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NOTE ON TOTALLY REAL PISOT-NUMBERS IN THE SUCCESSIVE DERIVED SETS OF PISOT-NUMBERS

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ABSTRACT. Let S denote the set of Pisot (or P-V) numbers and let θ be an element of S of degree d whose conjugates are real. It is shown that θ belongs to the kth derived set of S where k is the integer part of an exponential function of d. This improves the known result that θ belongs to the first derived set of S [4 and 2].

INTRODUCTION

A real algebraic integer $\theta > 1$ is said a Pisot (or $P-V$) number if all its conjugates over the field of rational numbers $\mathbb{Q}$ lie strictly within the unit circle.

In [5], Salem has shown that S is a closed subset of the real line. Then, J. Dufresnoy and C. Pisot have studied the successive derived sets $S^{(k)}$ of S. One of their results [4] shows that the smallest element of $S^{(k)}$ is $\geq k^{\frac{1}{4}}$ (this bound has been improved by D. W. Boyd [3] to $(k+1)^{\frac{1}{2}}$). Therefore for every Pisot number θ, the greatest rational integer k_θ such that $\theta \in S^{(k_\theta)}$ is defined. The next result [2, theorem 6.3.3] gives a lower bound for k_θ :

Theorem A. *Let θ be a Pisot number and k be a rational integer. If the absolute value of the minimal polynomial of θ over $\mathbb{Q}$ is greater on the unit circle than the sum of the absolute values of k polynomials with rational integer coefficients , then $k_\theta \geq k$.*

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The inequality $\min S^{(k)} \leq k+1$ is immediate from theorem A. It has been improved in the same period by D. W. Boyd [3] who has shown that $\limsup(\frac{\min S^{(k)}}{k}) < 1$ and by M. J. Bertin [1] who has proved that $\min S^{(k)} \leq \frac{k}{2}+1$ (resp. $\min S^{(k)} \leq \frac{k+1+\sqrt{(k+1)^2+16}}{4}$) when k is even (resp. when k is odd).

Another corollary [4 and 2, theorem 6.2.3] of theorem A , shows that if the conjugates of θ are real, then $k_\theta \geq 1$. If we combine the inequalities obtained for this case in [6]

$$\theta \geq (1.4954...)^{d-1}, \tag{1}$$

where d is the degree of θ over $\mathbb{Q}$ and

$$\theta \geq (1.5366...)^{d-1}. \tag{2}$$

When d is large, we obtain

Theorem. *Let θ be a Pisot number of degree d over $\mathbb{Q}$ whose conjugates are real, then there exists a constant $c_d \geq 1.1892$ such that $k_\theta \geq E(c_d^{d-1})$ where E is the integer part function. Furthermore if $d \geq 6$ (resp. d is large), then $c_d \geq 1.2$ (resp. $c_d \geq 1.2395$).*

Proof. As in [4] and in the proof of the theorem 6.2.3 of [2], let α be a real number and z be a complex number of modulus 1, then

$$|z-\alpha| \geq |z-1| \left|\frac{1+\alpha}{2}\right|. \tag{3}$$

If we change the signs of z and α in (3) we obtain

$$|z-\alpha| \geq |z+1| \left|\frac{1-\alpha}{2}\right|. \tag{4}$$

Let d be the degree of the minimal polynomial P of the Pisot number θ whose conjugates are real. From (3) and (4) we have

$$|P(z)| > |z-1|^p \, |z+1|^n \, |z-\theta| \sqrt{\frac{|P(0)|}{\theta}}, \tag{5}$$

where p (resp. n) is the number of positive (resp. negative) conjugates of θ not equal to θ.

From (3) and (5) we obtain

$$|P(z)| > |z-1|^{(p+1)}\, |z+1|^n \, \frac{\sqrt{|P(0)|}}{2}(\sqrt{\theta}+\frac{1}{\sqrt{\theta}}). \tag{6}$$

Combined with (5), the inequality $|z-\theta| \geq \theta - 1$ yields

$$|P(z)| > \; |z-1|^p \, |z+1|^n \sqrt{|P(0)|}\left(\sqrt{\theta}-\frac{1}{\sqrt{\theta}}\right). \tag{7}$$

Let $Q(z)$ be the polynomial $(z-1)^{p+1}(z+1)^n$ (resp. the polynomial $(z-1)^p(z+1)^n$) and k be the integer part of $\frac{\sqrt{|P(0)|}}{2}(\sqrt{\theta}+\frac{1}{\sqrt{\theta}})$ (resp. of $\sqrt{|P(0)|}\left(\sqrt{\theta}-\frac{1}{\sqrt{\theta}}\right)$), then we obtain from (6) and (7) the inequality $|P(z)| > k\,|Q(z)|$ which, using theorem A, gives

$$k_\theta \geq \max\left\{E(\frac{\sqrt{|P(0)|}}{2}(\sqrt{\theta}+\frac{1}{\sqrt{\theta}})), E(\sqrt{|P(0)|}(\sqrt{\theta}-\frac{1}{\sqrt{\theta}}))\right\}. \tag{8}$$

As $(\sqrt{\theta}+\frac{1}{\sqrt{\theta}}) > 2$, the inequality (8) gives for $1 \leq d \leq 5$,

$$k_\theta \geq E\left(\frac{\sqrt{|P(0)|}}{2}\left(\sqrt{\theta}+\frac{1}{\sqrt{\theta}}\right)\right) \geq 1 = E(1.1892^{d-1}).$$

The result follows immediately from inequalities (1) (resp. (2)) and (8) when $d \geq 6$ (resp. d is large).

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