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LARGE TIME ASYMPTOTIC BOUNDS OF L^∞ SOLUTIONS FOR SOME REACTION-DIFFUSION EQUATIONS

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ABSTRACT. We are concerned with the large time asymptotic bounds for solutions of the reaction-diffusion system $u_t = a\Delta u - uv^m$, $v_t = b\Delta u + d\Delta v + uv^m$, $0 < a < d$, $b < 0$, in $\mathbb{R}^n \times (0,\infty)$, $n \geq 1$ with $m \geq 2$ an even nonnegative integer and bounded uniformly continuous nonnegative initial data u_0, v_0. In case $d > a = 1$, $b = 0$, and $m \geq 1$ a nonnegative integer, P. Collet and J. Xin [5] proved the existence of global classical solutions and showed that the L^∞ norm of v cannot grow faster than $O(\ln \ln t)$ for any space dimension. In the present work, we show that if $0 < a < d, b < 0$ and $v_0 \geq \frac{b}{a-d} u_0$, the L^∞ norm of v cannot grow faster than $O(\ln t)$ for any space dimension.

1. INTRODUCTION

In this paper, we are concerned with the large time asymptotic bounds of the reaction-diffusion system

$$u_t = a\Delta u - uv^m, \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.1a}$$

$$v_t = b\Delta u + d\Delta v + uv^m, \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.1b}$$

with the initial data

$$u(x,0) = u_0(x) \quad \text{and} \quad v(x,0) = v_0(x), \; x \in \mathbb{R}^n. \tag{1.2}$$

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In (1.1), the constants a, b and d are such that $a > 0, d > 0, b \neq 0$. As for m it is assumed to be an even nonnegative integer. Also we suppose that $4ad \geq b^2$; this condition reflects the parabolicity of the system. Δ is the Laplace operator in x. The initial data u_0, v_0 are nonnegative functions in $C_{UB}(\mathbb{R}^n)$, the space of uniformly bounded continuous functions on $\mathbb{R}^n$.

In what follows, we use the following notation:
(i) $\|\cdot\|$ denotes the supremum norm on $\mathbb{R}^n$, i.e., $\|u\| = \sup_{x\in\mathbb{R}^n} |u(x)|$
(ii) For any $\theta \in L^1(\mathbb{R}^n)$, we write $\int \theta \equiv \int_{\mathbb{R}^n} \theta(x)dx$.

The reaction-diffusion system

$$(\star) \qquad \begin{cases} u_t = a\Delta u + f(u,v) \\ v_t = b\Delta u + d\Delta v + g(u,v \end{cases}$$

is well-studied in the literature. See [2], [3], [4], [5], [6], [7], [8], [9] and the references therein.

On a bounded domain, $b = 0$ and $g(u,v) = -f(u,v) \equiv uv^m$ with $m \geq 1$, under homogeneous Dirichlet or Neumann boundary conditions, the large time behavior of solutions is that (u,v) converges uniformly in $\overline{\Omega}$ to a constant vector (k_1, k_2), where $k_1 \geq 0$, $k_2 \geq 0$ and $k_1 k_2 = 0$ (see K. Masuda [9]).

On a bounded domain, $b \neq 0$ and $g(u,v) = -f(u,v) \equiv uh(v)$, where the function $h(s)$ is continuously differentiable and satisfies $\lim_{s\to\infty} \frac{1}{s}\ln(1+h(s)) = 0$, this system was studied by M. Kirane. In [7], M. Kirane showed that if $a > d > 0$, $b \geq 0$, $b^2 < 4ad$, under homogeneous Neumann boundary conditions, the solution (u,v) converges uniformly in $\overline{\Omega}$ to a constant vector (k_1, k_2) such that $k_1 \geq 0$, $k_2 \geq 0$ and $k_1 h(k_2) = 0$.

Motivated by thermal-diffusive models with Arrenius reactions [1], Berlyand and Xin [3] considered system $(\star)$ with $b = 0$ and $m = 2$ for a class of small initial data in $(L^1 \cap L^\infty(\mathbb{R}))^2$ and showed that u, v are bounded from above and below by self-similar upper and lower solutions. The results of [3] show that u decays to zero in time with an algebraic rate faster than $t^{-\frac{1}{2}-\delta}$, for some $\delta > 0$, and v decays to zero like $O(t^{-\frac{1}{2}})$.

More recently, J. Bricmont, A. Kupiainen and J. Xin [4] have studied the system $(\star)$ where $b = 0$, $g(u,v) = -f(u,v) \equiv uv^2$ and $x \in \mathbb{R}$. For non-negative spatially decaying initial data of arbitrary size, and for any positive constant d, they showed that if the initial data decay to zero sufficiently fast at infinity, the solution (u,v) converges to a self-similar solution of the reduced system

$$(\star') \qquad \begin{cases} u_t = -uv^2 \\ v_t = uv^2 \end{cases}$$

in the large time limit. In particular, u decays to zero like $O(t^{-\frac{1}{2}-\delta})$, where $\delta > 0$ is an anomalous exponent depending on the initial data, and v decays to zero with normal rate $O(t^{-\frac{1}{2}})$.

Also, P. Collet and J. Xin [5] have considered the system $(\star)$ with $x \in \mathbb{R}^n$, $d > a = 1$ and $m \geq 1$ is an arbitrary real constant. They proved the existence of global classical solutions if the initial data are bounded uniformly, continious and non-negative. Moreover, they showed that the L^∞ norm of v cannot grow faster than $O(\ln\ln t)$ for any space dimention n.

The system (1.1)-(1.2) with $a > 0$, $d > 0$, $b \neq 0$, m assumed to be an even nonnegative integer and $4ad \geq b^2$, was studied by S. Badraoui [2]. He proved that if the initial data u_0, v_0 are nonegative functions in $C_{UB}(\mathbb{R}^n)$, then

(i) if $b > 0$, $a > d$ and $v_0 \geq \frac{b}{a-d} u_0 \geq 0$, the problem (1.1)-(1.2) admits global classical bounded solutions.

(ii) If $b < 0, 0 < a < d$ and $v_0 \geq \frac{b}{a-d} u_0 \geq 0$, the problem (1.1)-(1.2) also admits global classical solutions which satisfy

$$\text{(1.3a)} \qquad \|u(t)\| \leq \|u_0\|,$$

$$\text{(1.3b)} \qquad \|v(t)\| \leq \frac{b}{a-d}\|u_0\| + \|v_0\| + \left(1 + \frac{b}{a-d}\right) \omega e^{\frac{\sigma}{p}t} \left(\frac{2q}{2q+n} t^{\frac{n}{2q}+1} + \frac{2p}{2p-n} t^{1-\frac{n}{2p}} \right),$$

for all $t \geq 0$, $p > \frac{n}{2}$ and some constants $\omega = \omega(n, d, u_0, v_0) > 0$ and $\sigma > 0$.

The last estimates are based on the following nonlinear functional of the solution (u,v)

$$\text{(1.4)} \qquad L(u,v) = (\alpha + 2u - \ln(1+u))\, e^{\varepsilon v},$$

where

$$(1.5)\quad \alpha \geq \ln(1+\|u_0\|) + \max\left\{\frac{(1+\|u_0\|)^2}{ad}\left(1+4(a+d)^2\right), (1+2\|u_0\|)^2\right\},$$

and

$$(1.6)\quad \varepsilon \leq \min\left\{\frac{1+2\|u_0\|}{2(\alpha+2\|u_0\|)(1+\|u_0\|)}, \frac{1}{|b|(\alpha+2\|u_0\|)}, \frac{a}{2|b|(1+\|u_0\|)^2}\right\}.$$

We proved that

$$(1.7)\quad \frac{d}{dt}\int \varphi L \leq \int (\varphi_t + d\Delta\varphi) L - \frac{1}{2}\int \varphi L_1 u v^m + \int ((d-a)L_1 + bL_2)\nabla\varphi\nabla u - \frac{1}{2}\int \varphi\left(\frac{a}{2}L_{11}|\nabla u|^2 + dL_{22}|\nabla v|^2\right)$$

for any smooth nonnegative function φ with exponential spatial decay at infinity. Here,

$$L_1 = \frac{\partial L}{\partial u}, L_2 = \frac{\partial L}{\partial v}, L_{11} = \frac{\partial^2 L}{\partial u^2}, L_{12} = \frac{\partial^2 L}{\partial u \partial v}, L_{22} = \frac{\partial^2 L}{\partial v^2}.$$

We also proved that there exist two constants $\beta, \sigma > 0$ such that

$$(1.8)\quad \int \varphi L \leq \beta e^{\sigma t},$$

for all $t > 0$, where $\varphi(x) = \frac{1}{(1+|x-x_0|^2)^n}$.

In the present work, we prove that the the L^∞ norm of v cannot grow faster than $O(\ln t)$ for any space dimention n.

2. LARGE TIME ASYMPTOTIC BOUNDS OF SOLUTIONS

In this section, we improve the L^∞ estimates (1.3b) from exponential growth in time to the order of logarithmic growth.

Theorem 2.1. *If $b < 0$, there exists a positive constant $\gamma = \gamma(n, d, m)$ such that*

$$(2.1)\quad \|v(t)\| \leq \gamma \|(u_0, v_0)\| \ln\left(\|(u_0, v_0)\|^m t + e\right),$$

for all $t > 0$, where $\|(u_0, v_0)\| = \max\{\|u_0\|, \|v_0\|\}$.

For the proof we need some lemmas.

Lemma 2.1 (See [5]). . *Let $T > 0$ be a positive number. The solution Φ of the backward heat equation*

$$\Phi_t + d\Delta\Phi = 0, \quad (x,t) \in \Omega \times (0,T). \tag{2.2}$$

is given by

$$\Phi \equiv \Phi(x;t,T) = (4\pi d)^{-\frac{n}{2}} (T-t)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{4d(T-t)}), \tag{2.3}$$

for $0 \le t < T$. Moreover, the function Φ has the following properties

$$\Delta\Phi = -\frac{c_1}{T-t}\Phi + \frac{1}{4d^2}\frac{|x|^2}{(T-t)^2}\Phi, \tag{2.4}$$

and

$$\frac{|\nabla\Phi|^2}{\Phi^2} = \frac{c_2\,|x|^2}{(T-t)^2}, \tag{2.5}$$

for some two positives constants $c_1 = c_1(n,d) > 0$ and $c_2 = c_2(n,d) > 0$.

Lemma 2.2. *For the nonlinear functional $L(u,v) = (\alpha + 2u - \ln(1+u))\, e^{\varepsilon v}$ and the test function Φ defined above, there is a positive constant $c_3 = c_3(n,d)$ > 0 such that*

(2.6)

$$\frac{d}{dt}\int \Phi L \le 2(d-a)c_1 \int \Phi u \frac{1}{T-t} \exp\left\{\varepsilon\,(8(d-a)c_1)^{\frac{1}{m}} \left(\frac{1}{T-t}\right)^{\frac{1}{m}}\right\} dx$$
$$-\frac{1}{8}\int \Phi e^{\varepsilon v} u v^m$$

for any $0 \le t < T$.

Proof. Define the following functions

$$g(u) = \alpha + 2u - \ln(1+u), \tag{2.7}$$

$$g_0(u) = 2u - \ln(1+u) \tag{2.8}$$

and

$$G(u) = (\alpha+1)\,u + u^2 - (1+u)\ln(1+u), \tag{2.9}$$

wherer α is the constant given in (1.5). The function G is an antiderivative of the function g. It is clear that the functions g, g_0, and G are nonnegative for all $u \ge 0$.

As Φ is a smooth nonnegative space-time function with exponential spacial decay at infinity, we can take $\varphi = \Phi$ in the relation (1.7). With this choice we then get

$$\frac{d}{dt}\int \Phi L \leq (d-a)\int L_1 \nabla\Phi\nabla u + b\int L_2\nabla\Phi\nabla u - \frac{1}{2}d\int \Phi L_{22}\left|\nabla v\right|^2 - \frac{1}{2}\int \Phi L_1 u v^m. \tag{2.10}$$

Integrating by parts, we find

$$\int L_1\nabla\Phi\nabla u = -\int e^{\varepsilon v} g_0 \Delta\Phi - \varepsilon\int e^{\varepsilon v} g_0 \nabla v\nabla\Phi \equiv I_1 + I_2. \tag{2.11}$$

From (2.4), it appears that

$$I_1 = \frac{C_1}{T-t}\int \Phi g_0(u) e^{\varepsilon v} - \frac{1}{4d^2}\int \frac{|x|^2}{(T-t)^2}\Phi g_0(u) e^{\varepsilon v}. \tag{2.12a}$$

On the other hand, using (2.5) and

$$\begin{aligned} \varepsilon\nabla v\nabla\Phi &\leq \sqrt{\frac{2g_0}{g\Phi}}\left|\nabla\Phi\right| .\varepsilon\sqrt{\frac{g\Phi}{2g_0}}\left|\nabla v\right| \\ &\leq \frac{g_0}{g\Phi}\left|\nabla\Phi\right|^2 + \frac{\varepsilon^2 g\Phi}{4g_0}\left|\nabla v\right|^2 \end{aligned}$$

obtained by the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |I_2| &\leq \int \frac{g_0^2}{g}\frac{\left|\nabla\Phi\right|^2}{\Phi} e^{\varepsilon v} + \frac{\varepsilon^2}{4}\int \Phi g\left|\nabla v\right|^2 e^{\varepsilon v} \\ &\leq c_2\int \frac{g_0^2}{g}\Phi\frac{|x|^2}{(T-t)^2} e^{\varepsilon v} + \frac{\varepsilon^2}{4}\int \Phi g\left|\nabla v\right|^2 e^{\varepsilon v}. \end{aligned} \tag{2.12b}$$

Integrating by parts again, we see that

$$\int bL_2\nabla\Phi\nabla u = -\varepsilon b\int G\Delta\Phi e^{\varepsilon v} - \varepsilon^2 b\int G\nabla v\nabla\Phi e^{\varepsilon v}.$$

As $b < 0$, we can write

$$\int bL_2\nabla\Phi\nabla u = \varepsilon\left|b\right|\int G\Delta\Phi e^{\varepsilon v} + \varepsilon^2\left|b\right|\int G\nabla v\nabla\Phi e^{\varepsilon v} \equiv I_3 + I_4. \tag{2.13}$$

From (2.4), we conclude that

$$I_3 = -\frac{\varepsilon\left|b\right| C_1}{T-t}\int \Phi G e^{\varepsilon v} + \frac{\varepsilon\left|b\right|}{4d^2}\int \frac{|x|^2}{(T-t)^2}\Phi G e^{\varepsilon v}. \tag{2.14a}$$

Similarly, using (2.5) and

$$\begin{aligned} \varepsilon^2 \nabla v \nabla \Phi &\le \varepsilon^{\frac{1}{2}} \sqrt{\frac{2}{\Phi}} \left|\nabla \Phi\right| \varepsilon^{\frac{3}{2}} \sqrt{\frac{\Phi}{2}} \left|\nabla v\right| \\ &\le \frac{\varepsilon}{\Phi} \left|\nabla \Phi\right|^2 + \frac{\varepsilon^3 \Phi}{4} \left|\nabla v\right|^2, \end{aligned}$$

obtained by the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} I_4 &\le \varepsilon^{\frac{1}{4}} \left|b\right| \int G \frac{\left|\nabla \Phi\right|^2}{\Phi} e^{\varepsilon v} + \frac{|b|}{4} \varepsilon^{\frac{9}{4}} \int \Phi G \left|\nabla v\right|^2 e^{\varepsilon v} \\ &\le \varepsilon^{\frac{1}{4}} \left|b\right| c_2 \int \Phi G \frac{|x|^2}{(T-t)^2} e^{\varepsilon v} + \varepsilon^{\frac{9}{4}} \frac{|b|}{4} \int \Phi G \left|\nabla v\right|^2 e^{\varepsilon v}. \end{aligned} \tag{2.14b}$$

Taking into account (2.12) and (2.14) in (2.10) and the fact that $L_1 = g'(u) e^{\varepsilon v}$, $L_{22} = \varepsilon^2 g e^{\varepsilon v}$, we get

(2.15)

$$\begin{aligned} \frac{d}{dt} \int \Phi L \le& \int \Phi e^{\varepsilon v} \frac{|x|^2}{(T-t)^2} \left(-\frac{d-a}{4d^2} g_0 + (d-a) c_2 \frac{g_0^2}{g} + \frac{\varepsilon |b|}{4d^2} G + \varepsilon^{\frac{1}{4}} |b| c_2 G \right) \\ &+ \int \Phi e^{\varepsilon v} \left|\nabla v\right|^2 \left(\varepsilon^2 \frac{(d-a)}{4} g + \varepsilon^{\frac{9}{4}} \frac{|b|}{4} G - \frac{1}{2} \varepsilon^2 d g \right) \\ &+ \int \Phi e^{\varepsilon v} \left(g_0 \frac{(d-a) c_1}{T-t} - \frac{\varepsilon |b| c_1}{T-t} G - \frac{1}{2} g'(u) u v^m \right) \\ \equiv& J_1 + J_2 + J_3. \end{aligned}$$

As $0 \le u \le \|u_0\|$, we can easily show that

$$G \le \lambda g_0, \tag{2.16}$$

for all $\lambda \ge \max\left\{ \frac{1}{2}\alpha + 1, \alpha + 2 \|u_0\|^2 \right\}$. Then,

$$J_1 \le \int g_0 \Phi e^{\varepsilon v} \frac{|x|^2}{(T-t)^2} \left(-\frac{d-a}{4d^2} + (d-a) c_2 \frac{g_0}{g} + \frac{\varepsilon |b|}{4d^2} \lambda + \varepsilon^{\frac{1}{4}} |b| c_2 \lambda \right).$$

We see that if α is chosen large enough (which is possible according to (1.5)) and ε sufficiently small (this is possible according to (1.6)), then

$$J_1 \le 0. \tag{2.17}$$

As $G \le \lambda g_0$, then a fortiori we have

$$G \le \lambda g.$$

We deduce then that

$$\begin{aligned} J_2 &= \int \Phi e^{\varepsilon v} |\nabla v|^2 \left(-\varepsilon^2 \frac{a+d}{4} g + \varepsilon^{\frac{9}{4}} \frac{|b|}{4} G \right) \\ &\leq \int g \Phi e^{\varepsilon v} |\nabla v|^2 \varepsilon^2 \left(-\frac{a+d}{4} + \varepsilon^{\frac{1}{4}} \frac{|b|}{4} \lambda \right). \end{aligned}$$

Clearly, if ε is small enough, then

$$J_2 \leq 0. \tag{2.18}$$

Now, consider J_3; as

$$g'(u) = 2 - \frac{1}{1+u} \geq 1$$

and

$$-\frac{\varepsilon |b| c_1}{T-t} G \leq 0,$$

we have

$$\begin{aligned} g_0 \frac{(d-a)c_1}{T-t} - \frac{\varepsilon |b| c_1}{T-t} G - \frac{1}{2} g'(u) u v^m & \\ \leq (2u - \ln(1+u)) \frac{(d-a)c_1}{T-t} - \frac{1}{2} u v^m & \\ \leq 2(d-a)c_1 u \left(\frac{1}{T-t} - \frac{1}{4(d-a)c_1} v^m \right). & \end{aligned} \tag{2.19}$$

If we put

$$\Omega = \left\{ x \in \mathbb{R}^n : \frac{1}{8(d-a)c_1} v^m \leq \frac{1}{T-t} \right\},$$

then (2.19) implies that

$$\begin{aligned} J_3 &\leq 2(d-a)c_1 \int_\Omega \Phi u e^{\varepsilon v} \left(\frac{1}{T-t} - \frac{1}{8(d-a)c_1} v^m \right) dx - \frac{1}{8} \int \Phi e^{\varepsilon v} u v^m \\ &\leq 2(d-a)c_1 \int \Phi u \frac{1}{T-t} \exp \left\{ \varepsilon \left(8(d-a)c_1 \right)^{\frac{1}{m}} \left(\frac{1}{T-t} \right)^{\frac{1}{m}} \right\} \\ &\quad - \frac{1}{8} \int \Phi e^{\varepsilon v} u v^m. \end{aligned} \tag{2.20}$$

The inequality (2.6) follows now from the insertion of (2.17), (2.18) and (2.20) into (2.15). □

Lemma 2.3. *Over the interval* $[0, T-1)$, *we have*

$$\int_0^{T-1} dt \int \Phi(x-\xi; t, T) e^{\varepsilon v} u v^m d\xi \leq c_3 (1 + \ln T), \tag{2.21}$$

where $c_3 = c_3(n, d, u_0, v_0) > 0$.

Proof. As $\frac{1}{T-t} \le 1$ and $\int \Phi(x;0,T)dx = 1$, we have from (2.6)

$$\frac{d}{dt}\int \Phi L \le \frac{2(d-a)c_1}{T-t}\|u_0\| \exp\left\{\varepsilon\left(8(d-a)c_1\right)^{\frac{1}{m}}\right\} - \frac{1}{8}\int \Phi e^{\varepsilon v} u v^m,$$

which implies that

$$\frac{d}{dt}\int \Phi(x;t,T)L(u,v) \le \frac{c_4}{T-t} - \frac{1}{8}\int \Phi(x;t,T)e^{\varepsilon v}uv^m, \tag{2.22}$$

where $c_4 = 2(d-a)c_1\|u_0\|\exp\{\varepsilon(8(d-a)c_1)\} \equiv c_4(n,a,d,u_0)$. Integrating (2.22) over $(0,\, t)$, for $t \in [0, T-1)$, we get

$$\int \Phi(x;t,T)L(u,v) - \int \Phi(x;0,T)L(u_0,v_0) \le c_4 \ln T - \frac{1}{8}\int_0^t d\tau \int \Phi(x;\tau,T)e^{\varepsilon v}uv^m dx.$$

This inequality yields

$$\frac{1}{8}\int_0^t d\tau \int \Phi(x;\tau,T)e^{\varepsilon v}uv^m dx + \int \Phi(x;t,T)L(u,v) \le \int \Phi(x;0,T)L(u_0,v_0) + c_4\ln T. \tag{2.23}$$

It follows from (2.23) that

$$\int_0^t d\tau \int \Phi(x;\tau,T)e^{\varepsilon v}uv^m dx \le 8\|L(u_0,v_0)\| + 8c_4 \ln T.$$

Thus,

$$\int_0^t d\tau \int \Phi(x;\tau,T)e^{\varepsilon v}uv^m dx \le c_3(1+\ln T), \tag{2.24}$$

where $c_3 = \max\{8\|L(u_0,v_0)\|, 8c_4\} > 0$. By choosing $\Phi = \Phi(x-\xi;t,T)$, for any $\xi \in \mathbb{R}$, we also arrive at (2.24) with the same constant c_3. (2.21) is then proved. □

Now, we consider $t \in [T-1, T]$.

Lemma 2.4 (See [5]). *Let $T \ge e$. Then for any $t \in [T-1,T]$ we have*

$$\int \Phi(x-\xi;t,T)e^{\varepsilon v}uv^m d\xi \le c_5\left((T-t)^{\frac{n}{2q}} + (T-t)^{-\frac{n}{2p}}\right)\left((1+[s])!\right)^{\frac{pm}{p(1+[s])}} \varepsilon^{-\frac{s}{p}} (\ln T)^{\frac{pm}{p(1+[s])}}, \tag{2.25}$$

where $p > \max\left\{1, \frac{n}{2}\right\}$ *and* q *is its conjugate,* $s = pm$, $[s]$ *is the integer part of* s *and* $c_5 = c_5(n, d, u_0, v_0) > 0$.

Lemma 2.5. *There exists a constant* $c_6 = c_6(n, d, u_0, v_0)$ *such that*

$$\int_{T-1}^{T} d\tau \int \Phi(x-\xi;\tau,T) uv^m(\xi,\tau) d\xi \leq c_6 (\ln T)^{\frac{1}{p}}, \tag{2.26}$$

where $p > \max\left\{1, \frac{n}{2}\right\}$.

Proof. An integration of (2.25) from $T-1$ to T shows that

$$\int_{T-1}^{T} dt \int \Phi(x-\xi;t,T) uv^m(\xi,t) d\xi$$
$$\leq c_5 \left(\frac{2q}{n+2q} + \frac{2p}{2p-n}\right) ((1+[s])!)^{\frac{pm}{p(1+[s])}} \varepsilon^{-\frac{s}{p}} (\ln T)^{\frac{1}{p}}.$$

If we put

$$c_6 = c_5 \left(\frac{2q}{n+2q} + \frac{2p}{2p-n}\right) ((1+[s])!)^{\frac{pm}{p(1+[s])}} \varepsilon^{-\frac{s}{p}},$$

then we get (2.26). □

Proof of theorem 2.1

The function v (see [2]) satisfies the integral equation

$$\begin{aligned} v(x,t) = {} & \frac{b}{a-d} S_1(t) u_0 + S_2(t) \left(v_0 - \frac{b}{a-d} u_0\right) \\ & - \frac{b}{a-d} \int_0^t S_1(t-\tau) u(\tau) v^m(\tau) d\tau \\ & + \int_0^t S_2(t-\tau) \left(u(\tau)v^m(\tau) + \frac{b}{a-d} u(\tau) v^m(\tau)\right) d\tau, \end{aligned}$$

where $S_1(t)$, $S_2(t)$ are the semigroups generated by the operators $a\Delta$, $d\Delta$ in $C_{UB}(\mathbb{R})$ respectively. Whence

(2.27)

$$\begin{aligned} v(x,T) \leq {} & \frac{b}{a-d} \|u_0\| + \|v_0\| + \left(1 + \frac{b}{a-d}\right) \int_0^{T-1} S_2(T-\tau) uv^m(\tau) d\tau \\ & + \left(1 + \frac{b}{a-d}\right) \int_{T-1}^{T} S_2(T-\tau) uv^m(\tau) d\tau. \end{aligned}$$

One can compute that

$$\int_0^t S_2(t-\tau)uv^m(\tau)d\tau = \int_0^t d\tau \int \Phi(x-\xi;\tau,t)uv^m(\xi,\tau)d\xi. \tag{2.28}$$

Also from (2.21) it is easy to see that

$$\int_0^{T-1} d\tau \int \Phi(x-\xi;\tau,t)uv^m(\xi,\tau)d\xi \le c_3\,(1+\ln T)\,. \tag{2.29}$$

Then, taking into account (2.26), (2.28) and (2.29) in (2.27) we get for any $T \ge e$

$$v(x,T) \le A + B\ln T. \tag{2.30}$$

Here,

$$A = \frac{b}{a-d}\,\|u_0\| + \|v_0\| + c_3\left(1+\frac{b}{a-d}\right),$$

and

$$B = (c_3+c_6)\left(1+\frac{b}{a-d}\right).$$

From the estimates (2.30) and the estimate (1.3a) of the global existence, we can prove that there exists a constant $c_7 = c_7(n,a,d,u_0,v_0) > 0$ such that

$$v(x,T) \le c_7 \ln(T+e) \tag{2.31}$$

for all $T \ge 0$. As in [5], we can drop the dependence of c_7 in (2.31) on u_0, v_0 to get finally (2.1). □

Consider more general nonlinear reaction terms of the form $uf(v)$, where $f(v)$ is nonnegative and continuous in $v \in \mathbb{R}$ and nondecreasing in $v \ge 0$, $f(0) = \lim\limits_{v\searrow 0^+} f(v) = 0$, and $\lim\limits_{v\nearrow\infty} f(v) > 0$, $\lim\limits_{v\nearrow\infty} \frac{1}{v}\ln(f(v)) = 0$. Then

Corollary 2.6. *Under the same assumptions in theorem 2.1, there exists a positive constant* $C = C(n,a,d,\|(u_0,v_0)\|, f)$ *such that*

$$\|v(t)\| \le C\ln(t+e)\,, \tag{2.32}$$

for all $t \ge 0$.

In particular, this form includes the Arrhenius reaction, i.e.

$$f(u,v) = uv^m \exp(-\frac{E}{v}),$$

for any m *an even nonnegative integer and* $E > 0$.

Proof. We replace everywhere v^m by $f(v)$. Hence in (2.6)

$$\exp\left\{\varepsilon\left(8(d-a)c_1\right)^{\frac{1}{m}}\left(\frac{1}{T-t}\right)^{\frac{1}{m}}\right\}$$

is replaced by

$$\exp\left\{\varepsilon f^{-1}\left(\frac{8(d-a)c_1}{T-t}\right)\right\}.$$

The condition $f(v)$being nondecreasing in v is used to derive an analogue to inequality (2.22). The remaining estimates are carried out without significant changes. □

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REFERENCES

1. J. D. Avrin, Qualitative theory for a model of laminar flames with arbitrary nonnegative initial data, J. Differential Equations **84**, pp 290-308, 1990.

2. S. Badraoui, Existence of global solutions for systems of reaction-diffusion equations on unbounded domains, E.J.D.E., Vol. 2002, No. **74**, pp 1-10, 2002.

3. L. Berlyand and J. Xin, Large time asymptotics of solutions to a model combustion system with critical nonlinearity, Nonlinearity **8**, pp 161-178, 1995.

4. J. Bricomt, A. Kupiainen, J. Xin, Global large time self-similarity of a thermal-diffusive combustion system with critical nonlinearity, J.D.E., Vol. **130**, No. 1, pp 9-35, 1996.

5. P. Collet & J. Xin, Global Existence and large time asymptotic bounds of L^∞ solutions of thermal diffusive Ccombustion systems on $\mathbb{R}^n$, Ann. Scuola Norm. Sup. **23**, pp 625-642, 1996.

6. S. Hollis, R. Martin and M. Pierre, Global existence and boundednessin reaction-diffusion systems, SIAM J. Math. Anal. **18**, No. 3, pp 744-761, 1987.

7. M. Kirane, Global bounds and asymptotics for a system of reaction-diffusion equations, J. Math. Anal. App. 138, pp. 328-342, 1989.

8. R. Martin and M. Pierre, Nonlinear reaction-diffusion systems, in "Nonlinear Equations in the Applied Sciences" (W. F. Ames and C. Rogers, Eds.), Academic Press, Boston, 1992.

9. K. Masuda, On the global existence and asymptotic behavior of solutions of reaction-diffusion equations, Hokkaido Math. J.**12**, pp 360-370, 1983.

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