$. In the final section, we extend these results from the Riemannian manifold R to the Brelot harmonic space Ω in the axiomatic potential theory.

2. Preliminary Remarks

The harmonic functions in the Riemannian manifold R defined locally as the $\mathbb{C}^\infty$ solutions of the Laplace-Beltrami equation $\Delta h = 0$, have the sheaf property, solve locally the Dirichlet problem and possess the Harnack property; that is, they satisfy the axioms 1,2,3 of Brelot in the axiomatic potential theory [11, pp.13 – 14]. Consequently, we are justified in using the results from the Brelot axiomatic theory in the context of the Riemannian manifold. Also, as remarked earlier in the Euclidean space, a theorem due to H.Cartan says that in a ball B in $\Re^n$, a set is polar if and only if its outer (Newtonian) capacity is 0 (see Brelot [10, p.57]). A bounded set E in $\Re^2$ is polar if and only if its outer logarithmic capacity is zero; if E is a polar subset of $\Re^2$, then E is totally disconnected (Armitage-Gardiner[4, p.154]). To prove a similar result in the Brelot axiomatic potential theory, Hervé [16] starts with a Brelot harmonic space Ω with postive potentials; then, using the notion of reduced functions, a strong Choquet capacity φ is constructed. It is then proved that a set e in Ω is polar if and only if its outer capacity defined by φ is 0. (For a summary of these developments, see Brelot [11, pp.45 – 47].)

With this type of characterisation of polar sets as sets of (some suitable) capacity 0, a relevant result for our article here is given in Sario and Oikawa [22,p.261] (see also Carleson [12, Section VII]):*Let E be a totally disconnected compact set in a Riemann surface W^* and let $W = W^* \setminus E$. Let U^* be a regular subregion of W^*, $E \subset U^*$. Let $U = U^* \setminus E$. Then E has logarithmic capacity* 0 *if and only if any bounded harmonic function u on a neighbourhood of $\overline{U} \cap W$ is harmonically extendable to U^*.*

In this note, we ask the question: In a Riemannian manifold R, with or without positive potentials, if e is a closed (not necessarily compact) set having the above-mentioned harmonic extension property, then is e a polar set? This investigation leads to some interesting theorems about the existence of positive harmonic functions with special properties on subdomains of R. We show also that these results in a Riemannian manifold R can be extended to the Brelot axiomatic potential theory. The proof of the sufficient condition for a closed set in R to be polar is relatively straightforward in the case of hyperbolic manifolds when we use the reduced functions (Brelot [11, p.36]):

Recall the definitions of reduced functions and balayage: Suppose in R, $s > 0$ is a superhrmonic function and e is a subset. Then, the reduced function is defined as $R^e_s(x) = \inf_{u \in \Im} u(x)$, where $\Im$ is the family of all positive superharmonic functions u on R that majorise s on e. The reduced function may not be a lower semi-continuous function, so that it may not be superharmonic on R. The balayage $\hat{\mathrm{R}}^e_s$ is the semi-continuous regularisation defined by $\hat{\mathrm{R}}^e_s(x) = \lim\inf_{y \to x} R^e_s(y)$, which is superharmonic. It is known, as a particular case of a general result in the axiomatic potential theory, that a closed set e is polar in a hyperbolic manifold R if and only if $\hat{\mathrm{R}}^e_s \equiv 0$ in R. But in the case of a parabolic Riemannian manifold, we need the following Lemma 2.1. We say that a compact set k in R is outer-regular if every point of ∂k is regular for the Dirichlet problem in $\omega \setminus k$ where $\omega \supset k$ is a relatively compact open set. Let $G(x, y)$ be the Green function for R if it is hyperbolic. If R is parabolic, let $E(x, y)$ be the Evans function for R as defined by M.Nakai (see Bagby and Blanchet [6, p.60]); the Evans function plays the role of $log|x - y|$ on $\Re^2$. We use the term "near infinity" to mean "outside a compact set".

Lemma 2.1. *Let k be a compact set, ω an open set, $k \subset \omega \subset R$. Let u be harmonic on $\omega \setminus k$. Then there exist a harmonic function t on ω and a harmonic function s on $R \setminus k$ such that $u = s - t$ on $\omega \setminus k$. Here s can be chosen so as to have the following additional property:*

(1)If R is hyperbolic and if $y \in R$, then for some constant α, $|s(x)| \leq \alpha G_y(x)$ near infinity.

(2)If R is parabolic and if $y \in R$, then for some constant α, $s(x) - \alpha E_y(x)$ is bounded near infinity

Proof. 1) First assume that ω is a domain. Suppose k_0 is an outer-regular compact set and ω_0 is a regular domain such that $k \subset \overset{\circ}{k_0} \subset k_0 \subset \omega_0 \subset \overline{\omega_0} \subset \omega$. By [3, Lemma 1], which has been proved in the general context of the Brelot axiomatic potential theory, there exist a harmonic function s_0 on $R \setminus k_0$ and a harmonic function t_0 on ω_0 such that $u = s_0 - t_0$ on $\omega_0 \setminus k_0$; let s_1 be the function $u + t_0$ on $\omega_0 \setminus k$ extended by s_0 on $R \setminus \omega_0$; let t_1 be the function $s_0 - u$ on $\omega \setminus k_0$ extended by t_0 on k_0. Then s_1 is harmonic on $R \setminus k$ and t_1 is harmonic on ω such that $u = s_1 - t_1$ on $\omega \setminus k$.

2)Suppose now ω is just an open set. In this case, we need only to consider the case when $\omega = \bigcup_{i=1}^{m} \omega_i$ where ω_i are disjoint domains and $k = \bigcup_{i=1}^{m} k_i$ where k_i is compact $\subset \omega_i$. Then for each i, by the step 1) there exist s_i harmonic on $R \setminus k_i$ and t_i harmonic on ω_i such that $u = s_i - t_i$ on $\omega_i \setminus k_i$. Define $s' = \sum_{i=1}^{m} s_i$ on $R \setminus k$ and t' on ω such that $t' = t_i + \sum_{j=1, j \neq i}^{m} s_j$ on ω_i. Then s' is harmonic on $R \setminus k$ and t' is harmonic on ω such that $u = s' - t'$ on $\omega \setminus k$.

3)Finally we prove the additional property. For that, remark that given any harmonic function s' outside a compact set in R, there exists a harmonic function f on R such that $|s' - f| \leq \alpha G_y$ near infinity if R is hyperbolic, and $s' = \alpha E_y + f +$ (a bounded harmonic function) near infinity if R is parabolic (see [2,Thorme 1.19 and Proposition 3.33]). Now in step 2), write $s = s' - f$ and $t = t' - f$ to obtain the desired decomposition. □

3. Characterisation of Closed Polar Sets in R

If e is a closed set in R and if every bounded harmonic function on $R \setminus e$ extends as a harmonic function on R, then we prove in this section that e is polar provided $R \in O_{HB}$ (that is, every bounded harmonic function on R is a constant), $\overset{\circ}{e} = \emptyset$, or e is compact.

Lemma 3.1. *Let e be a proper closed subset of R. Suppose every bounded harmonic function on $R \setminus e$ is a constant. Then $\overset{\circ}{e} = \emptyset$.*

Proof. Suppose $\overset{\circ}{e} \neq \emptyset$. Choose an outer-regular compact set $k \subset \overset{\circ}{e}$ such that $R \setminus k$ is connected. For any continuous function f on ∂k, let Bf denote the generalised Dirichlet solution on $R \setminus k$ with boundary values f on ∂k and 0 at infinity (the existence of Bf by the Perron-Wiener-Brelot Method is proved in [2]). ∂k cannot be a single point (since each point in R is polar). It is possible, therefore, to choose a finite continuous function f on ∂k such that f is not constant . Consider Bf on $R \setminus k$. Bf is bounded harmonic on $R \setminus k$, not constant since Bf tends to f on ∂k. Hence Bf is not constant in any domain δ of $R \setminus e$ (by Aronszajn-Cordes theorem, see [6, p.52]). Thus, there exists a non-constant bounded harmonic function on $R \setminus e$, a contradiction. □

It is known (proved for example as in Brelot [10, p.36]) that if e is a closed polar set in a Riemannian manifold R, then for any open set $\omega \supset e$, any bounded harmonic function on $\omega \setminus e$ extends as a bounded harmonic function on ω and if R is parabolic, then every lower bounded superharmonic function on $R \setminus e$ is constant. In the converse direction we prove the following.

Theorem 3.2. *Let e be a proper closed set in $R \in O_{HB}$. Then e is polar if and only if any bounded harmonic function on $R \setminus e$ is a constant.*

Proof. Let e be polar. Suppose u is a bounded harmonic function on$R \setminus e$. Then u extends as a bounded harmonic function on R, as noted above. Since $R \in O_{HB}$, u is a constant.

To prove the converse, let us consider the hyperbolic and the parabolic cases separately .

a) Let $R \in O_{HB}$ be hyperbolic. Suppose that every bounded harmonic function on $R \setminus e$ is constant. If e is not polar, then we can choose a compact $k \subset e$ that is not polar. Then $u = \hat{\mathrm{R}}_1^k > 0$ is a potential on R which is bounded harmonic outside k and hence by hypothesis u is a constant α on $R \setminus e$. This implies that $u = \alpha$ on $R \setminus k$, since $\overset{\circ}{e} = \emptyset$ by Lemma 2.1 and u is continuous on $R \setminus k$. But u being a potential on R, if u is a constant outside a compact set, then $u \equiv 0$ on R, a contradiction. Hence e is polar.

b) Let R be parabolic and suppose that every bounded harmonic function on $R \setminus e$ is constant. If e is not polar, choose a compact set $k \subset e$ that is not polar. Let ω be a relatively compact domain, $\omega \supset k$. Let $u = (\hat{\mathrm{R}}_1^k)_\omega$ where $(R_1^k)_\omega = \inf \{s \geq 0$ superharmonic on $\omega, s \geq 1$ on $k\}$ and $(\hat{\mathrm{R}}_1^k)_\omega$ is the lower semicontinuous regularization of $(R_1^k)_\omega$. Then u is bounded harmonic on $\omega \setminus k$ which does not extend harmonically on ω. This means that k cannot reduce to a single point, since each point in R is polar; choose two points x_i , $i = 1, 2$, in k such that u does not extend harmonically to any neighbourhood of x_i. Hence we can find two compact sets k_i which are not polar and two open sets ω_i such that $k_1 \cup k_2 \subset k, \omega_1 \cap \omega_2 = \phi$ and $k_i \subset \omega_i \subset \overline{\omega_i} \subset \omega$.

Let $u_i = (\hat{\mathrm{R}}_1^{k_i})_\omega$. Then u_i is a positive superharmonic function on ω, harmonic on $\omega \setminus k_i \supset \omega \setminus k$, but not harmonic on ω . Then by Lemma 2.1, for $y \in R$, we can write $u_i = s_i - t_i$ on $\omega \setminus k$ where s_i is harmonic on $R \setminus k$, t_i is harmonic on ω, and $(s_i - \alpha_i E_y)$ is bounded near infinity. Here $\alpha_i \neq 0$. For, otherwise the function

$$v_i = \begin{cases} s_i & \text{in } R \setminus k \\ u_i + t_i & \text{in } \omega \end{cases}$$

will be a non-constant bounded superharmonic function in the parabolic space R, which is not possible.

Let $s = \alpha_2 s_1 - \alpha_1 s_2$ on $R \setminus k$. Since s is bounded harmonic on $R \setminus k \supset R \setminus e$, by the assumption s is a constant c in $R \setminus e$. Thus s is harmonic on $R \setminus k$ and $s = c$ on $R \setminus e$; since $\overset{\circ}{e} = \emptyset$, we conclude that $s = c$ on $R \setminus k$ also. Hence on $\omega \setminus k$, $\alpha_2 s_1 = s + \alpha_1 s_2$ extends harmonically on ω_1. This means that u_1 extends harmonically on ω, a contradiction.Thus in each case, e is polar in R. □

In Theorem 3.2, the restriction that $R \in O_{HB}$ was placed mainly to prove that $\overset{\circ}{e} = \emptyset$. However, if we assume to start with that $\overset{\circ}{e} = \emptyset$ which anyway is a necessary condition for e to be polar, we have the following general result (which is proved in [19] in the general context of a Brelot harmonic space).

Theorem 3.3. *A closed set e in R is polar if and only if $\overset{\circ}{e} = \emptyset$ and for one (and hence any) open set $\omega \supset e$, every bounded harmonic function in $\omega \setminus e$ extends harmonically on ω .*

If e is a closed set contained in an open set ω, we shall say that a harmonic function u on $\omega \setminus e$ extends superharmonically (respectively harmonically) on ω if there exists a superharmonic (respectively harmonic) function s on ω such that $s = u$ on $\omega \setminus e$.

Theorem 3.4. *Let e be a k-analytic subset of R . Suppose for any compact $k \subset e$, every bounded harmonic function on $R \setminus k$ extends superharmonically on R. Then e is polar.*

Proof. Since e is k-analytic, it is enough to prove that every compact set k in e is polar (Bauer [7]). We shall consider the two cases of R being hyperbolic or parabolic separately.

1) Let R be hyperbolic. If k is not polar, let $s = 1 - \hat{\mathrm{R}}_1^k$. Then $0 \le s \le 1$ is harmonic in $R \setminus k$, $s \le 1$, $s \not\equiv 0$, $s \not\equiv 1$. By hypothesis, there exists a superharmonic function t on R such that $t = s$ on $R \setminus k$. Let R_n be a regular exhaustion of R, $R_n \supset k$ for every n; and for a fixed x_0, let $\rho^n_{x_0}$ be the harmonic measure on ∂R_n when $x_0 \in R_n$. Then $t(x_0) \ge \int t(x) d\rho^n_{x_0} = 1 - \int \hat{\mathrm{R}}_1^k d\rho^n_{x_0}$ when n is large. This implies, since $\hat{\mathrm{R}}_1^k$ is a potential, that $t(x_0) \ge 1$. Since x_0 is arbitrary, $t \ge 1$ on R and hence $\hat{\mathrm{R}}_1^k \equiv 0$,a contradiction.

2) Let R be parabolic. The assumption in this case implies that every bounded harmonic function on $R \setminus k$ is constant. For, if u is bounded harmonic on $R \setminus k$, the superharmonic function t on R such that $t = u$ on $R \setminus k$ is bounded outside a compact set and hence is a constant. Consequently, Theorem 3.2 can be used to prove that k is polar. □

4. Positive harmonic functions on domains in a Riemannian manifold

In this section, we give other characterisations of closed polar sets e in a Riemannian manifold R by the existence of certain types of positive harmonic functions on $\omega = R \setminus e$ with specified boundary limits. Extending a classical result (due to Evans), Bauer [7] proved in an axiomatic case with positive potentials that if e is a closed polar set in a harmonic space Ω , then there exists a potential on Ω such that $e = \{x : p(x) = \infty\}$. This result is extended to the case of a harmonic space without positive potentials [2,Thorme 3.21], by replacing p by a superharmonic funtion s on Ω. In particular, if e is a closed polar set in R, then there exists a real-valued superharmonic function s on $\omega = R \setminus e$ such that $\lim_{x \to y} s(x) = \infty$, for every $y \in \partial\omega$; conversly, of course, if there exists some superharmonic function s on ω such that $\lim_{x \to y} s(x) = \infty$, for every $y \in \partial\omega$, then e is polar. We improve this result in Theorem 4.1. Actually, the problem treated in this section is: Suppose ω is a proper subdomain of a Riemannian manifold R, hyperbolic or parabolic. When can we assert the existence of non-constant harmonic functions on ω that are bounded or at least positive?

Theorem 4.1. *Let e be a closed set in R. Let $\Omega = R \setminus e$. Consider the following statements:*

(1) e is polar.

(2) There exists a positive harmonic function $u(x)$ on Ω such that $\lim_{x \to y} u(x) = \infty$ for every $y \in \partial\Omega$.

(3) There exists a (not necessarily positive) harmonic function $v(x)$ on Ω such that $\lim_{x \to y} v(x) = \infty$ for every $y \in \partial\Omega$.

Then,(a) if R is hyperbolic, all the three statements are equivalent; and (b) if R is parabolic, the statements (1) and (3) are equivalent .

Proof. a) Let us first consider the case of a hyperbolic Riemannian manifold R of dimension $d \geq 2$. We shall actually assume that R is of dimension 3 (and use the kernal $\frac{1}{|x-y|}$ in the discussion); all other dimensions can be treated similarly. The harmonic and the superharmonic functions on R will be determined by means of the Laplace-Beltrami operator L, which will be called the L-harmonic and the L-superharmonic functions on R; for their properties we refer to Hervé [16, Chapter VII].

(1) $\Rightarrow$ (2). Let e be a closed polar set in R. If B is the parametric ball in R determined as the preimage of $\{x : |x - x_0| < r\}$, we shall denote by ω the preimage of $\{x : |x - x_0| < \frac{r}{2}\}$. Cover e by a countable number of such ω_n's. Let $e_n = e \cap (\overline{\omega_n} \setminus \bigcup_{i=1}^{n-1} \omega_i)$. Then e_n is compact and $e = \bigcup_n e_n$.

If (B_n, φ) is a chart, $\varphi(e_n)$ is a compact polar set contained in the ball $\varphi(B_n)$. Since $\varphi(e_n)$ is L-polar, by [16, Théorème 36.1] it is Δ-polar; here Δ refers to the Laplacian in the classical case $\Re^3$, using the local coordinates. Hence there exists a Δ-potential $q_n(x) = \int_{\varphi(e_n)} \frac{1}{|x-y|} d\mu(y)$ in $\Re^3$ with its associated measure μ having compact support in $\varphi(e_n) = \{x : x \in B_n, q_n(x) = \infty\}$.

By [16, Proposition 35.1], there exist constants c_1 and c_2 depending on n such that $\frac{c_2}{|x-y|} \leq G_y^n(x) \leq \frac{c_1}{|x-y|}$ for all x,y in $\varphi(e_n)$, where $G_y^n(x)$ is the Green L-potential in $\varphi(B_n)$. Hence $u_n(x) = \int_{\varphi(e_n)} G_y^n(x) d\mu(y)$ is an L-superharmonic function in $\varphi(B_n)$ and $\varphi(e_n) = \{x : x \in \varphi(B_n), u_n(x) = \infty\}$. This, together with Lemma 2.1, shows that there is an L-potential s_n in R, L-harmonic in $R \setminus e_n$ and $e_n = \{x : s_n(x) = \infty\}$.

Now, e being closed polar, $\Omega = R \setminus e$ is connected. Fix $x_0 \in \Omega$ and define $u_n(x) = \frac{s_n(x)}{n^2 s_n(x_0)}$ on R, to conclude that $u(x) = \sum u_n(x)$ is a positive L-superharmonic function on R, L-harmonic on Ω and $e = \{x : u(x) = \infty\}$. In particular, $u(x)$ is a positive L-harmonic function on Ω such that $\lim_{x \to y} u(x) = \infty$ for every $y \in \partial\Omega$.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1).Extend the harmonic function $v(x)$ on Ω by ∞ on e, to obtain a superharmonic function $v_1(x)$ on R. Consequently, e is polar .

b) Let us assume now R is parabolic .

(1) $\Rightarrow$ (3). Let e be a closed polar set in R. Let ω and B be parametric balls such that $\omega \subset \overline{\omega} \subset B \subset \Omega = R \setminus e$. Since each connected component of $R\setminus\overline{\omega}$ is a Green domain, there exists a superharmonic function $p(x) > 0$ in each component δ of $R\setminus\overline{\omega}$ such that p is harmonic on $\delta\setminus e$ and $\delta \cap e = \{x : p(x) = \infty\}$. Piecing together such functions p, we get a superharmonic function $q(x) > 0$ on $R \setminus \overline{\omega}$ such that $e = \{x : q(x) = \infty\}$.

By Lemma 2.1, there exists a harmonic function $s(x)$ on $R \setminus \overline{\omega}$ and a harmonic function $t(x)$ on B such that $q(x) = s(x) - t(x)$ on $B \setminus \overline{\omega}$. If we define

$$v(x) = \begin{cases} q(x) - s(x) & \text{on } R \setminus \overline{\omega} \\ -t(x) & \text{on } B, \end{cases}$$

then $v(x)$ is superharmonic on R, harmonic on Ω and $e = \{x : v(x) = \infty\}$. In particular, $v(x)$ is harmonic on Ω such that $\lim_{x \to y} v(x) = \infty$ for every $y \in \partial\Omega$.

(3) $\Rightarrow$ (1). Even if $v(x)$ is a superharmonic function on Ω tending to ∞ on $\partial\Omega$, $v(x)$ extended by ∞ on e is a superharmonic function $v_1(x)$ on R such that $e \subset \{x : v_1(x) = \infty\}$. Hence e is polar. □

Remarks 4.2. *The proof of the above theorem can be used to obtain an analogous result in the following situation considered by Hervé [16]. Let* $L = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}$ *be a second order elliptic differential operator defined on a domain* X *in* $\Re^n, n \geq 2$, *where* $a_{ij} = a_{ji}$ *and the quadratic form* $\sum_{i,j} a_{ij}(x)\xi_i\xi_j$ *is positive definite for* $x \in X$; $a_{ij}(x)$ *and* $b_i(x)$ *are locally Lipschitz functions on* X. *Then the* $\mathbb{C}^2$*-solutions of* L *determine a harmonic space* X *in the sense of Brelot's [11] and in this harmonic space the notions of L-harmonic, L-potentials, L-polar sets etc. are defined .*

Let now e be a closed L-polar set in X. Then there exists a positive L-harmonic function (respectively an L-harmonic function) $u(x)$ on $X \setminus e$ such that $\lim_{x \to y} u(x) = \infty$ for every $y \in e$, if X has positive L-potentials (respectively if there is no positive L-potential in X)

Remarks 4.3. *It is of interest in this context to refer to Kilpeläinen's papers [17], [18] which include some results on polar sets in $\mathbb{R}^n$ defined by supersolutions of an operator with degenerated ellipticity a degenerate elliptic operator.*

The above results are valid in particular when $R = \Re^n$, $n \geq 2$. In this case, we have also the property that if e is a closed polar set in $\Re^n$, then the Lebesgue measure of e is 0. As a consequence, we observe that if the Lebesgue measure of a closed set e in $\Re^n$, $n \geq 2$, is not 0, then there exists a bounded non-constant harmonic function on $\Re^n \setminus e$ (see Theorem 3.2).

To prove a similar result in a Riemannian manifold R, we can think of replacing the Lebesgue measure by the harmonic measure. This leads to the question whether there exists a positive harmonic function $h(x)$ on any domain Ω in R such that $h(x)$ is unbounded near every point of $\partial\Omega$. In $\Re^n$, $n \geq 3$, it is known (see Axler et al.[5, p.207]) that the answer is yes; and this proof can be easily adapted to show that in a hyperbolic Riemannian manifold of dimension $d \geq 2$, given any domain Ω, there exists a positive harmonic function $h(x)$ in Ω such that $\limsup_{x \to y} h(x) = \infty$ for every $y \in \partial\Omega$. But concerning $\Re^2$, there is only a remark in the above reference [5] saying that it may not be possible to construct a positive harmonic function on $\Omega \subset \Re^2$ that is unbounded near every point of $\partial\Omega$, with the example of $\Omega = \Re^2 \setminus \{0\}$ where every positive harmonic function is constant.The following theorem is of some interest in this context.

Theorem 4.4. *Let Ω be a proper domain in a parabolic Riemannian manifold R.Then the following are equivalent.*

(1)There exists a bounded non-constant harmonic function on Ω.

(2) There exists a positive harmonic function $h(x)$ on Ω such that $\limsup_{x \to y} h(x) = \infty$ *for every $y \in \partial\Omega$.*

(3) $R \setminus \Omega$ is not polar.

Proof. (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3). If $R \setminus \Omega$ were polar, every positive harmonic function and hence every bounded harmonic function on Ω should be constant.

(3) $\Rightarrow$ (1). Suppose every bounded harmonic function on Ω is constant. Then Theorem 3.2 says that $R \setminus \Omega$ is polar, contradicting the hypothesis (3).

(3) $\Rightarrow$ (2). Recall that on a domain ω in a parabolic space R, the Green potential exists if and only if ω^c is not polar. Let $y \in \partial\Omega$. Let ω_n be a decreasing sequence of domains such that $\{y\} = \bigcap_n \omega_n$. Writing $e = R \setminus \Omega$, suppose $e \cap \omega_n^c$ is polar for every n. Then $\bigcup_n (e \cap \omega_n^c)$ is polar and hence $e \setminus \{y\} = e \cap (\bigcap_n \omega_n)^c = e \cap (\bigcup_n \omega_n^c) = \bigcup_n (e \cap \omega_n^c)$ is polar. Since $\{y\}$ is polar, $e = (e \setminus \{y\}) \cup \{y\}$ also is polar, a contradiction to the hypothesis (3). Hence for some ω_m, $e \cap \omega_m^c$ is not polar. That is, if $\Omega_1 = \Omega \cup \omega_m$, then Ω_1 is a domain in R such that Ω_1^c is not polar. Hence Ω_1 is a Green domain and consequently we can choose a function $p_y(x)$ on Ω_1 such that $p_y(x)$ is positive harmonic on $\Omega_1 \setminus \{y\} \supset \Omega$ and $\lim_{x \to y} p_y(x) = \infty$.

Thus, if we choose a countable dense set $\{y_n\}$ of $\partial\Omega$, for each n we have a positive harmonic function $u_n(x)$ on Ω such that $\lim_{x \to y_n} u_n(x) = \infty, x \in \Omega$. Fix $x_0 \in \Omega$, and let $h_n(x) = \dfrac{u_n(x)}{n^2 u_n(x_0)}$. Then $h(x) = \sum_n h_n(x)$ is a positive harmonic function on Ω, tending to ∞ at every $y_n \in \partial\Omega$. Since $\{y_n\}$ is dense in $\partial\Omega$, $\limsup_{x \to y} h(x) = \infty$ for every $y \in \partial\Omega$. □

Corollary 4.5. *Let Ω be an open set, not connected in a parabolic Riemannian manifold R. Then there exists a positive harmonic function h in Ω such that* $\limsup_{x \to y} h(x) = \infty$ *for every $y \in \partial\Omega$.*

Corollary 4.6. *Let ω be a regular domain in a parabolic Riemannian manifold R. Let e be a closed set in R and $\Omega = R \setminus e$. Suppose the harmonic measure* *of* $e \cap \partial\omega$ *with respect to* ω *is not* 0. *Then there exists a non-constant bounded harmonic function* $b(x)$ *on* Ω *and also a positive harmonic function* $h(x)$ *on* Ω *such that* $\limsup_{x \to y} h(x) = \infty$ *for every* $y \in \partial\Omega$.

In view of Theorems 4.1 and 4.4 we have the following result.

Theorem 4.7. *Let* Ω *be a proper subdomain in a parabolic Riemannian manifold* R. *Then we have the following two mutually exclusive cases: either there exists a harmonic function* $v(x)$ *on* Ω *such that* $\lim_{x \to y} v(x) = \infty$ *for every* $y \in \partial\Omega$ *or there exists a positive harmonic function* $h(x)$ *on* Ω *such that* $\limsup_{x \to y} h(x) = \infty$ *for every* $y \in \partial\Omega$ *which is equivalent to saying that there exists a non-constant bounded harmonic function on* Ω.

In the hyperbolic case, we have the following analogous results.

Theorem 4.8. *Let* Ω *be a proper subdomain in a hyperbolic Riemannian manifold* R. *Then,*

(1) There exists a positive harmonic function $h(x)$ *on* Ω *such that* $\limsup_{x \to y} h(x) = \infty$ *for every* $y \in \partial\Omega$; *and*

(2) If every bounded harmonic function in R *is constant, then we have the following two mutually exclusive cases: either there exists a positive harmonic function* $u(x)$ *on* Ω *such that* $\lim_{x \to y} u(x) = \infty$ *for every* $y \in \partial\Omega$ *or there exists a bounded non-constant harmonic function on* Ω.

5. Extensions to the Brelot axiomatic case

In this section, we indicate (without proofs) how the above results in a Riemannian manifold of dimension $d \geq 2$ can be extended to a (Brelot) harmonic space X in the axiomatic potential theory. Let us make the following assumptions. a) X is a locally compact, connected space with a countable base, provided with a harmonic sheaf satisfying the axioms 1,2,3 of Brelot [11, pp.13-14]; the constants are harmonic on X but there may or may not be any positive potential on X.

b) Every point in X is polar.

c) If a harmonic function h on a domain $\Omega \subset X$ vanishes in a neighbourhood of a point in Ω, then $h \equiv 0$ on Ω (the property of analyticity, see de la Pradelle [19, p.391]).

Recall that we say that a set e in X is polar if there exists a superharmonic function s on X such that $s = \infty$ on e; s can be taken as a potential if there are positive potentials on X. As mentioned earlier, it is proved in [19] that a closed set e in X is polar if and only if $\overset{\circ}{e} = \emptyset$ and for one (and hence any) open set $\omega \supset e$, any bounded harmonic function on $\omega \setminus e$ extends harmonically on ω. Moreover, in such a harmonic space X, the following theorems can be proved as in the case of a Riemannian manifold.

Theorem 5.1. *Assume that there are positive potentials on X and let Ω be a domain in X, $\Omega \neq \phi, \Omega \neq X$. Then,*

(1) There always exists a positive superharmonic function $u(x)$ on Ω such that $\limsup_{x \to y} u(x) = \infty$ for every $y \in \partial\Omega$.

(2) If we suppose further that $X \in O_{HB}$, then either there exists a bounded non-constant harmonic function on Ω or there exists a finite-valued positive superharmonic function $s(x)$ on Ω such that $\lim_{x \to y} s(x) = \infty$ for every $y \in \partial\Omega$.

Theorem 5.2. *Suppose there are no positive potentials on X and let Ω be a domain in X, $\Omega \neq \phi, \Omega \neq X$. Then either there exists a finite-valued superharmonic function $s(x)$ on Ω such that $\lim_{x \to y} s(x) = \infty$ for every $y \in \partial\Omega$ or there is a bounded non-constant harmonic function on Ω which is equivalent to saying that there is a positive superharmonic function $u(x)$ on Ω such that $\limsup_{x \to y} u(x) = \infty$ for every $y \in \partial\Omega$.*

Remarks 5.3. *Suppose in the above assumptions a), b) and c) made on X , we replace b) by a stronger assumption b)': To every point y in X, there is an open neighbourhood N of y and a potential $p_y(x)$ in N with point harmonic support y such that $\lim_{x \to y} p_y(x) = \infty$. Remark that this stronger condition b)' is satisfied not only in Riemannian manifolds of dimension $d \geq 2$ but also* *in the harmonic spaces defined by the* $\mathbb{C}^2$ *-solutions of certain second order elliptic differential operators (see Hervé [16, Proposition* 35.1*]). If X satisfies the assumptions a), b)' and c), then in the above two theorems, we can replace the positive superharmonic function* $u(x)$ *on* Ω *by a positive harmonic function* $h(x)$ *on* Ω *such that* $\lim_{x\to y}\sup h(x) = \infty$ *for every* $y \in \partial\Omega$.

References

(1) H. Aikawa and M. Essén: *Potential Theory-Selected Topics*, Lecture Notes 1633, Springer-Verlag, 1996.

(2) V. Anandam: *Espaces harmoniques sans potentiel positif*, Ann. Inst. Fourier, 22 (1972), 97 – 160.

(3) V. Anandam: *Sur une propriété de la fonction harmonique*, Bull. Sc. Math., France, 101(1977), 255 – 263.

(4) D. H. Armitage and S .J. Gardiner: *Classical Potential Theory*, Springer Monographs in Mathematics, 2001.

(5) S. Axler, P. Bourdon and W. Ramey: *Harmonic function theory*, Springer-Verlag, N.Y., 1992.

(6) T. Bagby and P. Blanchet: *Uniform harmonic approximation on Riemannian manifolds*, J. d'Analyse Math., 62 (1994), 47 – 76.

(7) H. Bauer: *Propriétés fines des fonctions hyperharmoniques dans une théorie axiomatique du potentiel*, Ann. Inst. Fourier, 15 (1965), 137 – 154.

(8) G. Bouligand: *Sur le problème de Dirichlet*, Ann. Soc. Polonaise, 4 (1926), 59 – 112.

(9) M. Brelot: *Sur la théorie autonome des fonctions sousharmoniques*, Bull.Sci.Math.France, 65 (1941), 78 – 98.

(10) M. Brelot: *Éléments de la théorie classique du potentiel*, Les Cours de Sorbonne, CDU, Paris, 1965.

(11) M. Brelot: *Axiomatique des fonctions harmoniques*, Les Presses de l'Université de Montréal, 1966 .

(12) L. Carleson: *Selected topics on exceptional sets*, Van Nostrand Math. Studies 13, Van Nostrand, 1968.

(13) H. Cartan: *Capacité extérieure et suites convergentes de potentiels*, C.R.Acad.Sci., Paris, 214(1942), 944 – 946.

(14) H. Cartan: *Théorie du potentiel newtonien: énergie, capacité, suites de potentiels*, Bull.Soc.Math., France, 73(1945), 74 – 106.

(15) J. L. Doob: *Classical Potential Theory and Its Probablistic Counterpart*,Grundlehren 262, Springer-Verlag, 1984.

(16) R. M. Hervé: *Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel*, Ann. Inst. Fourier, 12 (1962), 415 – 571.

(17) T. Kilpeläinen: *Potential theory for supersolutions of degenerate elliptic equations*, Indiana Univ. Math.Journal, 38(1989), 253 – 275.

(18) T. Kilpeläinen: *Singular Solutions to p-Laplacian type equations*, Ark. Math., 37 (1999), 275 – 289.

(19) S. I. Othman and V. Anandam: *Singularities of bounded harmonic functions*, Real Analysis Exchange, 25 (1999-2000),1 – 5.

(20) S. C. Port and C. J. Stone: *Brownian Motion and Classical Potential Theory*, Academic Press, 1978.

(21) A. de la Pradelle: *Approximation et caractére de quasi-analyticité dans la théorie axiomatique des fonctions harmoniques*, Ann. Inst.Fourier,17 (1967), 383 – 399.

(22) L. Sario and K. Oikawa: *Capacity functions*, Springer-Verlag, N.Y.,1969.

1. Victor Anandam
Ramanujan Institute of Mathematics, University of Madras,
Chennai-600 005.
E-Mail: bhickooa@yahoo.co.uk

2. Suad AlHemedan
Dept.of Mathematics, King Saud University, P.O.Box No 2455,
Riyadh 11451, Saudi Arabia.
E-mail: suad_alhemedan@hotmail.com

Date received August 15, 2008