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APPROXIMATION OF CONJUGATES OF ALMOST LIPSCHITZ FUNCTIONS BY MATRIX-CESÀRO SUMMABILITY METHOD

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ABSTRACT. In this paper a new theorem on the degree of approximation of conjugates of functions in an almost Lip α class by the Matrix-Casàro $(T)C_1$ summability method of the conjugate series of a Fourier series has been proved.

1. DEFINITIONS AND NOTATIONS

Let $\sum\limits_{n=0}^{\infty} u_n$ be an infinite series with n^{th} partial sum $s_n = \sum\limits_{\nu=0}^{n} u_\nu$. If

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s, \quad \text{as} \quad n \to \infty$$

then $\sum\limits_{n=0}^{\infty} u_n$ is said to be the $(C,1)$ summable to s.

Let $T = (a_{n,k})$ be an infinite triangular matrix satisfying the Silverman-Toeplitz [9] conditions of regularity, i.e.,

$$\sum_{k=0}^{n} a_{n,k} \to 1 \quad \text{as} \quad n \to \infty, \quad a_{n,k} = 0 \ \text{ for } k > n$$

and

$$\sum_{k=0}^{n} |a_{n,k}| \leq M, \text{ a finite constat.}$$

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The sequence-to-sequence transformation

$$m_n = \sum_{k=0}^{n} a_{n,k} s_k$$

defines the sequence $\{m_n\}$ of matrix means of the sequence $\{s_n\}$ generated by the sequence of coefficients $(a_{n,k})$. The series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the sum s by matrix method if $\lim_{n\to\infty} m_n$ exists and equal to s. If

$$t_n = \sum_{k=0}^{n} a_{n,k}\sigma_k \to s, \quad \text{as} \quad n \to \infty$$

then we say that $\sum_{n=0}^{\infty} u_n$ is summable to s by Matrix-Cesàro means $(T)\,C_1$. It is denoted by $t_n \to s((T)C_1)$, as $n \to \infty$.

Thus, if the method of summability (T) is superimposed on the Cesàro means of order 1, another method of summability $(T)\,C_1$ i.e. Matrix-Cesàro summability is obtained.

The important particular cases of Matrix-Cesàro summability method are given by

1. $(C, \delta)\,C_1$, if $a_{n,k} = \dfrac{\binom{n-k+\delta-1}{\delta-1}}{\binom{n+\delta}{\delta}}$, $\delta > 0$
2. $(H, 1)\,C_1$, if $a_{n,k} = \frac{1}{(n-k+1)\log n}$
3. $(N, p_n)\,C_1$ if $a_{n,k} = \frac{p_{n-k}}{p_n}$
4. $(N, p, q)\,C_1$ if $a_{n,k} = \frac{p_{n-k}q_k}{R_n}$ where $R_n = \sum_{k=0}^{n} p_k q_{n-k}$.

Let $0 < \alpha \leq 1$ and let $f : R \to R$ be almost Lipschitz function of order α, $f \in \underset{a}{\mathrm{Lip}}\,\alpha$, in the sense that there is a constant $M = M_f \geq 0$ and for each $x \in R$, there is a subset $A_x \subset \left[0, \frac{\pi}{2}\right]$ of measure zero, such that $t \in \left[0, \frac{\pi}{2}\right] \setminus A_x$ implies

$$|f(x+2t) - f(x-2t)| = M\,t^{\alpha}$$

Remark: Almost Lipschitz class of order α, denoted by $\overset{a}{\mathrm{Lip}}\,\alpha$, generalizes greatly the class $\mathrm{Lip}\,\alpha$. Every $\mathrm{Lip}\,\alpha$ function is trivially $\overset{\alpha}{\mathrm{Lip}}\,\alpha$ but the converse is not true. For example, let g denote the characteristic function of the irrationals. Take

$$A_x = \left\{t \in \left[0, \frac{\pi}{2}\right] : \text{at least one of } (x+2t) \text{ and } (x-2t) \text{ is rational}\right\},$$

so that A_x being countable has measure zero.

For each x and each $t \in \left[0, \frac{\pi}{2}\right] \setminus A_x$ both $(x+2t)$ and $(x-2t)$ are irrational and so $|g(x+2t) - g(x-2t)| = 0$. Hence g is $\overset{a}{\mathrm{Lip}}\,\alpha$ for every α. But, obviously, g is not Lipschitz function of any non-zero order.

If f is $\overset{a}{\mathrm{Lip}}\,\alpha$ function, 2π-periodic on R and Lebesgue integrable on $[-\pi, \pi]$ then its Fourier series is given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u) \cos nu\, du, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u) \sin nu\, du$$

and the conjugate series of the Fourier series (1.1) is the series

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx). \tag{1.2}$$

For $0 < t \le \frac{\pi}{2}$, since $\sin t \ge \frac{2t}{\pi}$ so for each $x \in R$, we have

$$|\psi_x(t) \cot t| \le M t^{\alpha} \frac{\pi}{2t} = M\frac{\pi}{2} t^{\alpha-1}, \qquad t \in \left[0, \frac{\pi}{2}\right] \setminus A_x,$$

where

$$\psi_x(t) = f(x+2t) - f(x-2t).$$

Since A_x has measure zero, it follows at once that f has its conjugate function $\overline{f}$, (Zygmund [10]), defined and finite for each $x \in R$ by the improper Lebesgue integral

$$\overline{f}(x) = -\frac{1}{\pi}\int_0^{\pi/2} \psi_x(t) \cot t\, dt = -\frac{1}{\pi}\lim_{\varepsilon \to 0}\int_{\varepsilon}^{\pi/2} \psi_x(t) \cot t\, dt.$$

The degree of approximation of a function $f : R \to R$ by a trigonometric Polynomial t_n of order n is defined by, (Zygmund [10]),

$$\|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\}.$$

2. MAIN THEOREM

The degree of approximation of a function $f \in \operatorname{Lip}\alpha$ by Cesàro means, Nörlund means, Generalized Nörlund means, K^λ-means and (e.c.) means has been studied by Alexits [1], Siddiqi [7], Qureshi [5], Sahney and Goel [6], Chandra [2], Lal [4] and Srivastava and Verma [8]. Working in this direction Gát and Goginava [3] discussed some general methods with respect to summability of Nörlund-logarithmic means of Lipschitz type functions. But till now nothing seems to have been done for the approximation of conjugate function $\overline{f}$ of the function $f \in \overset{a}{\operatorname{Lip}}\,\alpha$ by product summability means of the form Matrix-Cesàro $(T)\,C_1$. Matrix-Cesàro summability method includes $(C,\delta)\,C_1$, $(H,1)\,C_1$, $(N,p_n)\,C_1$, $(N,p,q)\,C_1$ means as particular cases. In an attempt to make an advance in this paper, we study the degree of approximation of the conjugate functions $\overline{f}$ of the function $f \in \overset{a}{\operatorname{Lip}}\,\alpha$ by Matrix-Cesàro $(T)\,C_1$ product means of the conjugate Fourier series of f in the following form:

Theorem 2.1. *Let $T = (a_{n,k})$ be a lower triangular matrix with finite norms and $\sum_{k=0}^{n} \frac{a_{n,k}}{k+1} = O\left(\frac{1}{n+1}\right)$. If $f : R \to R$ is 2π-periodic and Lebesgue integrable on $[-\pi,\pi]$ and it is almost $\operatorname{Lip}\alpha$, $f \in \overset{a}{\operatorname{Lip}}\,\alpha$, then the degree of approximation of its conjugate function $\overline{f}$ by Matrix-Cesàro product means $\overline{t}_n = \sum_{k=0}^{n} a_{n,k}\overline{\sigma}_k$ of the conjugate Fourier series (1.2) satisfies, for $n = 0,1,2,\cdots$,*

$$\left\|\overline{t}_n(x) - \overline{f}\right\|_\infty = \begin{cases} O\left(\frac{1}{(n+1)^\alpha}\right), & 0<\alpha<1 \\ O\left(\frac{\log(n+1)}{(n+1)}\right), & \alpha = 1. \end{cases}$$

3. PROOF OF THE THEOREM

The r^{th} partial sum of the conjugate series (1.2) can be written as

$$\overline{s}_r(x) = \overline{f}(x) + \frac{1}{\pi}\int_0^{\pi/2} \psi_x(t)\frac{\cos(2r+1)t}{\sin t}dt$$

so the $(C,1)$ means of the series (1.2) are

$$\begin{aligned}
\overline{\sigma}_k(x) &= \frac{1}{k+1}\sum_{r=0}^{k}\overline{s}_r(x), \qquad k = 0,1,2,\cdots \\
&= \overline{f}(x) + \frac{1}{(k+1)\pi}\int_0^{\pi/2}\psi_x(t)\frac{\sum_{r=0}^{k}\cos(2r+1)t}{\sin t}dt \\
&= \overline{f}(x) + \frac{1}{(k+1)\pi}\int_0^{\pi/2}\psi_x(t)\cdot\frac{\cos(k+1)t\sin(k+1)t}{\sin^2 t}dt, \\
&= \overline{f}(x) + \frac{1}{\pi}\int_0^{\pi/2}\psi_x(t)\frac{\cos(k+1)t\sin(k+1)t}{(k+1)\sin^2 t}dt,
\end{aligned}$$

$$\left(\text{as}\quad \sum_{r=0}^{k}\cos(2r+1)t = \frac{\cos(k+1)t\sin(k+1)t}{\sin t}\right).$$

Therefore, the Matrix-Cesàro $(T)C_1$ product means of the series (1.2) are

$$\begin{aligned}
\overline{t}_n(x) &= \sum_{k=0}^{n} a_{n,k}\overline{\sigma}_k(x) \\
&= \overline{f}(x) + \frac{1}{\pi}\int_0^{\pi/2}\psi_x(t)\sum_{k=0}^{n}\frac{a_{n,k}}{(k+1)}\frac{\sin(k+1)t\cos(k+1)t}{\sin^2 t}dt
\end{aligned}$$

or,

$$\begin{aligned}
\overline{t}_n(x) - \overline{f}(x) &= \frac{1}{\pi}\int_0^{\pi/2}\psi_x(t)\sum_{k=0}^{n}\frac{a_{n,k}}{(k+1)}\cdot\frac{\sin(k+1)t\cos(k+1)t}{\sin^2 t}dt \\
&= \frac{1}{\pi}\int_0^{\frac{1}{n+1}}\psi_x(t)\sum_{k=0}^{n}\frac{a_{n,k}}{(k+1)}\cdot\frac{\sin(k+1)t\cos(k+1)t}{\sin^2 t}dt \\
&\quad + \frac{1}{\pi}\int_{\frac{1}{n+1}}^{\pi/2}\psi_x(t)\sum_{k=0}^{n}\frac{a_{n,k}}{(k+1)}\cdot\frac{\sin(k+1)t\cos(k+1)t}{\sin^2 t}dt \\
&:= I_1 + I_2 \qquad (3.1)
\end{aligned}$$

Since, here $\sin t \geq \frac{2t}{\pi}$, $|\sin\theta| \leq \theta$ and $|\cos\theta| \leq 1$, if follows that

$$|I_1| \leq \frac{1}{\pi}\int_0^{\frac{1}{n+1}} |\psi_x(t)| \sum_{k=0}^{n} \frac{a_{n,k}}{k+1} \frac{|\sin(k+1)t|\,|\cos(k+1)t|}{|\sin^2 t|} dt$$

$$\leq \frac{1}{\pi}\int_0^{\frac{1}{n+1}} Mt^{\alpha}\frac{\pi^2}{4t^2}\left(\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}(k+1)t\right)dt$$

$$= \frac{M\pi}{4}\int_0^{\frac{1}{n+1}} t^{\alpha-1}dt$$

$$= \frac{M\pi}{4\alpha(n+1)^{\alpha}} \tag{3.2}$$

Also,

$$|I_2| = \frac{1}{\pi}\int_{\frac{1}{n+1}}^{\pi/2} Mt^{\alpha}\cdot\frac{\pi^2}{4t^2}\sum_{k=0}^{n}\left(\frac{a_{n,k}}{k+1}\right)dt$$

$$= \frac{M\pi}{4(n+1)}\int_{\frac{1}{n+1}}^{\pi/2} t^{\alpha-2}dt, \qquad \text{by the hypothesis of the theorem}$$

$$= \begin{cases} \frac{M\pi}{4(1-\alpha)}\left[\frac{1}{(n+1)^{\alpha}} - \frac{1}{(n+1)}\left(\frac{2}{\pi}\right)^{1-\alpha}\right], & 0<\alpha<1 \\ \frac{M\pi}{4(n+1)}\left[\log(n+1)\frac{\pi}{2}\right], & \alpha=1 \end{cases} \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we have

$$\left|\bar{t}_n(x) - \bar{f}(x)\right| = \begin{cases} \frac{M\pi}{4\alpha(n+1)^{\alpha}} + \frac{M\pi}{4(1-\alpha)}\left[\frac{1}{(n+1)^{\alpha}} - \frac{1}{(n+1)}\left(\frac{2}{\pi}\right)^{1-\alpha}\right], & 0<\alpha<1 \\ \frac{M\pi}{4}\frac{1}{(n+1)} + \frac{M\pi}{4(n+1)}\log(n+1)\frac{\pi}{2}, & \alpha=1 \end{cases}$$

$$\leq \begin{cases} \frac{M\pi}{4}\left(\frac{1}{\alpha(1-\alpha)} + \left(\frac{2}{\pi}\right)^{1-\alpha}\right)\frac{1}{(n+1)^{\alpha}}, & 0<\alpha<1 \\ \frac{M\pi}{4}\left[1+\log(n+1)\frac{\pi}{2}\right], & \alpha=1 \end{cases}$$

Thus, we obtained that

$$\left\|\bar{t}_n(x) - \bar{f}(x)\right\|_{\infty} = \sup\left\{\left|\bar{t}_n(x) - \bar{f}(x)\right|\right\}$$

$$= \begin{cases} O\left(\frac{1}{(n+1)^{\alpha}}\right), & 0<\alpha<1 \\ O\left(\frac{\log(n+1)}{n+1}\right), & \alpha=1 \end{cases}$$

This completes the proof of the theorem.

4. FINAL REMARK

The following corollary can be derived from the main theorem.

Corollary 1. *If $f \in \operatorname{Lip}\alpha$, then the degree of approximation of its conjugate function $\overline{f}$ by matrix-Cesàro means is given by*

$$\left\|\bar{t}_n(x) - \overline{f}(x)\right\|_\infty = \begin{cases} O\left(\frac{1}{(n+1)^\alpha}\right), & 0 < \alpha < 1 \\ O\left(\frac{\log(n+1)}{n+1}\right), & \alpha = 1 \end{cases}$$

Remarks:

1. Independent proof of the corollary can be developed along the same line as the theorem
2. Results similar to main theorem and corollary may be derived for product summability methods $(C,\delta)\,C_1$, $(H,1)\,C_1$, $(N,p_n)\,C_1$ and $(N,p,q)\,C_1$.

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