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HYPERSURFACES IN THE EUCLIDEAN SPACE R^4

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ABSTRACT. For a compact and connected hypersurface M in the Euclidean space R^4, it is proved that, if the mean curvature is nowhere zero and the scalar curvature S satisfies $||\psi||^2 S = 6$, where ψ is the position vector field of M in R^4, then M is isometric to a sphere. It is also proved that if the Ricci curvature of the hypersurface M satisfies $0 < Ric \leq \frac{2}{3}\lambda_1$, where λ_1 is the first nonzero eigenvalue of the Laplacian on M, then M is isometric to a sphere.

1. INTRODUCTION

In the geometry of compact hypersurfaces of a Euclidean space R^n, one of the interesting questions is to obtain conditions under which the hypersurface is isometric to a sphere in R^n. This problem becomes more interesting for the hypersurfaces of even-dimensional Euclidean spaces R^{2n} owing to the complex geometry of R^{2n} and, in particular, of R^4 which has quaternion structure. The motivation for the present paper comes from the following considerations:

Let $S^n(c)$ be the n-sphere of constant curvature c in R^{n+1} centered at the origin and $\psi : S^n(c) \to R^{n+1}$ be the inclusion map. Then the mean curvature α and the scalar curvature S of S^n satisfy:

$$\alpha = -n\sqrt{c} < 0 \quad \text{and} \quad ||\psi||^2 S = n(n-1).$$

This raises the question: Is a compact and connected immersed hypersurface $\psi : M \to R^{n+1}$ whose mean curvature α is nowhere zero and

1991 Mathematics Subject Classification: Primary 53C40, Secondary 53C42, 53C20 whose scalar curvature S satisfies $\|\psi\|^2 S = n(n-1)$, necessarily isometric to a sphere in R^{n+1}?

Also the Ricci curvature Ric of $\psi : S^n(c) \to R^{n+1}$ satisfies $0 < \mathrm{Ric} = \frac{(n-1)}{n}\lambda_1$, where λ_1 is the first nonzero eigenvalue of the Laplacian on $S^n(c)$. This raises another question: Is a compact and connected immersed hypersurface $\psi : M \to R^{n+1}$ whose Ricci curvature Ric satisfies $0 < \mathrm{Ric} = \frac{(n-1)}{n}\lambda_1$, where λ_1 is the first nonzero eigenvalue of the Laplacian on M, necessarily isometric to a sphere in R^{n+1}?

In this paper we answer these questions in the affirmative for compact and connected hypersurfaces of R^4. Indeed, we prove the following theorems:

Theorem 1. Let $\psi : M \to R^4$ be a compact and connected immersed hypersurface. If the mean curvature α of M is nowhere zero and the scalar curvature S of M satisfies $\|\psi\|^2 S = 6$, then M is isometric to a sphere.

Theorem 2. Let $\psi : M \to R^4$ be a compact and connected immersed hypersurface. If the Ricci curvature of M satisfies $0 < \mathrm{Ric} \leq \frac{2}{3}\lambda_1$, where λ_1 is the first nonzero eigenvalue of the Laplacian on M with respect to the induced metric, then M is isometric to a sphere.

The proof of Theorem 2 depends heavily on the quaternion structure of R^4. However, it is surprising to note that we use no quaternion structure of R^4 in the proof of Theorem 1 though the proof works only for this dimension and cannot be extended to $\psi : M \to R^{n+1}$.

2. PRELIMINARIES

Let $< \cdot, \cdot >$ be the inner product on R^4 and $\bar{\nabla}$ be the Euclidean connection on R^4. Let $\psi : M \to R^4$ be an orientable hypersurface with unit normal vector field N. We denote by g and ∇ the induced metric and the Riemannian connection on M, respectively. Then we have

$$(2.1) \quad \bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \bar{\nabla}_X N = -AX, \quad X, Y \in \mathfrak{X}(M),$$

where A is the shape operator of M and $\mathfrak{X}(M)$ is the Lie algebra of vector fields on M. The shape operator A satisfies

$$(\nabla_X A)(Y) = (\nabla_Y A)(X), \tag{2.2}$$

$$g((\nabla_X A)(Y), Z) = g((\nabla_X A)(Z), Y), \quad X, Y, Z \in \mathfrak{X}(M). \tag{2.3}$$

For a local orthonormal frame $\{e_1, e_2, e_3\}$ on M, the mean curvature α is given by $\alpha = \frac{1}{3}\sum_{i=1}^{3} g(Ae_i, e_i)$ and it satisfies

$$X(\alpha) = \frac{1}{3}\sum_{i=1}^{3} g((\nabla_{e_i} A)(e_i), X), \quad X \in \mathfrak{X}(M). \tag{2.4}$$

The Ricci curvature tensor Ric and the scalar curvature S of M are given by

$$\text{Ric}\,(X, Y) = 3\alpha g(AX, Y) - g(AX, AY), \;\; S = 9\alpha^2 - \|A\|^2, \quad X, Y \in \mathfrak{X}(M), \tag{2.5}$$

where $\|A\|^2 = trA^2$.

For the orientable hypersurface $\psi : M \to R^4$, we can treat ψ as the position vector field of M in R^4 and therefore it can be expressed as

$$\psi = t + \rho N, \tag{2.6}$$

where $t \in \mathfrak{X}(M)$ and $\rho =< \psi, N >$ is called the *support function* of M. Using (2.1) and (2.6) one immediately obtains

$$\nabla_X t = X + \rho AX, \qquad X(\rho) = -g(AX, t), \quad X \in \mathfrak{X}(M). \tag{2.7}$$

If M is compact, we have Minkowski's formula

$$\int_M (1 + \rho\alpha) dv = 0. \tag{2.8}$$

We denote by J_1, J_2 and J_3 the complex structures on R^4 which define the quaternion structure of R^4. Then we have

$$J_1J_2 = -J_2J_1 = J_3, \quad J_2J_3 = -J_3J_2 = J_1, \quad J_3J_1 = -J_1J_3 = J_2; \tag{2.9}$$

$$\bar{\nabla} J_i = 0, \qquad < J_i, J_i > = <,>, \quad i = 1,2,3. \tag{2.10}$$

Define the vector fields ξ_1, ξ_2, ξ_3 on M by $\xi_i = -J_i N$, $i = 1,2,3$ and set $J_i X = \phi_i X + \eta_i(X)N$, where ϕ_i is a (1,1)-tensor field on M and η_i is a 1-form dual to ξ_i on $M, i = 1,2,3$. It is easy to verify that each triplet (ϕ_i, ξ_i, η_i) satisfy

$$\phi_i^2 = -I + \eta_i \otimes \xi_i, \quad \phi_i(\xi_i) = 0, \quad \eta_i \circ \phi_i = 0, \tag{2.11}$$

$$g(\phi_i X, \phi_i Y) = g(X,Y) - \eta_i(X)\eta_i(Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.12}$$

For a local unit vector field e on M satisfying $g(e, \xi_i) = 0$ for a fixed i, $\{e, \phi_i(e), \xi_i\}$ is a local orthonormal frame and such a local frame will be referred to as an adapted frame. Using (2.1) and (2.10) with $\xi_i = -J_i N$, we obtain $\nabla_X \xi_i = \phi_i AX$, $X \in \mathfrak{X}(M)$. From this equation, using an adapted frame, we get

$$div\, \xi_i = 0, \quad i = 1,2,3. \tag{2.13}$$

Using the complex structures J_i and the position vector field ψ of M in R^4, we define the smooth functions $\rho_i : M \to R$ by $\rho_i = < J_i\psi, N >$ and the vector fields $t_i \in \mathfrak{X}(M)$ by setting $J_i\psi = t_i + \rho_i N$. Using (2.1), (2.10) and $J_i\psi = t_i + \rho_i N$, we now obtain

$$\nabla_X t_i = \phi_i X + \rho_i AX, \quad X(\rho_i) = -g(At_i, X) + \eta_i(X), \quad X \in \mathfrak{X}(M). \tag{2.14}$$

We also have

$$g(t_i, \xi_i) = < J_i\psi - \rho_i N, \xi_i > = - < \psi, J\xi_i > = - < \psi, N > = -\rho. \tag{2.15}$$

3. PROOF OF THEOREM 1

Let $\psi : M \to R^4$ be a compact and connected immersed hypersurface of R^4. We define $F : M \to R$ by $F = \frac{1}{2}\|\psi\|^2$. Then, using (2.6) and (2.7), we obtain

$$\Delta F = 3(1 + \rho\alpha), \tag{3.1}$$

where Δ is the Laplacian operator on M. Using (2.6) and (2.7), we compute $\Delta\rho$ to obtain

$$\Delta\rho = -3t(\alpha) - 3\alpha - \rho\|A\|^2. \tag{3.2}$$

We have $X(F) = g(t, X)$, $X \in \mathfrak{X}(M)$, and consequently grad $F = t$. Also from the second equation in (2.7), we get grad $\rho = -At$. Thus, using (3.1) and (3.2), we obtain

$$\begin{aligned} \Delta(\rho F) &= F\Delta\rho + \rho\Delta F + 2g(\text{grad } F, \text{grad}\rho) \\ &= -3Ft(\alpha) - 3\alpha F - \rho F\|A\|^2 \\ &\quad +3\rho(1+\rho\alpha) - 2g(At, t) \end{aligned}$$

Since $div\,(fX) = X(f) + f\,div\,X$, $X \in (M), f \in c^\infty(M)$ and $t(\rho) = -g(At, t)$, we have

$$\begin{aligned} \Delta(\rho F) &= -3[div\,(F\alpha t) - \alpha\,div(Ft)] \\ &\quad -3\alpha F - F\rho\|A\|^2 + 3\rho + 3\alpha\rho^2 + 2[div(\rho t) - \rho div t] \\ &= -3div(F\alpha t) + 2div(\rho t) + 3\alpha[t(F) + F div\, t] \\ &\quad -3\alpha F - F\rho\|A\|^2 + 3\rho + 3\alpha\rho^2 - 2\rho div t \\ &= -3div(F\alpha t) + 2div(\rho t) + 3\alpha\|t\|^2 + F\rho[g\alpha^2 - \|A\|^2] \\ &\quad +6F\alpha - 3\rho - 3\rho^2\alpha, \end{aligned}$$

where we have used $t(F) = \|t\|^2$ and $div t = 3(1+\rho\alpha)$. Consequently, with $\|\psi\|^2 = \|t\|^2 + \rho^2$, we have

$$\begin{aligned} \Delta(\rho F) + 3div(F\alpha t) - 2div(\rho t) &= 3\alpha(\|t\|^2 - \rho^2 + 2F) + F\rho S - 3\rho \\ &= 6\alpha\|t\|^2 + \frac{\rho}{2}[\|\psi\|^2 - 6]. \end{aligned}$$

Integrating this last equation over M and using the hypothesis of the theorem we get

$$\int_M \alpha\|t\|^2 dv = 0.$$

Since α is nowhere zero and M is connected we must have $\alpha < 0$ (as there is a point where all eigenvalues of A are negative; indeed this point is where the height function of M attains its maximum). Hence the above integral gives $t = 0$ on M. Then, the equations in (2.7) yield $\rho AX = -X$

and $\rho =$ constant. That $\rho \neq 0$ follows from $\psi = \rho N$ (as $t = 0$). Hence $A = \frac{-1}{\rho} I$, that is, M is a totally umbilical hypersurface of R^4 and, as such, it is isometric to a sphere.

Remark: If we take M to be a hypersurface of R^{n+1} and proceed with the computation as in the above proof, we arrive at

$$\Delta(\rho F)+3div(F\alpha t)-2div(\rho t) = n\alpha(\|t\|^2(\frac{n+1}{2})+\rho^2\frac{(n-3)}{2})+\rho(FS-n),$$

which suggests that the conclusion of the theorem works only for $n = 3$, and therefore this cannot give information beyond dimension 3, unless some additional assumptions are made. It will be an interesting problem to generalise the theorem for a compact and connected hypersurface of R^{n+1}.

4. PROOF OF THEOREM 2.

First we prove the following

Proposition. Let $\psi : M \to R^4$ be a compact and connected immersed hypersurface of R^4 with non-negative Ricci curvature. If $\rho_i = 0$, $i = 1, 2, 3$, then M is isometric to a sphere in R^4.

Proof. Note that for an adapted frame $\{e, \phi_i(e), \xi_i\}$, for a fixed i, we can use (2.11) to obtain $\phi_i^2(e) = -e$ as $e \perp \xi_i$. Thus if $\{e_1, e_2, e_3\}$ is such a frame, we have

$$(4.1)\quad \sum_{j=1}^{3} g(\phi_i(e_j), A(e_j) = g(\phi_i(e), Ae) + g(\phi_i(\phi_i(e)), A(\phi_i(e)) = 0,$$

Since $\rho_1 = \rho_2 = \rho_3 = 0$, the second equation in (2.14) gives $At_i = \xi_i$, $i = 1, 2, 3$. As ξ_i are globally defined unit vector fields on M and A is a linear operator, it follows from $At_i = \xi_i$, that t_i are nowhere zero on M. Moreover, using (2.9), we have

$$g(t_1, t_2) =< J_1\psi, J_2\psi >= - < \psi, J_1 J_2 \psi >= - < \psi, J_3\psi >= 0,$$

and similarly $g(t_2, t_3) = 0$, $g(t_3, t_1) = 0$, that is, the vector fields t_1, t_2, t_3 are mutually orthogonal. Thus the unit vector fields $\hat{t}_i$ along t_i give the orthonormal frame $\{\hat{t}_1, \hat{t}_2, \hat{t}_3\}$. Using (2.13), and $At_i = \xi_i$, we get

$$0 = div(At_i) = \sum_{j=1}^{3} g(\nabla_{e_j} At_i, e_j) = \sum_{j=1}^{3} [g((\nabla_{e_j} A)(t_i), e_j) + g(\nabla_{e_j} t_i, Ae_j)].$$

Using (2.4), (2.14) and (4.1) in the above equation we get $3t_i(\alpha) = 0$ or $\hat{t}_i(\alpha) = 0$, $i = 1, 2, 3$. This proves that the mean curvature α is a constant.

Then the equation (3.2), after integration, gives

$$\int_M (3\alpha + \rho tr A^2) dv = 0. \tag{4.2}$$

The integral formula (2.8) with α a constant can be restated as

$$\int_M (3\alpha + 3\rho\alpha^2) dv = 0 \tag{4.3}$$

The integrals (4.2) and (4.3) give

$$\int_M \rho(3\alpha^2 - tr A^2) dv = 0. \tag{4.4}$$

We use (2.5) and $At_i = \xi_i$, to arrive at Ric $(t_1, t_1) = 3\alpha g(t_i, \xi_i) - \|\xi_i\|^2$, or Ric $(t_1, t_1) = -3\alpha\rho - 1 \geq 0$, as the Ricci curvature is non-negative from the hypothesis. This last inequality suggests that there is no point $p \in M$ such that $\rho(p) = 0$. Thus, M being connected, we have either $\rho > 0$ or $\rho < 0$. Moreover the Schwarz inequality states that $3\alpha^2 \leq tr A^2$, with equality holding at a point if and only if it is an umbilic point. The integral (4.4) gives $3\alpha^2 = tr A^2$, proving that M is an umbilical hypersurface of R^4, and this proves the proposition.

Now we proceed to prove Theorem 2. Let $\psi : M \to R^4$ be a compact and connected immersed hypersurface of R^4. We assume that the center of mass of M is at the origin of R^4 (for otherwise an isometry $\phi : R^4 \to R^4$ can be chosen which maps the center of mass of M to the origin of R^4, and then $\psi' = \phi \circ \psi$ will be the desired immersion). Thus, using the minimal principle with $\int_M \psi dv = 0$, we get

$$\lambda_1 \leq 3vol(M) / \int_M \|\psi\|^2 dv,$$

where λ_1 is the first nonzero eigenvalue of the Laplacian operator on M. Consequently we have

$$\int_M \|\psi\|^2 dv \leq \frac{3vol(M)}{\lambda_1}. \tag{4.5}$$

We use (2.3), (2.4), (2.14) and (4.1) to compute $div(At_i)$:

$$div(At_i) = \sum_{j=1}^{3} g(\nabla_{e_j} At_i, e_j) = 3t_i(\alpha)+\rho_i\|A\|^2 = 3[div(\alpha t_i)-\alpha div t_i]+\rho_i\|A\|^2.$$

This gives

$$div(At_i) = 3div(\alpha t_i) + \rho_i[\|A\|^2 - 9\alpha^2], \tag{4.6}$$

where we have used $div(t_i) = 3\rho_i\alpha$ as a result of (2.14). Now the second equations in (2.14) and (2.15), yield

$$\begin{aligned} div(\rho_i\alpha t_i) &= \rho_i div(\alpha t_i) + \alpha t_i(\rho_i) \\ &= \rho_i div(\alpha t_i) - \alpha g(At_i, t_i) - \alpha\rho, \end{aligned}$$

that is,

$$3\rho_i div(\alpha t_i) = 3div(\rho_i\alpha t_i) + 3\alpha g(At_i, t_i) + 3a\rho. \tag{4.7}$$

Finally we use (2.14), (4.6) and (4.7) to compute $div(\rho_i At_i)$ and obtain

$$\begin{aligned} div(\rho_i At_i) &= At_i(\rho_i) + \rho_i div(At_i) \\ &= -g(At_i, At_i) + \eta_i(At_i) + 3\rho_i div(\alpha t_i) + \rho_i^2[\|A\|^2 - 9\alpha^2] \\ &= -\|At_i\|^2 + g(At_i, \xi_i) + 3div(\rho_i\alpha t_i) + 3\alpha g(At_i, t_i) \\ &\quad +3\alpha\rho - \rho_i^2 S. \end{aligned}$$

Since $g(At_i, \xi_i) = -[-g(At_i, \xi_i)+\eta_i(\xi_i)-1] = -[\xi_i(\rho_i)-1] = -div(\rho_i\xi_i)+1$, (where we have used (2.13)), the above equation becomes

$$div[\rho_i At_i - 3\rho_i\alpha t_i + \rho_i\xi_i] = \text{Ric}\ (t_i, t_i) - \rho_i^2 S + 3\rho\alpha + 1. \tag{4.8}$$

Let $\hat{t}_i$ be the unit vector field defined on the open subset of M where $t_i \neq 0$. Using $\|\psi\|^2 = \|t_i\|^2 + \rho_i^2$, which follows from $J_i\psi = t_i + \rho_i N$, in equation (4.8), we arrive at

$$div[\rho_i At_i - 3\rho_i\alpha t_i + \rho_i\xi_i] = -\rho_i^2[\text{Ric}(\hat{t}_i, \hat{t}_i) + S] + \|\psi\|^2\ \text{Ric}(\hat{t}_i, \hat{t}_i) + 3\alpha\rho + 1.$$

From the hypothesis of the theorem that $0 < \text{Ric} \leq \frac{2}{3}\lambda_1$, we obtain

$$div[\rho_i A t_i - 3\rho_i \alpha t_i + \rho_i \xi_i] \leq -\rho_i^2[\text{Ric}(\hat{t}_i, \hat{t}_i) + S] + \frac{2}{3}\lambda_1 \|\psi\|^2 + 3\alpha\rho + 1.$$

Integrating the above inequality and using formula (2.8), we get

$$\int_M \rho_i^2[\text{Ric}(\hat{t}_i, \hat{t}_i) + S]dv \leq -2vol(M) + \frac{2}{3}\lambda_1 \int_M \|\psi\|^2 dv.$$

This, together with (4.5), give

$$\int_M \rho_i^2[\text{Ric}(\hat{t}_i, \hat{t}_i) + S]dv \leq 0.$$

Since Ric > 0, this integral inequality gives $\rho_i = 0$ $i = 1, 2, 3$, and the above proposition completes the proof.

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