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APPROXIMATION OF $W(L^p, \xi(t))$ FUNCTION BY $(N,p,q)C_1$ MEANS OF ITS FOURIER SERIES

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ABSTRACT. In this paper, the degree of approximation of a function belonging to $W(L^p, \xi(t))$ class by product summability method $(N,p,q)C_1$ of its Fourier series has been determined.

1. INTRODUCTION

Hardy [7] established a theorem on $(C,\alpha), (\alpha > 0)$ summability of the series. Harmonic summability is weaker than (C,α) summability. Iyengar [10] proved a theorem on harmonic summability of a Fourier series. The result of Iyengar [10] has been generalized by several researchers like Siddiqi [21], Pati [16], Singh [26], Rajagopal [20], Hirokawa [8], Hirokawa and Kayashima [9], Tripathi and Singh [28] and Singh [25] for Nörlund means. In 1959 Varshney [29], for the first time, studied the sequnce $\{nB_n(x)\}$ by product summability of the form $(H,1)C_1$. Later on $(N,p_n)C_1$ summability of sequence $\{nB_n(x)\}$ has been studied by number of researchers like Sharma ([23] & [24]), Prasad [17], Dwivedi [6], Dikshit [5] and Lal [12]. Working in the same direction, in 1996, Bhatt and Kathal [2] obtained interesting results on $(C,1)$ $(E,1)$ summability of Fourier series and its conjugate series. These results are recently generalized by Lal & Verma [14]. Here $(N,p,q)C_1$ summability is considered. $(N,p,q)C_1$ summability reduces to $(N,p_n)C_1$ if $q_n = 1$ $\forall n$ and $\left(\overline{N}, q_n\right) C_1$ if $p_n = 1$ $\forall n$. The degree of approximation by Cesàro means , Nörlund means, (N,p,q) means and matrix means of a function $f \in Lip\alpha$, $Lip(\alpha,p)$, $Lip(\xi(t),p)$ and $W(L^p, \xi(t))$ has been studied by number of researchers like Alexits [1], Sahney

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and Goel [22], Chandra [4], Qureshi [18], Qureshi & Nema [19] and Khan [11]. But till now no work seems to have been done to obtain the degree of approximation of the function $f \in W(L^p, \xi(t))$ by product summability means of the form $(N,p,q)C_1$. In an attempt to make an advance in this direction, in this paper, we study the degree of approximation of a function belonging to $W(L^p, \xi(t))$ class by $(N,p,q)C_1$ means of its Fourier series.

2. DEFINITIONS AND NOTATIONS

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series whose n^{th} partial sum is S_n. Write

$$\sigma_n = \frac{1}{n+1}\sum_{v=0}^{n} S_v = (C,1) \tag{2.1}$$

means of the sequence $\{S_n\}$.
For any two sequences $\{p_n\}$ and $\{q_n\}$ of real numbers such that $p_0 > 0$, $q_0 > 0$, we write

$$t_n^{p,q} = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k S_k$$

where $R_n = (p * q)_n = \sum_{k=0}^{n} p_{n-k}\, q_k \neq 0\ \forall n$. The generalized Nörlund transform (N,p,q) of the sequence $\{S_n\}$ is the sequence $\{t_n^{p,q}\}$. If $t_n^{p,q} \to S$ as $n \to \infty$, then the sequence $\{S_n\}$ is said to be summable by generalized Nörlund method (N,p,q) to S (Borwein [3]) and is denoted by $t_n^{p,q} \to S(N,p,q)$, $n \to \infty$.

The (N,p,q) transform of the $(C,1)$ transform defines the $(N,p,q)C_1$ transform of the partial sum $\{S_n\}$ of the series $\sum_{n=0}^{\infty} u_n$. Thus if

$$t_n^{p,q,C_1} = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k \sigma_k = \frac{1}{R_n}\sum_{k=0}^{n} \frac{p_{n-k} q_k}{k+1}\sum_{v=0}^{k} S_v \tag{2.2}$$

tends to S as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or $\{S_n\}$ is said to be summable by$(N,p,q)\,C_1$ to S. It is denoted by $t_n^{p,q,C_1} \to S((N,p,q)\,C_1)$ as $n \to \infty$.

The necessary and sufficient conditions for the (N, p, q) method to be regular are

$$\sum_{k=0}^{n} |p_{n-k}q_k| = O(R_n) \tag{2.3}$$

and

$$p_{n-k} = o(R_n), \text{ as } n \to \infty, \text{ for every fixed } k > 0, \text{ for which } q_k \neq 0. \tag{2.4}$$

Cesàro summability of order 1 or $(C, 1)$ summability is also regular. Let us verify the regularity conditions of $(N, p, q)C_1$ method.

$$t_n^{p,q,C_1} = \frac{1}{R_n}\sum_{k=0}^{n} \frac{p_{n-k}q_k}{k+1}\sum_{v=0}^{k} S_v = \sum_{k=0}^{n} C_{n,k}\sigma_k$$

where

$$C_{n,k} = \begin{cases} \frac{1}{R_n}\left(\frac{p_{n-k}q_k}{k+1}\right)\sum\limits_{v=0}^{k} 1, & k \leq n \\ 0, & k > n \end{cases}$$

and so

$$C_{n,k} = \frac{1}{R_n}\left(\frac{p_{n-k}q_k}{k+1}\right)(k+1) = \frac{p_{n-k}q_k}{R_n} \qquad \text{for } k \leq n.$$

Now,

(i) $\sum_{k=0}^{\infty} |C_{n,k}| = \frac{1}{R_n}(\sum_{k=0}^{n} p_{n-k}q_k) = 1,$

(ii) $C_{n,k} = \frac{p_{n-k}q_k}{k+1} \to 0$ as $n \to \infty$, for fixed k, by (2.4),

(iii) $\sum_{k=0}^{\infty} |C_{n,k}| = 1.$

Thus $s_n \to s$ implies $\sigma_n \to s$ as $n \to \infty$. Consequently $t_n^{p,q,C_1} \to s$ as $n \to \infty$. Hence $(N, p, q)C_1$ summability method is regular.

Some important particular cases $(N, p, q)C_1$ means are

(i) $(N, p_n)C_1$, if $q_n = 1\ \forall n$,

(ii) $(\bar{N}, q_n)C_1$, if $p_n = 1\ \forall n$,

(iii) $(C, \alpha)C_1$, if $p_n = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$, $q_n = 1\ \forall n$.

Let $f(t)$ be a periodic function with period 2π, integrable in the sense of Lebesgue over $(-\pi,\pi)$. The Fourier series of $f(t)$ is given by

$$f(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt). \tag{2.5}$$

We define the norm $\|\ \|_p$by

$$\| f \|_p=\left\{\int_0^{2\pi}|f(x)|^p dx\right\}^{\frac{1}{p}} ;\ p\geq 1,$$

and let the degree of approximation $E_n(f)$ be given by (Zygmund [30])

$$E_n(f)=Min\ \| f-t_n \|_p\ ,$$

where $t_n(x)$ is some n^{th} degree trigonometric polynomial.

A function $f\in Lip\alpha$ if $f(x+t)-f(x)=O(t^\alpha)$, for $0<\alpha\leq 1$. $f\in Lip(\alpha,p)$ if $\left\{\int_0^{2\pi}|(x+t)-f(x)|^p dx\right\}^{\frac{1}{p}}=O(t^\alpha)$, $0<\alpha\leq 1$, $p\geq 1$ (def. 5.38 of McFadden [15]).

Given a positive increasing function $\xi(t)$ and an integer $p>1$, we find that $f\in Lip(\xi(t),p)$ if $\left\{\int_0^{2\pi}|(x+t)-f(x)|^p dx\right\}^{\frac{1}{p}}=O(\xi(t))$. When $\xi(t)=t^\alpha$, the class $Lip(\xi(t),p)$ coincides with $Lip(\alpha,p)$. $f\in W(L^p,\xi(t))$ (Lal [13]) if $\left\{\int_0^{2\pi}|(x+t)-f(x)|^p\sin^{\beta p}t\ dt\right\}^{\frac{1}{p}}=O(\xi(t))$, $\beta\geq 0$. In case $\beta=0$, we notice that $W(L^p,\xi(t))$ concides with $Lip(\xi(t),p)$.

We shall use the following notations:

$$\phi(t)=f(x+t)+f(x-t)-2f(x) \tag{2.6}$$

$$(NC)_1^{p,q}(t)=\frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_{n-k}q_k}{k+1}\frac{\sin^2(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}} \tag{2.7}$$

3. MAIN THEOREM

Quite good amount of work is known on degree of approximation of function belonging to $Lip\alpha$, $Lip(\alpha,p)$, $Lip(\xi(t),p)$, $W(L^p,\xi(t))$ class by Cesàro's means, Nörlund means, (N,p,q) means and matrix summability method. The purpose of this paper is to determine the degree of approximation of a function $f \in W(L^p, \xi(t))$ class by $(N,p,q)\,C_1$ means of its Fourier series. In fact, in this paper, we shall prove following theorem:

Theorem 3.1. *Let $\{p_n\}$ be a non-negative, non-increasing sequence and $\{q_n\}$ be a non-negative, non-decreasing sequence such that*

$$\sum_{k=0}^{n} \frac{p_{n-k}q_k}{k+1} = O\left(\frac{R_n}{n+1}\right).$$

If $f : R \to R$ is 2π-periodic, Lebesgue integrable on $[-\pi,\pi]$ and $W(L^p, \xi(t))$ class function then the degree of approximation of function f by $t_n^{p,q,C_1} = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}q_k\sigma_k$ i.e. $(N,p,q)\,C_1$-summability means of the Fourier series (2.5) *is given by*

$$\left\|t_n^{p,q,C_1}(x) - f(x)\right\|_p = O\left((n+1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right), \tag{3.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t\,|\phi(t)|}{\xi(t)}\right)^p \sin^{\beta p} t\, dt\right\}^{\frac{1}{p}} = O\left(\frac{1}{n+1}\right) \tag{3.2}$$

$$\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}\,|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} = O\left((n+1)^{\delta}\right) \tag{3.3}$$

where δ is an arbitrary number such that $q(1-\delta) - 1 > 0$, conditions (3.2) *and* (3.3) *hold uniformly in x.*

4. LEMMAS

For the proof of our theorem following lemmas are required

Lemma 4.1. *Let $(NC)_1^{p,q}(t)$ be given by* (2.7), *then*

$$(NC)_1^{p,q}(t) = O(n+1), \ \text{for}\ 0 < t \le \frac{1}{n+1}.$$

Proof.

$$\begin{aligned}(NC)_1^{p,q}(t) &= \frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_{n-k}q_k}{k+1}\frac{\sin^2(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\\ &\le \frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_{n-k}q_k}{k+1}(k+1)^2\frac{\sin^2\frac{t}{2}}{\sin^2\frac{t}{2}}\\ &= \frac{1}{2\pi R_n}\sum_{k=0}^{n}p_{n-k}q_k(k+1)\\ &\le \frac{(n+1)}{2\pi R_n}\sum_{k=0}^{n}p_{n-k}q_k\\ &= \frac{(n+1)}{2\pi}\\ &= O(n+1)\end{aligned}$$

□

Lemma 4.2. *Let* $(NC)_1^{p,q}$ *be given by* (2.7) *then*

$$(NC)_1^{p,q}(t) = O\left(\frac{1}{(n+1)t^2}\right), \quad \text{for } \frac{1}{n+1} < t \le \pi.$$

Proof.

$$\begin{aligned}(NC)_1^{p,q}(t) &= \frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_{n-k}q_k}{k+1}\frac{\sin^2(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\\ &= \frac{1}{4\pi R_n}\sum_{k=0}^{n}\frac{p_{n-k}q_k}{k+1}\frac{(1-\cos(k+1)t)}{\sin^2\frac{t}{2}}\\ &\le \frac{1}{4\pi R_n}\sum_{k=0}^{n}\frac{p_{n-k}q_k}{k+1}\frac{\pi^2}{t^2} \quad \left(\because \frac{1}{\sin\frac{t}{2}} \le \frac{\pi}{t}, \text{ by Jordan's lemma}\right)\\ &= \frac{\pi}{4R_n t^2}\sum_{k=0}^{n}\frac{p_{n-k}q_k}{k+1}\\ &= \frac{\pi}{4R_n t^2}O\left(\frac{R_n}{n+1}\right), \text{ by the hypothesis of the theorem}\\ &= O\left(\frac{1}{(n+1)t^2}\right),\end{aligned}$$

□

5. PROOF OF THE THEOREM

Following Titchmarsh [27], the n^{th} partial sum S_n of the Fourier series (2.5) at $t = x$ is given by

$$S_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt.$$

The $(C,1)$ is transform i.e. σ_n of $S_n(x)$ is given by

$$\frac{1}{n+1}\sum_{k=0}^{n}\left(S_k(x) - f(x)\right) = \frac{1}{2(n+1)\pi}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\left(\sum_{k=0}^{n}\sin\left(k+\frac{1}{2}\right)t\right)dt$$

or

$$\sigma_n(x) - f(x) = \frac{1}{2(n+1)\pi}\int_0^{\pi}\frac{\sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\phi(t)\,dt.$$

Denoting (N,p,q) transform of σ_n i.e. the $(N,p,q)\,C_1$ transform of $S_n(x)$ by t_n^{p,q,C_1}, we have

$$\frac{1}{R_n}\sum_{k=0}^{n}p_{n-k}q_k\left\{\sigma_k(x) - f(x)\right\} = \int_0^{\pi}\frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_{n-k}q_k}{k+1}\frac{\sin^2(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\phi(t)\,dt$$

or

$$\begin{aligned} t_n^{p,q,C_1}(x) - f(x) &= \int_0^{\pi}(NC)_1^{p,q}(t)\,\phi(t)\,dt \\ &= \int_0^{\frac{1}{n+1}}(NC)_1^{p,q}(t)\,\phi(t)\,dt + \int_{\frac{1}{n+1}}^{\pi}(NC)_1^{p,q}(t)\,\phi(t)\,dt \\ &= I_1 + I_2, \quad \text{say.} \end{aligned} \tag{5.1}$$

Applying Hölder's inequality and the fact that $\phi(t) \in W(L^p, \xi(t))$, we have

$$\begin{aligned} |I_1| &\le \left\{\int_0^{\frac{1}{n+1}}\left(\frac{t\,|\phi(t)|}{\xi(t)}\right)^p \sin^{\beta p}t\,dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{t\sin^{\beta}t}(NC)_1^{p,q}(t)\right\}^q dt\right\}^{\frac{1}{q}} \\ &= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left(O(n+1)\frac{\xi(t)}{t^{\beta+1}}\right)^q dt\right]^{\frac{1}{q}}, \end{aligned}$$

by Lemma (4.1) and cond. (3.2)

$$= O(1)\left[\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)}{t^{\beta+1}}\right)^q dt\right]^{\frac{1}{q}}$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right)\left[\int_\epsilon^{\frac{1}{n+1}}\frac{dt}{t^{q(\beta+1)}}\right],$$

by the mean value theorem, where $0 < \varepsilon < \frac{1}{n+1}$

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right)\left[\left\{\frac{t^{-q(\beta+1)+1}}{-q(\beta+1)+1}\right\}_\varepsilon^{\frac{1}{n+1}}\right]^{\frac{1}{q}}$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right)\left[(n+1)^{(\beta+1)q-1}\right]^{\frac{1}{q}}$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)(n+1)^{\left(\beta+1-\frac{1}{q}\right)}\right)$$

$$(5.2)\quad = O\left((n+1)^{\left(\beta+\frac{1}{p}\right)}\cdot\xi\left(\frac{1}{n+1}\right)\right)\qquad\left(\text{as } \frac{1}{p}+\frac{1}{q}=1\right).$$

Similarly, as above, we have

$$|I_2| \leq \int_{\frac{1}{n+1}}^{\pi}(NC)_1^{p,q}(t)\,\phi(t)\,dt$$

$$\leq \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\sin^\beta t\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)(NC)_1^{p,q}(t)}{t^{-\delta}\sin^\beta t}\right)^q dt\right\}^{\frac{1}{q}}$$

$$= \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{-\delta+\beta}}\cdot\frac{1}{(n+1)t^2}\right\}^q dt\right\}^{\frac{1}{q}},$$

by lemma (4.2) & cond. (3.3).

$$= O(n+1)^{\delta-1}\left[\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{2+\beta-\delta}}\right)^q\right]^{\frac{1}{q}}$$

$$= \left(O(n+1)^{\delta-1}\xi\left(\frac{1}{n+1}\right)\right)\left[\int_{\frac{1}{n+1}}^{\pi}\frac{dt}{t^{q(2+\beta-\delta)}}\right]^{\frac{1}{q}}$$

$$= \left(O(n+1)^{\delta-1}\xi\left(\frac{1}{n+1}\right)\right)\left[\frac{t^{-q(2+\beta-\delta)+1}}{-q(2+\beta-\delta)+1}\right]_{\frac{1}{n+1}}^{\pi}$$

$$= O\left((n+1)^{\delta-1}\xi\left(\frac{1}{n+1}\right)\right)\left((n+1)^{q(2+\beta-\delta)-1}\right)^{\frac{1}{q}}$$
$$= O\left((n+1)^{\beta+1-\frac{1}{q}}\xi\left(\frac{1}{n+1}\right)\right)$$
$$= O\left((n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right)\quad\left(\text{as } \frac{1}{p}+\frac{1}{q}=1\right). \tag{5.3}$$

By (5.1), (5.2), and (5.3), we have

$$\left|t_n^{p,q,C_1}(x)-f(x)\right| = O\left((n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right)$$

or

$$\begin{aligned}\left\|t_n^{p,q,C_1}(x)-f(x)\right\|_p &= O\left[\int_0^{2\pi}\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\}^p dx\right]^{\frac{1}{p}}\\ &= O\left((n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right)\left(\int_0^{2\pi}dx\right)^{\frac{1}{p}}.\\ &= O\left((n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right).\end{aligned}$$

This completes the proof of the theorem.

6. COROLLARIES

The following corollaries can be derived from the theorem.

Corollary 1. *If $\beta = 0$ and $\xi(t) = t^\alpha$, then the degree of approximation of a function $f \in Lip(\alpha, p)$ is given by* $\left|t_n^{p,q,C_1}(x)-f(x)\right| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right)$

Proof.

$$\begin{aligned}\left|t_n^{p,q,C_1}(x)-f(x)\right| &= O\left((n+1)^{\beta+\frac{1}{p}}\cdot\xi\left(\frac{1}{(n+1)}\right)\right)\\ &= O\left((n+1)^{\frac{1}{p}}\cdot\left(\frac{1}{n+1}\right)^{\alpha}\right)\\ &= O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right),\end{aligned}$$

which completes the proof. □

Corollary 2. *If $p \to \infty$ in Corollary 1, then we have, for $0 < \alpha < 1$,*

$$\left|t_n^{p,q,C_1}(x) - f(x)\right| = O\left(\frac{1}{(n+1)^{\alpha}}\right).$$

Remarks:

(1) An independent proof of Corollary 1 can be developed along the same lines as the theorem.

(2) Results similar to the main theorem and Corollaries 1 and 2 may be derived for $f \in Lip(\alpha, p)$ and $Lip\alpha$ for $(N, p_n)\, C_1$ taking $q_n = 1\ \forall n$.

Example 6.1. Consider the infinite series.

$$1 + 4\sum_{n=1}^{\infty} n(-1)^n. \tag{6.1}$$

The n^{th} partial sum of (1) is given by

$$\begin{aligned} s_n &= 1 + 4\sum_{k=1}^{n} k(-1)^k \\ s_n &= (2n+1)(-1)^n \end{aligned}$$

and so

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n}(2k+1)(-1)^k = (-1)^n.$$

Therefore the series (6.1) is not $(C, 1)$ summable. Since $\{(-1)^n\}$ is (N, p, q) summable therefore, the series (6.1) is $(N, p, q)\, C_1$ summable. Hence the product summability $(N, p, q)\, C_1$ is more powerful than the individual methods (N, p, q) and $(C, 1)$. Consequently $(N, p, q)\, C_1$ mean gives better approximation than individual methods (N, p, q) and $(C, 1)$.

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