Arab J. Math. Sc.
Volume 4, Number 1, June 1998
pp. 39-45

# BIHARMONIC GREEN FUNCTIONS IN A RIEMANNIAN MANIFOLD

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ABSTRACT. In a Riemannian manifold $R$, a biharmonic Green function can be defined if and only if there exists a nonharmonic positive superharmonic function $s$ in $R$ such that $\Delta s$ is also a superharmonic function.

## 1. INTRODUCTION

Leo Sario defines the notions of the biharmonic Green functions [6] and the biharmonic measure of the ideal boundary [5] in a Riemannian manifold $R$ and proves that there exists a biharmonic Green function in $R$ if and only if the biharmonic measure of the Alexandrov point at infinity of $R$ is finite. In particular, he shows that the biharmonic Green functions exist on the Euclidean space $\mathbb{R}^n$ if and only if $n \geq 5$.

Now, another distinguishing feature of $\mathbb{R}^n$, $n \geq 5$, is that the fundamental solution of the operator $\Delta^2$ is proportional to $S_n = r^{-n+4}$ which is a potential and so is $\Delta S_n$, where $\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x^{i2}}$. This suggests the following result proved here: There exists a biharmonic Green function in a Riemannian manifold $R$ if and only if there exists a nonharmonic positive superharmonic function $s$ in $R$ such that $\Delta s$ is also superharmonic, $\Delta$ being the Laplace-Beltrami operator.

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*1991 Mathematics Subject Classification.*31C12,31B30.

## 2. PRELIMINARIES

Let $R$ be an oriented Riemannian manifold of dimension $n \geq 2$ with local parameters $x = (x^1, \cdots, x^n)$ and a $C^\infty$ metric tensor $g_{ij}$ such that $g_{ij}x^i x^j$ is positive definite. If $D$ is the determinant of $g_{ij}$, denote the volume element by $dx = D^{\frac{1}{2}} dx^1 \ldots dx^n$; $\Delta = d\delta + \delta d$ is the Laplace-Beltrami operator, acting on $R$ in the sense of distributions; in the Euclidean case, $\Delta$ reduces to the form $-\sum_{i=1}^{n} \frac{\partial^2}{\partial x^{i2}}$.

M. Brelot [2] shows that given a Radon measure $\mu \geq 0$ on open set $\omega$ in $\mathbb{R}^n$, $n \geq 2$, there exists a superharmonic function $s$ in $\omega$ with associated measure $\mu$ in a local Riesz representation. The same theorem can be proved (as Theorem 4.2 [1]) in the axiomatic potential theory of Brelot, by making use of an approximation theorem proved by A. De la Pradelle [4] in this context; now we can deduce from it (or prove directly using the Approximation Theorem 3.10 of T. Bagby and P. Blanchet [3] in a Riemannian manifold $R$) the following theorem:

**Theorem 2.1:** *Let $\mu \geq 0$ be a Radon measure defined on an open set $\omega$ in a Riemannian manifold $R$. Then there exists a superharmonic function $s$ in $\omega$ such that $\Delta s = \mu$ in $\omega$.*

**Remarks.** 1) Given a locally $dx$-integrable function $f$ on an open set $\omega$ in $R$, we write $\Delta u = f$ if $u$ is a $\delta$-superharmonic function in $\omega$ such that $\Delta u = \mu$ where $\mu$ is the signed measure defined by $d\mu(x) = f(x)dx$.

2) In the above theorem, $s$ will not be always positive. However, if $\mu \geq 0$ has compact support in $\omega$, and if $R$ is hyperbolic we easily see that $s$ can be taken as a potential in $\omega$.

**Lemma 2.2.** *Let $u \geq 0$ be a superharmonic function in an open set $\omega \subset R$, with associated measure $\sigma \geq 0$; that is $\Delta u = \sigma$. Let $\mu$ be a Radon measure in $\omega$ such that $0 \leq \mu \leq \sigma$. Then there exists a potential $p$ in $\omega$ such that $\Delta p = \mu$.*

*Proof.* Let $\nu = \sigma - \mu$. Then $\nu$ is a Radon measure $\geq 0$ in $\omega$ and by Theorem 2.1 there exist superharmonic functions $s$ and $t$ in $R$ such that

$\Delta s = \mu$ and $\Delta t = \nu$.

This implies that $s + t = u +$ ( a harmonic function $h$ ) in $\omega$ and thus $s$ majorizes the subharmonic function $h - t$ in $\omega$ (since $u \geq 0$).

Consequently, $s$ can be represented in $\omega$ as the sum of a potential $p$ and a harmonic function which may not be positive.

We have $\Delta p = \mu$.

## 3. BIHARMONIC GREEN FUNCTIONS

**Definition 3.1.** Given $y$ in a hyperbolic Riemannian manifold $R$, let $G_y(x)$ denote the Green function in $R$ with point harmonic support $\{y\}$ and flux $G_y$ at infinity equal to -1. If there exists a potential $g_y$ in $R$ such that $\Delta g_y = G_y$, we term $g_y$ as the *biharmonic Green function* in $R$ with point biharmonic singularity $\{y\}$.

**Note.** If $g_y$ exists, it is unique. For, suppose $g_1$ is another such potential. Then $\Delta g_y = \Delta g_1$ implies that $g_y = g_1 +$ (a harmonic function $h$) in $R$. Since $g_y$ and $g_1$ are potentials, $h \equiv 0$.

**Theorem 3.2.** *In a Riemannian manifold $R$, the following are equivalent:*

*1) There exists a nonharmonic superharmonic function $s \geq 0$ in $R$ such that $\Delta s$ is superharmonic.*

*2) There exists a potential $g > 0$ in $R$ such that $\Delta g$ is a potential.*

*3) Given $y \in R$, the biharmonic Green function with biharmonic singularity $\{y\}$ exists in $R$.*

*Proof.*

1) $\Rightarrow$ 2): Since $s$ is superharmonic, $\Delta s \geq 0$; since $\Delta s$ is superharmoic and $s$ is nonharmonic, $t = \Delta s > 0$ in $R$.

Let $A$ be a parametric ball in $R$. Since $R$ has to be hyperbolic by hypothesis, $p = \hat{R}_t^A$ is a potential in $R$; $\hat{R}_t^A$ is the lower semi- continuous regularisation of $R_t^A = \inf\{v : v \text{ superharmonic} > 0 \text{ in } R \text{ such that } v \geq t$ on $A\}$.

Since $0 < p \leq t$, by Lemma 2.2 there exists a potential $g$ in $R$ such that $\Delta g = p$.

2) $\Rightarrow$ 3): We assume that there exists a potential $g > 0$ in $R$ such that $p = \Delta g$ is also a potential in $R$.

Given $y$, choose an open parametric ball $B$ containing $y$. Let $G_y(x)$ be the Green function in $R$ with harmonic point support $\{y\}$. Then, (if $DG_y$ denotes the Dirichlet solution in $R\backslash\bar{B}$ with boundary values $G_y$ on $\partial B$ and 0 at the Alexandrov point at infinity of $R$) $DG_y = G_y$ in $R\backslash\bar{B}$. For, clearly $DG_y \leq G_y$ in $R\backslash\bar{B}$ and hence $G_y - DG_y$ extended by 0 on $\bar{B}$ is a subharmonic function majorized by the potential $G_y$ in $R$; hence $G_y - DG_y \equiv 0$.

Consequently, if $\max_{\partial B} G_y = \alpha \min_{\partial B} p$, $G_y \leq \alpha p$ in $R\backslash B$.

Let $0 \leq \varphi \leq 1$ be a continuous function with compact support in $R$ and equal to 1 on $\bar{B}$. Then, there exists a potential $g_1$ on $R$ such that $\Delta g_1(x) = \varphi(x) G_y(x)$.

Then $u = \alpha g + g_1$ is a potential in $R$ such that $\Delta u \geq G_y$ in $R$. Hence by Lemma 2.2, there exists a potential $g_y$ in $R$ such that $\Delta g_y = G_y$.

3) $\Rightarrow$ 1) Evident.

**Note.** This theorem incidentally establishes also that if the biharmonic Green function $g_y$ exists in $R$ for some one point $y$, then such a function exists with a point biharmonic singularity at any given point in $R$.

## 4. SARIO CRITERION FOR THE EXISTENCE OF BIHARMONIC GREEN FUNCTIONS

Given $y \in R$, let $\mathcal{F}$ be an increasingly filtered family of relatively compact domains $\omega$ in $R$, such that $y \in \omega$ and $R = \bigcup_{\mathcal{F}} \omega$.

For $\omega \in \mathcal{F}$, let $p_\omega$ denote the unique potential in $\omega$ with point support $\{y\}$ which is of the form $p_\omega(x) = e_y(x) -$ (a harmonic function in $\omega$); here $e_y(x)$ denotes the harmonic Green function or the Evans function (p. 60 [3]) with point singularity $\{y\}$, depending on whether $R$ is hyperbolic or parabolic. It's easy to see that $p_\omega \geq p_{\omega'}$ in $\omega' \subset \omega$.

For each $\omega \in \mathcal{F}$, we have a unique potential $g_\omega$ in $\omega$ such that $\Delta g_\omega = p_\omega$ constructed as follows: choose $\omega_1 \in \mathcal{F}$ such that $\bar{\omega} \subset \omega_1$ and a function $s$ such that $\Delta s = p_{\omega_1}$ in $\omega_1$. Clearly $s$ is superharmonic in $\omega_1$ and if $h$ is its g.h.m. in $\omega$, $\Delta(s-h) = p_{\omega_1} \geq p_\omega$ in $\omega$. Then by Lemma 2.2, there exists the unique potential $g_\omega$ in $\omega$ such that $\Delta g_\omega = p_\omega$ in $\omega$.

It's easy to see that $g_\omega \geq g_{\omega'}$ in $\omega' \subset \omega$. For $g_\omega = g_{\omega'} +$ a superharmonic function $s$ in $\omega'$; hence $g_{\omega'} \geq -s$ implies that $-s \leq 0$ in $\omega'$.

Consequently, $\sup_{\mathcal{F}} g_\omega$ is either superharmonic in $R$ or $\equiv \infty$.

**Definition 4.1.** We say that $R$ satisfies the *Sario criterion* for the existence of biharmonic functions if $\sup_{\mathcal{F}} g_\omega$ is a potential in $R$.

In Chapter VIII [7] it's proved that $R$ satisfies the Sario criterion if and only if the biharmonic measure of the Alexandrov point at infinity of $R$ is finite. Here is another characteristic of this criterion.

**Theorem 4.2.** *$R$ satisfies the Sario criterion if and only if there exists a potential $g > 0$ in $R$ such that $\Delta g$ is a potential (the second equivalent condition in Theorem 3.2).*

*Proof.* Assume that the equivalent conditions of Theorem 3.2 are satisfied. Then, given $y$ in $R$ there exists a potential $g_y$ in $R$ such that

$\Delta g_y = G_y$.

Since $G_y \geq p_\omega$ for every $\omega \in \mathcal{F}$, we find that $g_y \geq g_\omega$ in $\omega$. Hence $\sup_{\mathcal{F}} g_\omega \leq g_y$; that is, $R$ satisfies the Sario condition.

Conversely, suppose $g = \sup_{\mathcal{F}} g_\omega$ is a potential. Then, $\sup_{\mathcal{F}} p_\omega$ is a potential. For, if $\omega \in \mathcal{F}$, $g = g_\omega +$ a superharmonic function in $\omega$ and hence $\Delta g \geq \Delta g_\omega = p_\omega$, in $\omega$, which shows that $\sup_{\mathcal{F}} p_\omega \not\equiv \infty$ in $R$.

We'll complete the proof by showing that $\Delta g = p$.

Write $g = \sup_n g_{\omega_n}$ for an increasing sequence $\omega_n \in \mathcal{F}$, $R = \cup_n \omega_n$. Then, in the sense of distributions, $\Delta g = \lim_n \Delta g_{\omega_n} = \lim_n p_{\omega_n} = p$.

**Remark.** We've proved above that if $\sup_{\mathcal{F}} g_\omega$ is a potential, then $\sup_{\mathcal{F}} p_\omega$ also is a potential. But when $\sup_{\mathcal{F}} g_\omega \equiv \infty$, $\sup_{\mathcal{F}} p_\omega$ may be a potential (as in the case of the Euclidean spaces $\mathbb{R}^n$, $n = 3, 4$) or $\sup_{\mathcal{F}} p_\omega$ may be $\equiv \infty$ (as in $\mathbb{R}^2$).

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Date received September 20, 1997.