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BIHARMONIC POINT SINGULARITIES IN $\mathbb{R}^n$

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ABSTRACT. Biharmonic functions near a singular point in $\mathbb{R}^n$, $n \geq 2$, finite or infinite, are studied. A type of Liouville-Picard theorem for biharmonic functions is proved.

1. INTRODUCTION

We study in this note the behaviour of biharmonic functions in a neighbourhood of singular points in $\mathbb{R}^n, n \geq 2$, finite or infinite. The use of distributions and the generalized Kelvin transforms of biharmonic functions simplifies the calculations to a certain extent.

Among the results we have proved here that depend on the dimension n, we mention two:

(1) Let u be a biharmonic function defined outside a compact set in $\mathbb{R}^n$. (a) If $u(x) = o(|x|)$ when $|x| \to \infty$, then u has a finite limit at infinity if $n \geq 5$ and (b) if $u(x)$ is bounded, then u has a limit at infinity if $n \geq 4$.

(2) Let $u : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a continuous function. (a) If $u \geq 0, \Delta u \leq 0$ and $\Delta^2 u \leq 0$, then u is a constant if $2 \leq n \leq 4$, and (b) if $u \geq 0, \Delta u \leq 0$ and $\Delta^2 u = 0$, then u is a constant for all $n \geq 2$.

In the last section, we gather together similar results for polyharmonic functions in $\mathbb{R}^n$ which can be proved, as the results above for biharmonic functions, using the Poisson integrals and the Kelvin transforms.

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2. REMOVABLE BIHARMONIC POINT SINGULARITY IN $\mathbb{R}^n$

A bounded harmonic function $h(x)$ in $0 < |x| < 1$ in $\mathbb{R}^n$, $n \geq 2$, extends as a harmonic function in $|x| < 1$. However, it is shown in [2], that any bounded biharmonic function $u(x)$ in $0 < |x| < 1$ in $\mathbb{R}^n$ extends as a biharmonic function in $|x| < 1$ if and only if $n \geq 4$.

In this section, we give an improved version of this result by proving that if the biharmonic function $u(x)$ in $0 < |x| < 1$ is of order $o(|x|^{4-n})$ near 0 when $n \geq 5$ and of order $o(\log \frac{1}{|x|})$ near 0 when $n = 4$, then also $u(x)$ extends as a biharmonic function in $|x| < 1$. We obtain also a necessary and sufficient condition for the biharmonic function $u(x)$ in $0 < |x| < 1$ to extend as a biharmonic function in $|x| < 1$ when $n = 2$ or 3.

Lemma 2.1. *Let $u(x)$ be a biharmonic function in $0 < |x| < 1$ in $\mathbb{R}^2$, and $u(x) = o(\log \frac{1}{|x|})$ when $|x| \to 0$. Then $u(x)$ is of the form*

$$u(x) = B(x) + (\alpha|x|^2 + \alpha_1 x_1 + \alpha_2 x_2)\log|x| + \alpha_3 \cos 2\theta + \alpha_4 \sin 2\theta,$$

where $B(x)$ is biharmonic in $|x| < 1, x = (x_1, x_2)$ and $\tan\theta = \dfrac{x_2}{x_1}$.

Proof. Define $u(0) = \liminf_{|x|\to 0} u(x)$. Thus extended, u becomes a lower semicontinuous function in $|x| < 1$, bounded near 0 by $\log \frac{1}{|x|}$ and hence defines a distribution in $|x| < 1$. Then $\Delta^2\, u$ is a distribution with point support $\{0\}$, and consequently $\Delta^2 u$ can be represented in $|x| < 1$ as $\Delta^2 u = \sum_f a_\alpha \partial^\alpha \delta$ (where $\sum_f$ stands for a finite sum).

If $S(x) = \frac{1}{8\pi}|x|^2 \log|x|$,then $\Delta^2 S = \delta$ and hence $\Delta^2 u = \sum_f a_\alpha \partial^\alpha (\Delta^2 S)$. Thus $T = u - \sum_f a_\alpha \partial^\alpha S$ is a distribution in $|x| < 1$ such that $\Delta^2 T = 0$; consequently, there exists a biharmonic function $b(x)$ in $|x| < 1$ such that $T = b$ (a.e.) in $|x| < 1$. This implies that $u - \sum_f a_\alpha \partial^\alpha S = b$ in $0 < |x| < 1$ since all the functions in this equation are continuous in

$0 < |x| < 1$. Taking the derivatives of $S(x) = \frac{1}{8\pi}|x|^2 \log|x|$ in $0 < |x| < 1$ and taking into account that $u(x) = o(\log\frac{1}{|x|})$ near 0, we obtain

$$\begin{aligned} u(x) &= b(x) + a_1|x|^2\log|x| + a_2|x|\cos\theta(1+2\log|x|) \\ &+a_3|x|\sin\theta(1+2\log|x|) + a_4(1+2\log|x|+2\cos^2\theta) \\ &+a_5(1+2\log|x|+2\sin^2\theta) + a_6\sin 2\theta. \end{aligned}$$

In this equation, since $u(x)$ is $o(\log\frac{1}{|x|})$ and $b(x)$ is bounded near 0, we see that $a_4 + a_5 = 0$.

Write $B(x) = b(x) + a_2|x|\cos\theta + a_3|x|\sin\theta$. Then $B(x)$ is biharmonic in $|x| < 1$ and we have the representation

$$\begin{aligned} u(x) &= B(x) + (\alpha|x|^2 + \alpha_1 x_1 + \alpha_2 x_2)\log|x| + \alpha_3\cos 2\theta \\ &+\alpha_4 \sin 2\theta \text{ in } 0 < |x| < 1. \end{aligned}$$

Theorem 2.2. *Let $u(x)$ be a biharmonic function in $0 < |x| < 1$ in $\mathbb{R}^2$. Then u extends as a biharmonic function in $|x| < 1$ if and only if u and Δu are $o(\log\dfrac{1}{|x|})$ when $|x| \to 0$.*

Proof. By Lemma 2.1, u can be written as,

$u(x) = B(x) + (\alpha|x|^2 + \alpha_1 x_1 + \alpha_2 x_2)\log|x| + \alpha_3\cos 2\theta + \alpha_4\sin 2\theta$ in $0 < |x| < 1$. Since $B(x)$ is biharmonic in $|x| < 1, h(x) = \Delta B(x)$ is harmonic in $|x| < 1$.

Hence,

$$\begin{aligned} \Delta u(x) &= h(x) + \alpha(4 + 4\log|x|) + \frac{2\alpha_1\cos\theta}{|x|} + \frac{2\alpha_2\sin\theta}{|x|} \\ &-\frac{4\alpha_3\cos 2\theta}{|x|^2} - \frac{4\alpha_4\sin 2\theta}{|x|^2} \end{aligned}$$

If $\Delta u(x) = o(\log\frac{1}{|x|})$ when $|x| \to 0$, then $\alpha = 0$ and $\alpha_i = 0, i = 1,2,3,4$.

Hence $u(x) = B(x)$ in $0 < |x| < 1$. That is, u extends as a biharmonic function. The converse is obvious.

The following proposition is a slightly improved version of Theorem 1 in Sario-Wang [4], where it is shown that if u is a bounded biharmonic function in $|x| > 0$, it is generated by 1, $\cos 2\theta$ and $\sin 2\theta$.

Proposition 2.3. *Let $u(x)$ be a biharmonic function in $|x| > 0$ in $\mathbb{R}^2$. Suppose $u(x) = o(\log \frac{1}{|x|})$ near 0 and $u(x) = o(|x|)$ near the point at infinity. Then u is a linear combination of 1, $\cos 2\theta$ and $\sin 2\theta$.*

Proof. We shall use the representation of u near 0 as given in Lemma 2.1, since u is $o(\log \frac{1}{|x|})$ near 0. Here, since u is biharmonic in $|x| > 0, B(x)$ is biharmonic in $\mathbb{R}^2$. Hence by the Almansi expansion, $B(x) = |x|^2 h_1(x) + h_2(x)$ where $h_1(x)$ and $h_2(x)$ are harmonic in $\mathbb{R}^2$.

Taking the mean-values on $|x| = r$, we get $r^2 h_1(0) + h_2(0) =$ (a function of order $o(r)$ near infinity) $- \alpha r^2 \log\, r$, when $r \to \infty$; consequently, $\alpha = 0$, and in $|x| > 0$ we have

$$B(x) = u(x) - (\alpha_1 x_1 + \alpha_2 x_2) \log |x| - \alpha_3 \cos 2\theta - \alpha_4 \sin 2\theta. \tag{1}$$

Since by hypothesis $u(x) = o(|x|)$ near infinity, $B(x) = o(|x|^2)$ near infinity. Now, $B(x)$ being biharmonic in $\mathbb{R}^2$, this condition implies that $B(x)$ is of the form $c + c_1 x_1 + c_2 x_2$ in $\mathbb{R}^2$.

Substituting in (1) and checking the coefficients of x_1 and x_2 on both sides, we see that $\alpha_1 = 0$ and $\alpha_2 = 0$. Then again the growth of u near infinity implies that $c_1 = 0$ and $c_2 = 0$. Thus, from (1), $c = u(x) - \alpha_3 \cos 2\theta - \alpha_4 \sin 2\theta$ in $|x| > 0$. In other words, $u(x)$ is a linear combination of 1, cos 2θ and sin 2θ.

In the case of $\mathbb{R}^3$, using the representation of biharmonic functions with a point singularity (similar to the one given in $\mathbb{R}^2$), we can obtain the following improved versions of some results given in Anandam and Damlakhi [2] and Theorem 2 in Sario-Wang [4] which states that a bounded biharmonic function in $|x| > 0$ in $\mathbb{R}^3$ is generated by 1,

$\sin\theta\cos\psi$, $\sin\theta\sin\psi$ and $\cos\theta$.

Lemma 2.4. *Let $u(x)$ be a biharmonic function in $0 < |x| < 1$ in $\mathbb{R}^3$. If $u(x) = o(\frac{1}{|x|})$ when $|x| \to 0, u(x)$ is of the form $u(x) = b(x) + \alpha|x| + \frac{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}{|x|}$, where $b(x)$ is biharmonic in $|x| < 1$.*

Theorem 2.5. *Let $u(x)$ be a biharmonic function in $0 < |x| < 1$ in $\mathbb{R}^3$. Then u extends as a biharmonic function in $|x| < 1$ if and only if $u(x)$ and $\Delta u(x)$ are $o(\frac{1}{|x|})$ when $|x| \to 0$.*

Proposition 2.6. *Let $u(x)$ be a biharmonic function in $|x| > 0$ in $\mathbb{R}^3$. Suppose $u(x) = o(\frac{1}{|x|})$ when $|x| \to 0$, and $u(x) = o(|x|)$ near infinity. Then u is a linear combination of 1,$\frac{x_1}{|x|}, \frac{x_2}{|x|}$and $\frac{x_3}{|x|}$.*

In the sequel, we denote by $S_n(x)$ the fundamental solution of Δ^2 in $\mathbb{R}^n, n \geq 2$, that is $\Delta^2 S_n(x) = \delta$.

Lemma 2.7. *Let $u(x)$ be a biharmonic function in $0 < |x| < 1$ in $\mathbb{R}^n, n \geq 4$. Suppose $u(x) = o(|x|^{3-n})$ when $|x| \to 0$. Then there exists a biharmonic function $B(x)$ in $|x| < 1$ such that $u(x) = B(x) + \alpha S_n(x)$ in $0 < |x| < 1$. .*

Proof. Defining $u(0) = \liminf_{|x|\to 0} u(x)$, we obtain a distribution which can be represented as in the proof of Lemma 2.1 as $u(x) = B(x) + \sum_f a_\alpha \partial^\alpha(S_n(x))$ in $0 < |x| < 1$. Since $S_4(x) = c_4 \log|x|$ and $S_n(x) = c_n|x|^{-n+4}$ for $n \geq 5$, the condition $u(x) = o(|x|^{3-n})$ implies that $u(x) = B(x) + \alpha S_n(x)$ in $0 < |x| < 1$, for all $n \geq 4$.

Theorem 2.8. *Let $u(x)$ be a biharmonic function in $0 < |x| < 1$ in $\mathbb{R}^n$, $n \geq 4$.*

Then u extends as a biharmonic function in $|x| < 1$ if

i) $u(x) = o(|x|^{4-n})$ when $|x| \to 0$, for $n \geq 5$;

ii) $u(x) = o(\log \frac{1}{|x|})$ *when* $|x| \to 0$, *for* $n = 4$.

Proof. The stated conditions on $u(x)$ imply that $u(x) = o(|x|^{3-n})$ when $|x| \to 0$, for $n \geq 4$. Hence by Lemma 2.7, $u(x) = B(x) + \alpha S_n(x)$ in $0 < |x| < 1$. Consequently, the theorem is proved since $S_4(x) = c_4 \log |x|$ and $S_n|x| = c_n|x|^{-n+4}$ for $n \geq 5$.

3. BIHARMONIC FUNCTIONS OF ORDER $o(|x|)$ AT INFINITY

In this section we consider certain properties of biharmonic functions defined outside a compact set in $\mathbb{R}^n$, $n \geq 2$, (that is, functions with biharmonic singularity at infinity) which are of growth of order $o(|y|)$ when $|y| \to \infty$. In particular, we show that such functions have a finite limit at infinity if $n \geq 5$; and any bounded biharmonic function defined outside a compact set has a limit at infinity if $n \geq 4$.

As is customary (Nicolesco [3] p. 14) we say that a biharmonic function $u(y)$ defined in $|y| > r$ is biharmonic at infinity if its extended Kelvin transform $Ku(x) = |x|^{4-n}u(\frac{x}{|x|^2})$ which is biharmonic in $0 < |x| < \frac{1}{r}$ extends as a biharmonic function in $|x| < \frac{1}{r}$.

Lemma 3.1. *Let* u *be a biharmonic function defined outside a compact set in* $\mathbb{R}^n, n \geq 5$. *Then* u *is biharmonic at infinity if and only if* $\lim_{|y|\to\infty} u(y) = 0$.

Proof. If u is biharmonic at infinity, $\lim_{|x|\to 0} |x|^{4-n}u(\frac{x}{|x|^2})$ exists; this implies that $\lim_{|y|\to\infty} |y|^{n-4}u(y)$ is finite and hence $\lim_{|y|\to\infty} u(y) = 0$.

Conversely, suppose $\lim_{|y|\to\infty} u(y) = 0$. Then $Ku(x) = |x|^{4-n}u(\frac{x}{|x|^2}) = o(|x|^{4-n})$, when $|x| \to 0$. Hence by Theorem 2.8, $Ku(x)$ extends as a biharmonic function in a neighbourhood of 0.

Theorem 3.2. *Let* $u(y)$ *be a biharmonic function defined outside a compact set in* $\mathbb{R}^n$, $n \geq 5$. *Then the following are equivalent.*

1) *u is bounded near infinity.*

2) $u(y) = o(|y|)$ *when* $|y| \to \infty$.

3) *There exists an* α *such that* $u(y) - \alpha$ *is biharmonic at infinity.*

4) *u has a finite limit at infinity.*

Proof. 4)$\Rightarrow$ 1) $\Rightarrow$ 2) : Obvious.

2)$\Rightarrow$ 3) : $Ku(x) = |x|^{4-n}u(\frac{x}{|x|^2}) = o(|x|^{3-n})$ when $|x| \to 0$. Hence by Lemma 2.7, $Ku(x) = B(x) + \alpha|x|^{4-n}$ in $0 < |x| < r$ and $B(x)$ is biharmonic in $|x| < r$. This means that $K[u(y) - \alpha] = Ku(x) - \alpha|x|^{4-n}$ extends as a biharmonic function including $x = 0$; that is $u(y) - \alpha$ is biharmonic at infinity.

3)$\Rightarrow$ 4) :By Lemma 3.1, $\lim_{|y|\to\infty}(u(y) - \alpha) = 0$.

Theorem 3.3. *Let* $u(y)$ *be a biharmonic function defined outside a compact set in* $\mathbb{R}^4$. *Then the following are equivalent.*

1) *u is bounded near infinity.*

2) $u(y) = o(\log|y|)$ *when* $|y| \to \infty$.

3) $u(y)$ *is biharmonic at infinity.*

4) *u has a finite limit at infinity.*

Proof. 4)$\Rightarrow$ 1) $\Rightarrow$ 2) : Obvious.

2) $\Rightarrow$ 3) : $Ku(x) = u(\frac{x}{|x|^2}) = o(\log\frac{1}{|x|})$ when $|x| \to 0$; by Theorem 2.8, $Ku(x)$ extends as a biharmonic function in $|x| < r$. That is, $u(y)$ is biharmonic at infinity.

3) $\Rightarrow$4): Since $u(y)$ is biharmonic at infinity, $\lim_{|x|\to 0} u(\frac{x}{|x|^2}) = \lim_{|x|\to 0}$ $Ku(x)$ is a finite number α . Hence $\lim_{|y|\to\infty} u(y) = \alpha$.

Theorem 3.4. *The extended Kelvin transform in* $\mathbb{R}^n$, $n \geq 4$, *establishes a (1,1) correspondence between the biharmonic functions in* $|x| < 1$ *and the biharmonic functions* $b(y)$ *in* $|y| > 1$ *for which* $\lim_{|y|\to\infty} |y|^{n-4} b(y)$ *is finite.*

Proof. Let $b(y)$ be biharmonic in $|y| > 1$ for which $\lim_{|y|\to\infty} |y|^{n-4} b(y)$ is finite. Then $|b(y)| \leq \frac{A}{|y|^{n-4}}$ for $|y| > r$. Hence, $Kb(x) = |x|^{4-n} b(\frac{x}{|x|^2})$ is bounded near 0.

Since a bounded biharmonic function in $0 < |x| < \frac{1}{r}$ extends as a biharmonic function in $|x| < \frac{1}{r}$ when $n \geq 4$, $Kb(x)$ can be considered as a biharmonic function in $|x| < 1$.

Conversely, let $u(x)$ be a biharmonic function in $|x| < 1$. Then $Ku(y) = |y|^{4-n} u(\frac{y}{|y|^2})$ is biharmonic in $|y| > 1$ and $\lim_{|y|\to\infty} |y|^{n-4} Ku(y) = \lim_{|y|\to\infty} u(\frac{y}{|y|^2}) = \lim_{|x|\to 0} u(x)$ is a finite number.

More generally, the following theorem characterizes the extended Kelvin transform of biharmonic functions $u(y)$ defined outside a compact set and of growth of order $o(|y|)$ near infinity in $\mathbb{R}^n$,$n \geq 2$.

Theorem 3.5. *In* $\mathbb{R}^n$,$n \geq 2$, *let* Γ *be the class of biharmonic functions* $u(y)$ *in* $|y| > 1$ *such that* $u(y) = o(|y|)$ *when* $|y| \to \infty$ *and let* Γ_0 *be the class of biharmonic functions* $u(x)$ *in* $0 < |x| < 1$ *of the form* $u(x) = B(x) + \alpha S_n(x)$ *where* $B(x)$ *is biharmonic in* $|x| < 1$ *and moreover* $B(x) = o(|x|^{3-n})$ *when* $|x| \to 0$ *if* $n = 2$ *or* 3. *Then the extended Kelvin transform establishes a (1,1) correspondence between* Γ *and* Γ_0.

Proof. Let $u \in \Gamma$. Then $Ku(x) = |x|^{4-n} u(\frac{x}{|x|^2}) = o(|x|^{3-n})$ near 0 and hence $Ku(x) = B(x) + \alpha S_n(x)$, as a consequence of Lemma 2.7 (when $n \geq 4$), of Lemma 2.4 (when $n = 3$) and of Lemma 2.1 (when $n = 2$). Moreover, when $n = 2$ or 3, since $Ku(x) = o(|x|^{3-n})$ and also $S_n(x) = o(|x|^{3-n})$ we have $B(x) = o(|x|^{3-n})$ when $|x| \to 0$ and hence $Ku \in \Gamma_0$.

Conversely, let $v \in \Gamma_0$. That is $v(x) = B(x) + \alpha S_n(x)$ in $0 < |x| < 1$ where $B(x)$ is biharmonic in $|x| < 1$ and $B(x) = o(|x|^{3-n})$ when $|x| \to 0$, if $n = 2$ or 3. Then

$$Kv(y) = |y|^{4-n} v(\frac{y}{|y|^2}) = |y|^{4-n}[B(\frac{y}{|y|^2}) + \alpha S_n(\frac{y}{|y|^2})]$$

and hence

$$\lim_{|y|\to\infty} \frac{Kv(y)}{|y|} = \lim_{|y|\to\infty} [|y|^{3-n} B(\frac{y}{|y|^2}) + \alpha |y|^{3-n} S_n(\frac{y}{|y|^2})].$$

Since $\lim_{|y|\to\infty} |y|^{3-n} S_n(\frac{y}{|y|^2}) = 0$ for all n, we have

$$\lim_{|y|\to\infty} \frac{Kv(y)}{|y|} = \lim_{|y|\to\infty} |y|^{3-n} B(\frac{y}{|y|^2}) = \lim_{|x|\to 0} \frac{B(x)}{|x|^{3-n}} = 0$$

by hypothesis. Hence $Kv \in \Gamma$.

Since the inverse of the Kelvin transform is of the same form, the theorem is proved.

4. LIOUVILLE-PICARD THEOREM FOR BIHARMONIC FUNCTIONS IN $\mathbb{R}^n$

The Liouville-Picard theorem for harmonic functions can be stated as follows: A positive function $h(x)$ in $\mathbb{R}^n$, $n \geq 2$, is a constant if h is superharmonic when $n = 2$ or if h is harmonic when $n \geq 2$. In this section we obtain similar results for biharmonic functions in $\mathbb{R}^n$.

Let u be a biharmonic function in $\mathbb{R}^n$, $n \geq 2$. It is known (Nicolesco [3], pp. 19 and 20) that if $u(x) = o(|x|)$ near infinity, then u is a constant. We give below a slightly improved version of this result.

Lemma 4.1. *Let $u(x)$ be a biharmonic function in $\mathbb{R}^n$, $n \geq 2$, such that $\liminf_{|x|\to\infty} \frac{u(x)}{|x|} > -\infty$. Then $u(x) = c|x|^2 + h(x)$ where c is a positive constant and $h(x)$ is a harmonic function.*

Proof. By Almansi expansion $u(x) = |x|^2 h_1(x) + h(x)$. By hypothesis, $u(x) > A|x|$ when $|x| > r$. Let $z \in \mathbb{R}^n$ and $a > \max(r, |z|)$. If $d\rho_z^a$ represents the harmonic measure (the Poisson kernel)) on $|x| = a$, we have

$$h_1(z) = \int_{|x|=a} h_1(x) d\rho_z^a(x) \geq \int_{|x|=a} \left(\frac{A}{|x|} - \frac{h(x)}{|x|^2} \right) d\rho_z^a(x) = \frac{A}{a} - \frac{h(z)}{a^2}.$$

Allowing $a \to \infty, h_1(z) \geq 0$. Thus h_1 being a positive harmonic fucntion in $\mathbb{R}^n$ is a constant. Hence the lemma.

Proposition 4.2. *Let u be a biharmonic function majorizing a subharmonic function in $\mathbb{R}^n$, $n \geq 2$. Then u itself is a subharmonic function of the form $u(x) = c|x|^2 + h(x)$ where $c \geq 0$ and h is harmonic.*

Proof. Let $u(x) = |x|^2 h_1(x) + h(x) \geq s(x)$ in $\mathbb{R}^n$ where $s(x)$ is subharmonic. Then as in Lemma 4.1, for $z \in \mathbb{R}^n, |z| < |x| = a$, we have

$$a^2 h_1(z) + h(z) = \int_{|x|=a} u(x) d\rho_z^a(x) \geq \int_{|x|=a} s(x) d\rho_z^a(x) \geq s(z).$$

Dividing by a^2, and allowing $a \to \infty$, we see that $h_1(z) \geq 0$ except possibly on a polar set where $s(z) = -\infty$; however, h_1 being harmonic, $h_1 \geq 0$ in $\mathbb{R}^n$ and hence h_1 is a constant $c \geq 0$. Thus $u(x) = c|x|^2 + h(x)$ is a subharmonic function in $\mathbb{R}^n$.

Theorem 4.3. *Let $u(x)$ be a biharmonic function in $\mathbb{R}^n$, $n \geq 2$, such that* $\limsup_{|x|\to\infty} \frac{u(x)}{|x|^2} \leq 0 \leq \liminf_{|x|\to\infty} \frac{u(x)}{|x|}$. *Then u is a constant.*

Proof. By Lemma 4.1, $u(x) = c|x|^2 + h(x), c \geq 0$. Now $\limsup_{|x|\to\infty} \frac{u(x)}{|x|^2} \leq 0$ implies that $u(x) \leq \varepsilon - |x|^2$ if $|x| \geq r$. Then as in the above lemma we obtain, c$\leq \varepsilon - \frac{h(z)}{a^2}$ which implies that $c \leq 0$. Hence $u(x)$ is harmonic in $\mathbb{R}^n$. Since it satisfies the condition $\liminf_{|x|\to\infty} \frac{u(x)}{|x|} \geq 0$, it is a constant (see for example [1]).

Corollary 4.4 [3]. *Let u be a biharmonic function in $\mathbb{R}^n$, $n \geq 2$. If $u(x) = o(|x|)$ near infinity, in particular if $u(x)$ is bounded , then u is a constant.*

It can be proved also either using the Almansi expansion or as a consequence of the following Proposition 4.5 that a positive biharmonic function in $\mathbb{R}^n$, $n \geq 2$, is a constant if it is superharmonic.

Proposition 4.5. *Let $u(x)$ be a superharmonic function on $\mathbb{R}^n$, $n \geq 2$. Suppose $\Delta^2 u = 0$(Δ in the sense of distributions) and* $\liminf_{|x|\to\infty} \frac{u(x)}{|x|} \geq 0$. *Then u is a constant.*

Proof. Since $\Delta u \leq 0$ is harmonic in $\mathbb{R}^n$, Δu is a constant $c \leq 0$. Hence $\Delta(u(x) - \frac{c|x|^2}{2n}) = 0$. This means that there exists a harmonic function $h(x)$ in $\mathbb{R}^n$ such that $u(x) - \frac{c|x|^2}{2n} = h(x)$ a.e.

Since the superharmonic function $u(x)$ is majorized by the harmonic function $h(x)$ a.e., $u(x) \leq h(x)$ everywhere in $\mathbb{R}^n$. Then the assumed condition on u implies that $\liminf_{|x|\to\infty} \frac{h(x)}{|x|} \geq 0$, and hence h is a constant.

Now, if $c < 0$, then $\liminf_{|x|\to\infty} \frac{u(x)}{|x|} = -\infty$, a contradiction. Hence $c = 0$. That is, u is a constant.

In the above proposition, if the condition that u is biharmonic is replaced by the condition $\Delta^2 u \geq 0$, u need not be a constant as seen by taking $u(x) = |x|^{4-n}$ for $n \geq 5$. However we have

Theorem 4.6. *Let u be a superharmonic function in $\mathbb{R}^n$, $2 \leq n \leq 4$, for which $\Delta^2\, u \geq 0$. Suppose u satisfies an additional condition:*

1. *When $n = 2$,* $\liminf_{|x|\to\infty} \frac{u(x)}{\log|x|} > -\infty$.

2. *When $n = 3$,* $\liminf_{|x|\to\infty} \frac{u(x)}{|x|} \geq 0$.

3. *When* $n = 4$, $\liminf_{|x|\to\infty} \dfrac{u(x)}{\log|x|} \geq 0.$

Then $u(x)$ *is a constant.*

Proof. By Lemma 4.2 in [2], the above conditions would imply that $u(x)$ is harmonic outside the origin in $\mathbb{R}^n$ in all the three cases; that is, $u(x)$ is a superharmonic function with point (harmonic) support. Hence $\Delta u = c\delta$ in $\mathbb{R}^n$ where δ is the Dirac measure and c is a constant, $c \leq 0$.

But this is incompatible with the assumption that $\Delta^2 u \geq 0$ unless $c = 0$. Now, $c = 0$ implies that u is harmonic in $\mathbb{R}^n$, satisfying (by the assumption in the theorem) the condition $\liminf_{|x|\to\infty} \dfrac{u(x)}{|x|} \geq 0$. Hence u is a constant.

5. POLYHARMONIC FUNCTIONS OF FINITE ORDER

A C^{2p}-function u in an open set ω in $\mathbb{R}^n$, $n \geq 2$, is called a p-harmonic function if $\Delta^p u = 0$ in ω. Then the above results concerning biharmonic functions can readily be extended to p-harmonic functions in $\mathbb{R}^n$. We indicate below a few such extensions.

1. Let $u(x)$ be a p-harmonic function in $0 < |x| < 1, n \geq 2p$. Suppose $u(x) = o(|x|^{2p-1-n})$ when $|x| \to 0$. Then there exists a p-harmonic function $v(x)$ in $|x| < 1$ such that $u(x) = v(x) + \alpha e_{n,p}(x)$ in $0 < |x| < 1$ where $\Delta^p e_{n,p} = \delta$ in $\mathbb{R}^n$. Consequently, $u(x)$ extends as a p-harmonic function in $|x| < 1$, (a) if $u(x) = o(|x|^{2p-n})$ when $|x| \to 0$ for $n \geq 2p+1$ and (b) if $u(x) = o(\log \dfrac{1}{|x|})$ when $|x| \to 0$ for $n = 2p$.

2. Let $u(x)$ be a p-harmonic function defined outside a compact set in $\mathbb{R}^n$, $n \geq 2p$. Then,

   a) If $n \geq 2p + 1$, the following are equivalent: (i) $u(x)$ is bounded near infinity. (ii) $u(x) = o(|x|)$ near infinity. (iii) $u(x)$ has a finite limit at infinity.

b) If $n = 2p$, the following are equivalent: (i) $u(x)$ is bounded near infinity. (ii) $u(x) = o(\log|x|)$ near infinity. (iii) $u(x)$ has a finite limit at infinity.

3. Let $u \geq 0$ be a p-harmonic function in $\mathbb{R}^n$, for which $(-1)^m \Delta^m u \geq 0 (0 < m < p)$. Then u is a constant.

A p-harmonic function $u \geq 0$ for which $(-1)^m \Delta^m u \geq 0, 0 < m < p$, is called a completely superharmonic function and a representation for such a function in $\mathbb{R}^n$ is given on p. 21 [3] which together with the conditions $u \geq 0$ and $\Delta u \leq 0$ implies that u is a constant. Proposition 4.5 is a somewhat generalized version of this result for a biharmonic function in $\mathbb{R}^n$.

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