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New Oscillation criteria for nonlinear higher order neutral difference equation

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ABSTRACT. In this paper, we establish some new oscillation criteria for the oscillation of all solution of neutral difference equation of the form

$$\triangle^m(x_n - p_n x_{n-\tau}) + \sum_{i=1}^{r} Q_i(n) f(x_{n-\sigma_i})$$

for different ranges of p_n, where $m \geq 2$. The nonlinearity of f, the nature of m and the range of p_n are closely related for the results concerning this equation.

Keywords. Oscillation, Higher order, Non-linear, Neutral difference equation.
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1. Introduction

During the last few years there have been some activities concerning the oscillatory and non-oscillatory solutions of higher order neutral delay difference equations, see for example, the monographs [1] and [4] where numerous properties of their solution are studied and a detailed bibliography is given. The conditions assumed differ from author to author due to the different techniques they use and different type of equations they consider. It is interesting to note that the conditions assumed by different authors for similar types of equations are often not comparable. Among numerous papers dealing with the subject we refer in particular to [2, 5-6, 9-19] and the references cited therein.

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Recently, Guan, et. al. [7] considered the higher order nonlinear neutral difference equation

$$(1.1)\qquad \triangle^m\big[x_n - p_n\, x_{n-\tau}\big] + \sum_{i=1}^{r} Q_i(n) f\big(x_{n-\sigma_i(n)}\big) \quad n \in 0, 1, 2, ...$$

and obtained some results for the oscillation of solutions of (1.1). They proved that every solution of (1.1) oscillates if

$$\limsup_{n\to\infty} \sum_{s=n+\tau-\sigma}^{n} Q_i(s) > \frac{(m-1)!p}{\lambda\big(s-n+m-2\big)^{(m-1)}}$$

where λ is some positive constant. When $i = 1$, this equation (1.1) reduces to the form

$$(1.2)\qquad \triangle^m\big[x_n - p_n\, x_{n-\tau}\big] + q(n) f(x_{n-\sigma}).$$

For different ranges of p_n, R.P.Agawaral, E.Thandapani and P.J.Y.Wong [3] considered the Eqn. (1.2) and obtained sufficient conditions for the oscillation of solutions of (1.2).

In the present work, we have considered a more general equation of the form (1.2) with r variable terms and obtained some new oscillation results which are different from those for the equation (1.1).

We shall consider the oscillatory behavior of higher order difference equations of the form

$$(1.3)\qquad \triangle^m\big[x_n - p_n\, x_{n-\tau}\big] + \sum_{i=1}^{r} Q_i(n) f\big(x_{n-\sigma_i}\big) \quad n \in N = \{0, 1, 2, ...\}$$

where $\triangle$ is the forward difference operator defined by $\triangle x_n = x_{n+1} - x_n$ and for $i > 1$, $\triangle^i$ is the i^{th} order forward difference operator $\triangle^i x_n = \triangle(\triangle^{i-1} x_n)$. τ and σ_i, $1 \leq i \leq r$ r are fixed nonnegative integers. $\{p_n\}$ is a real sequence and $\{Q_i(n)\}_{i=1}^{r}$ is a nonnegative real sequence. Moreover $f : R \to R$ is non-decreasing function with $u\ f(u) > 0$ for all $u > 0$.

Let $\rho = max.\{\tau, \sigma_i\}, 1 \leq i \leq r$. Let N_0 be a fixed non negative integer. Then by a solution of equation (1.3), we mean a real sequence $\{x_n\}$ which is defined for all $n \geq N_0 - \rho$ and satisfies (1.3) for all $n \geq N_0$. A solution

$\{x_n\}$ of (1.3) is said to be eventually positive if $x_n > 0$ for all large n, and eventually negative if $x_n < 0$ for all large n. It is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called non-oscillatory. We will also say that equation (1.3) is oscillatory if every solution of (1.3) is oscillatory. For the sake of convenience, we define $\{z_n\}$ by

$$z_n = x_n - p_n\, x_{n-\tau}. \tag{1.4}$$

Regarding to equation (1.3), throughout we shall assume the following conditions.

- (C_1) $\{Q_i(n)\}_{i=1}^{r}$ is not eventually identically zero and $Q_i(n) > 0$ for $1 \leq i \leq r$ and $n \in N$.
- (C_2) For $u > 0$, and $v > 0$, there exists $\delta > 0$ such that $f(u) + f(v) \geq \delta f(u+v)$.
- $(C_{2'})$ For $u < 0$, and $v < 0$, there exists $\delta > 0$ such that $f(u) + f(v) \leq \delta f(u+v)$.
- (C_3) For $u > 0$, and $v > 0$ $f(uv) \leq f(u)f(v)$.
- (C_4) $f(-u) = -f(u), u \in R$.
- (C_5) $\liminf_{|u|\to\infty} \left\{\dfrac{f(u)}{u}\right\} > \alpha > 0$.
- (C_6) $\liminf_{|u|\to 0} \left\{\dfrac{f(u)}{u}\right\} > \beta > 0$.
- (C_7) $\displaystyle\int_0^k \frac{du}{f(u)} < \infty$ for every $k > 0$.
- (C_8) $\displaystyle\int_{\pm k}^{\pm\infty} \frac{du}{f(u)} < \infty$ for every $k > 0$.
- (C_9) $\displaystyle\sum_0^{\infty} \Big(\sum_{i=1}^{r} Q_i(n)\Big) = \infty$.
- (C_{10}) $\displaystyle\sum_0^{\infty} \Big(\sum_{i=1}^{r} Q_i^*(n)\Big) = \infty$.

where $Q_i^*(n) = \min\Big\{Q_i(n), Q_i(n-\tau)\Big\}_{i=1}^{r}$.

Remark 1.1. The prototype of f satisfying (C_2) and (C_4) is

$$f(u) = (a + b|u|^{\lambda})|u|^{\mu}\ sgn\ u,$$

where $a \geq 0$, $b \geq 0$, $\lambda \geq 0$ and $\mu \geq 0$ such that $a^2 + b^2 \neq 0$. It satisfies (C_3) if $a \geq 1$ and $b \geq 1$. Moreover, $f : R \to R$ is continuous such that $uf(u) > 0$ for $u \neq 0$ and f is non-decreasing. If $\lambda + \mu \geq 1$ and $b > 0$, then f satisfies (C_5). On the other hand, (C_6) holds if $\lambda + \mu \leq 1$ and $b > 0$. Further, $\lambda + \mu < 1$ and $b > 0$ imply that (C_7) holds, and $\lambda + \mu > 1$ and $b > 0$ imply that (C_8) holds because we may write $f(u) \geq b|u|^{\lambda+\mu}\ sgn\ u$. If $f_1(u) = |u|^{\gamma}\ sgn\ u$, where $\gamma > 0$, then $f_1 : R \to R$ is continuous with $uf_1(u) > 0$ for $u \neq 0$ and it is non-decreasing. Further, f_1 satisfies $(C_2) - (C_4)$. It satisfies (C_5) if $\gamma \geq 1$ and (C_8) if $\gamma > 1$. Further, it satisfies (C_6) if $\gamma \leq 1$ and (C_7) if $\gamma < 1$.

2. Oscillation of equation (1.3)

This section deals with the oscillation of solutions of equation (1.3). The results here differ substantially from those in [7, 8]. Different types of sublinear / superlinear f are considered in this paper. We need the following Lemmas for our work in the sequel:

Lemma 2.1. *[1] (**Discrete Kneser's Theorem**) Let z_n be defined for $n \geq a$ and $z_n > 0$ with $\triangle^m z_n$ of constant sign for $n \geq a$ and not identically zero. Then, there exits an integer $j, 0 \leq j \leq m$ with $(m + j)$ odd for $\triangle^m z_n \leq 0$ and $(m + j)$ even for $\triangle^m z_n \geq 0$, such that*

$$j \leq m - 1 \ implies\ (-1)^{j+i} \triangle^i z_n > 0 \ for\, all\ n \geq a, j \leq i \leq m - 1,$$

and

$$j \geq 1 \ implies\ \triangle^i z_n > 0 \ for\, all\, large\ n \geq a, 1 \leq i \leq j - 1,$$

Lemma 2.2. *[1] Assume that $Q(n) \geq 0$ for all $n \in N$, and*

$$\liminf_{n\to\infty} \left[\sum_{s=n-\sigma}^{n-1} Q(s) \right] > \left(\frac{\ell}{\ell = 1} \right)^{\ell=1}$$

Then,

(i) *$v_{n+1} - v_n + Q(n)v_{n-\sigma} \leq 0$, $n \in N$ cannot have an eventually positive solution.*
(ii) *$v_{n+1} - v_n + Q(n)v_{n-\sigma} \geq 0$, $n \in N$ cannot have an eventually negative solution.*

Lemma 2.3. *[1] If $Q(n)$ satisfies the conditions of Lemma 2.2, then*

(i) $v_{n+1} - v_n - Q(n)v_{n-\sigma} \geq 0,\ n \in N$ *cannot have an eventually positive solution.*

(ii) $v_{n+1} - v_n - Q(n)v_{n-\sigma} \leq 0,\ n \in N$ *cannot have an eventually negative solution.*

Lemma 2.4. *Suppose that $0 \leq p_n \leq p$, where p is a constant and (C_9) holds. If x_n is a solution of equation (1.3) with $x_n > 0$ for $n \geq n_0 \in N$, then either*

$$\lim_{n\to\infty} \triangle^k z_n = -\infty, \qquad k = 0, 1, 2, ..., m-1 \tag{2.1}$$

or

$$(-1)^{m+\ell}\triangle^\ell z_n \leq 0, \quad \ell = 0, 1, 2, ..., m-1 \qquad n \geq n_1 \geq n_0 + \rho \tag{2.2}$$

and

$$\lim_{n\to\infty} \triangle^\ell z_n = 0, \ \ \ell = 0, 1, 2, ..., m-1$$

Proof. From (1.3), we have

$$\triangle^m z_n = -\sum_{i=1}^{r} Q_i(n) f(x_{n-\sigma_i}) \leq 0 \qquad for\ n \geq n_0 + \rho. \tag{2.3}$$

Hence $z_n, \triangle z_n, ..., \triangle^{m-1} z_n$ are monotonic and are of constant sign for $n \geq n_1 \geq n_0 + \rho$. Further, either

$$\lim_{n\to\infty} \triangle^{m-1} z_n = -\infty$$

or

$$\lim_{n\to\infty} \triangle^{m-1} z_n = -L \in R$$

If the former holds, then ,

$$\lim_{n\to\infty} \triangle^k z_n = -\infty, \qquad k = 0, 1, 2, ..., m-1.$$

Suppose that the latter holds. Then from (2.3) we obtain

$$\sum_{n_1}^{\infty}\left[\sum_{i=1}^{r} Q_i(n) f(x_{n-\sigma_i})\right] < \infty \tag{2.4}$$

We now claim that $\liminf_{n\to\infty} x_n = 0$. If not, then $x_n > \alpha > 0$ for $n \geq n_2 \geq n_1$. Hence

$$f(\alpha)\sum_{n_2+\rho}^{\infty}\left[\sum_{i=1}^{r} Q_i(n)\right] \leq \sum_{n_2+\rho}^{\infty}\left[\sum_{i=1}^{r} Q_i(n) f(x_{n-\sigma_i})\right] < \infty$$

due to (2.4), a contradiction to (C_9). Thus our claim holds. Consequently, there exists a sequence $\{n_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} x_{n_k} = 0$ Since z_n is monotonic, $z_n \leq x_n$ and $z_{n+\tau} > -p\, x_n$, then $\lim_{n\to\infty} z_n = 0$ and hence (2.2) holds. Thus the lemma is proved. □

Lemma 2.5. *If the range of p_n in Lemma 2.4 is replaced by $0 \leq p_n \leq 1$, then only (2.2) holds.*

Proof. If (2.1) holds then $\lim_{n\to\infty} \triangle z_n = 0$. Since $z_n < 0$ for large n, then $x_n < p_n\, x_{n-\tau} \leq x_{n-\tau}$ and hence x_n is bounded. Consequently, z_n is bounded, a contradiction. Thus the lemma follows from Lemma 2.4. □

Theorem 2.1. *Let $-p \leq p_n \leq 0$, where $p > 0$ is a constant, and p_n be a monotonic decreasing sequence. Let $\tau < \sigma = \min\{\sigma_i : 1 \leq i \leq r\}$. If $(C_2) - (C_4)$, (C_7) and (C_{10}) hold, then every solution of equation (1.3) oscillates.*

Proof. If possible, let x_n be a non-oscillatory solution of equation (1.3) we may take $x_n > 0$ for $n \geq n_0 \geq N_0$ in view of (C_4). Setting z_n as in (1.4), we obtain $z_n > 0$ for $n \geq n_0 + \tau$. Proceeding as in the proof of Lemma 2.4, we obtain either

$$\lim_{n\to\infty} \triangle^{m-1} z_n = -\infty \qquad or \qquad \lim_{m\to\infty} \triangle^{m-1} z_n = L \in R.$$

If either the former holds or $L < 0$, then $z_n < 0$ for large n, a contradiction. Hence $0 \leq L \leq \infty$. Since $z_n > 0$ and $\triangle^m z_n \leq 0$ for $n \geq n_0 \geq \rho$, then by Lemma 1, there exists an integer $j \leq m-1$ and $n_1 \geq n_0 + \rho$ such that $m + j$ is odd,

$$\triangle^i z_n > 0 \ for\ i = 0, 1, 2, ..., j-1$$

and

$$(-1)^{j+i}\triangle^i z_n > 0 \; for \; i = j, j+1, ..., m-2 \, and \; n \geq n_1.$$

By the Taylor series expansion, we have for $n \geq n_1$,

$$z_n = z_{n-\tau} + \tau \triangle z_{n-\tau} + \frac{\tau^2}{2!} z_{n-\tau} + ... + \frac{r^{(j)}}{j!} \triangle^j z_y$$

where $\tau > 0$, (j) is the usual factorial notation and $n-\tau < y < n$. Since $\triangle^j z_n$ is decreasing, than

$$z_n > \frac{\tau^{(j)}}{j!} \triangle^j z_n$$

Another Taylor series expansion yields

$$\begin{aligned} \triangle^j z_n =& \triangle^j z_{n+\tau} + (-\tau)\triangle^{j+1} z_{n+\tau} + \frac{(-\tau)^{(2)}}{2!} \triangle^{j+2} z_{n+\tau} + ... \\ &+ \frac{(-\tau)^{(m-j-1)}}{(m-j-1)!} \triangle^{m-1} z_x \\ &> \frac{\tau^{(m-j-1)}}{(m-j-1)!} \triangle^{m-1} z_{n+\tau} \end{aligned}$$

because $\tau > 0$, $n < x < n+\tau$ and $\triangle^{m-1} z_n$ is monotonically decreasing. Hence, for $n \geq n_1 + \tau$

$$\begin{aligned} z_n &> \frac{\tau^{(j)}}{j!} \frac{\tau^{(m-j-1)}}{(m-j-1)!} \triangle^{m-1} z_{n+\tau} \\ &= \frac{\tau^{(j)} \tau^{(m-j-1)}}{(m-1)!} \triangle^{m-1} z_{n+\tau} \end{aligned} \tag{2.5}$$

The use of (C_2), (C_3) and (2.5) yields for $n \geq n_2 \geq n_1 + (\sigma - \tau) + \rho$

$$\begin{aligned} 0 &= \triangle^m z_n + \sum_{i=1}^{r} Q_i(n) f(x_{n-\sigma_i}) + f(-p_{n-\sigma_i}) \Big[\triangle^m z_{n-\tau} + \sum_{i=1}^{r} Q_i(n-\tau) f(x_{n-\sigma_i-\tau})\Big] \\ &\geq \triangle^m z_n + f(p)\triangle^m z_{n-\tau} + \sum_{i=1}^{r} Q_i^*(n) \Big[f(x_{n-\sigma_i}) f(-p_{n-\sigma_i}) f(x_{n-\sigma_i-\tau})\Big] \\ &\geq \triangle^m z_n + f(p)\triangle^m z_{n-\tau} + \delta \sum_{i=1}^{r} Q_i^*(n) f\Big[\frac{(\sigma_i-\tau)^{(j)}(\sigma_i-\tau)^{(m-j-1)}}{(m-1)!}\Big] \triangle^{m-1} z_{n-\tau} \\ &\geq \triangle^m z_n + f(p)\triangle^m z_{n-\tau} + \delta f\Big[\frac{(\sigma-\tau)^{(j)}(\sigma-\tau)^{(m-j-1)}}{(m-1)!} \triangle^{m-1} z_{n-\tau}\Big] \sum_{i=1}^{r} Q_i^*(n) \end{aligned}$$

Hence

$$\delta \sum_{i=1}^{r} Q_i(n) + \frac{\triangle^m z_n}{f(u)} + \frac{f(p)\triangle^m z_{n-\tau}}{f(v)} \leq 0$$

where

$$u = \frac{(\sigma-\tau)^{(j)}(\sigma-\tau)^{(m-j-1)}}{(m-1)!} \triangle^{m-1} z_n$$

and

$$v = \frac{(\sigma-\tau)^{(j)}(\sigma-\tau)^{(m-j-1)}}{(m-1)!} \triangle^{m-1} z_{n-\tau}$$

and the fact that $\triangle^{m-1} z_n$ is monotonically decreasing is used. Summing the above inequality, we obtain

$$\delta \sum_{n_2}^{\infty} \Big(\sum_{i=1}^{r} Q_i^*(n) \Big) + \frac{(m-1)!}{(\sigma-\tau)^{(j)}(\sigma-\tau)^{(m-j-1)}} \int_{c_1}^{L} \frac{du}{f(u)} +$$

$$\frac{f(p)(m-1)!}{(\sigma-\tau)^{(j)}(\sigma-\tau)^{(m-j-1)}} \int_{c_2}^{L} \frac{du}{f(v)} \leq 0$$

where

$$c_1 = \frac{(\sigma-\tau)^{(j)}(\sigma-\tau)^{(m-j-1)}}{(m-1)!} \triangle^{m-1} z_{n_2}$$

and

$$c_2 = \frac{(\sigma-\tau)^{(j)}(\sigma-\tau)^{(m-j-1)}}{(m-1)!} \triangle^{m-1} z_{n_2-\sigma}$$

This leads to a contradiction to (C_{10}) in view of (C_7). Thus the theorem is proved. □

Example 2.1. Consider the difference equation

$$(2.6) \quad \triangle^m \big[x_n + \frac{1}{2} x_{n-1}\big] + 3^{n/2+1} 2^{n/2} \big[x_{n-4}\big]^{1/2} + 3^{n/2+1} 2^{n/2+2} \big[x_{n-6}\big]^{1/2} = 0$$

where $m = n/2$, $p = -1/2$ and $f(x) = x^{1/2}$. It is noted that all hypothesis of Theorem 1 are fulfilled. In fact, (2.6)) has an oscillatory solution given by $\{x_n\} = \{(-1)^n 2^n\}$.

Theorem 2.2. *Let* $-1 \leq p_n \leq 0$. *If* (C_2), $(C_{2'})$, (C_7) *and* (C_{10}) *hold, then every solution of equation (1.3) oscillates, where* $\tau < \sigma = \min\{\sigma_i : 1 \leq i \leq r\}$.

Proof. Proceeding as in the proof of Theorem 2.1 we obtain (2.5), since $z_n \leq x_n + x_{n-\tau}$, then (C_2) and (10) yield, for $n \geq n_2 \geq n_1 + (\sigma - \tau) + \rho$,

$$\begin{aligned} 0 = \triangle^m z_n + \triangle^m z_{n-\tau} + \sum_{i=1}^{r} Q_i(n) f(x_{n-\sigma_i}) \\ + \sum_{i=1}^{r} Q_i(n-\tau) f(x_{n-\sigma_i-\tau}) \\ \geq \triangle^m z_n + \triangle^m z_{n-\tau} + \delta \sum_{i=1}^{r} Q_i^*(n) f(z_{n-\sigma_i}) \end{aligned}$$

The rest of the proof is similar to that of Theorem 2.1. Thus the proof of the theorem is complete. □

Theorem 2.3. *Let* $0 \leq p_n \leq 1$. *If* m *is odd and if* (C_7) *and* (C_9) *hold, then every solution of equation (1.3) oscillates.*

Proof. Let x_n be a non-oscillatory solution of equation (1.3) with $x_n > 0$ or $x_n < 0$ for $n \geq n_0 \geq N_0$. We consider the case $x_n > 0$ for $n \geq n_0$. The case $x_n < 0$ is similar. Setting z_n as in (1.4), we get $z_n \leq x_n$ for $n \geq n_0 + \tau$. Then (2.2) holds by Lemma 2.5. Since m is odd, then $z_n > 0$ for $n \geq n_0 + \rho$. Taylor series expansion yields, for $n \geq n_0$,

$$\begin{aligned} z_{n-\tau} = z_n + (-\tau) \triangle z_n + \frac{(-\tau)^{(2)}}{2!} \triangle z_n + \ldots \\ + \ldots + \frac{(-\tau)^{(m-1)}}{(m-1)!} \triangle^{m-1} z_x \\ \geq \frac{\tau^{(m-1)}}{(m-1)!} \triangle^{m-1} z_n \end{aligned} \tag{2.7}$$

because $\triangle^{m-1} z_n$ is monotonically decreasing, where $\tau > 0$, $n - \tau < x < n$. Hence for $n \geq n_1$,

$$\begin{aligned} 0 &= \triangle^m z_n + \sum_{i=1}^{r} Q_i(n) f(x_{n-\sigma_i}) \\ &\geq \triangle^m z_n + \sum_{i=1}^{r} Q_i(n) f(z_{n-\sigma_i}) \\ &\geq \triangle^m z_n + \sum_{i=1}^{r} Q_i(n) f\left(\frac{(\sigma_i)^{(m-1)}}{(m-1)!} \triangle^{m-1} z_x \right) \\ &\geq \triangle^m z_n + \sum_{i=1}^{r} Q_i(n) f\left(\frac{(\sigma)^{(m-1)}}{(m-1)!} \triangle^{m-1} z_x \right) \end{aligned}$$

where $\sigma = \min\{\sigma_i : 1 \leq i \leq r\}$. Proceeding as in the proof of Theorem 2.1 and using (C_7), we obtain a contradiction to (C_9). Hence the theorem is proved. □

Theorem 2.4. *Let $0 \leq p_n \leq 1$, m is odd and (C_6) holds. If*

$$\liminf_{x\to\infty} \sum_{n-\sigma}^{n-1} \left(\sum_{i=1}^{r} Q_i(s) \right) > \frac{(m-1)!}{\beta \in \sigma^{(m-1)}} \tag{2.8}$$

where $2\sigma = \min\{\sigma_i : 1 \leq i \leq r\}$, then every solution of equation (1.3) oscillates.

Proof. Suppose that x_n is a non-oscillatory solution of equation (1.3) with $x_n > 0$ for $n \geq n_0 \geq N_0$. The case $x_n < 0$ for $n \geq n_0$ is similar be dealt with. Then $z_n \leq x_n$ for $n \geq n_0 \geq \tau$, where z_n is same as in (1.4). We claim that (2.8) implies (C_9). Indeed if (C_9) fails, then

$$0 < \lambda = \sum_{0}^{\infty} \left(\sum_{i=1}^{r} Q_i(s) \right) < \infty.$$

Hence

$$\begin{aligned}
&\liminf_{n\to\infty} \sum_{n-\sigma}^{n-1} \Big(\sum_{i=1}^{r} Q_i(n) \Big) \\
&= \liminf_{n\to\infty} \left[\sum_{0}^{n-1} \Big(\sum_{i=1}^{r} Q_i(n) \Big) - \sum_{0}^{n-\sigma-1} \Big(\sum_{i=1}^{r} Q_i(n) \Big) \right] \\
&\le \liminf_{n\to\infty} \left[\sum_{0}^{n-1} \Big(\sum_{i=1}^{r} Q_i(n) \Big) \right] + \limsup_{n\to\infty} \left[- \sum_{0}^{n-\sigma-1} \Big(\sum_{i=1}^{r} Q_i(n) \Big) \right] \\
&\le \liminf_{n\to\infty} \left[\sum_{0}^{n-1} \Big(\sum_{i=1}^{r} Q_i(n) \Big) \right] - \liminf_{n\to\infty} \left[\sum_{0}^{n-\sigma-1} \Big(\sum_{i=1}^{r} Q_i(n) \Big) \right] \\
&= \lambda - \lambda = 0.
\end{aligned}$$

Which is a contradiction. Thus (2.2) holds by Lemma 2.5. Since m is odd, then $z_n > 0$ for $n \ge n_1 \ge n_0 + \tau$. Further, (C_6) yields $f(z_n) \ge \beta z_n$ for $n \ge n_1 \ge n_2$. Proceeding as in the proof of Theorem 2.3, we obtain (2.7). Hence, for $n \ge n_2 + \rho$,

$$\begin{aligned}
0 &= \triangle^m z_n + \sum_{i=1}^{r} Q_i(n) f(x_{n-\sigma_i}) \\
&\ge \triangle^m z_n + \sum_{i=1}^{r} Q_i(n) f(z_{n-\sigma_i}) \\
&\ge \triangle^m z_n + \beta \sum_{i=1}^{r} Q_i(n) z_{n-\sigma_i} \\
&\ge \triangle^m z_n + \beta \sum_{i=1}^{r} Q_i(n) z_{n-2\sigma_i} \\
&\ge \triangle^m z_n + \beta \frac{\sigma^{(m-1)}}{(m-1)!} \Big(\sum_{i=1}^{r} Q_i(n) \Big) \triangle^{m-1} z_{n-\sigma}
\end{aligned}$$

□

Example 2.2. Consider the difference equation

$$\triangle^5 \big[x_n - x_{n-1}\big] + 1 \cdot \big[x_{n-5}\big]^3 + 65 \big[x_{n-4}\big]^5 = 0,\ n \ge 5 \tag{2.9}$$

Clearly all the conditions of Theorem 2.4 are satisfied. The equation (2.9) has one such oscillatory solution $\{x_n\} = \{(-1)^n\}$.

Theorem 2.5. *Let $0 \leq p_n \leq 1$, where $p > 0$ is a constant. Let m be odd and $\tau > \sigma_1 = \max\{\sigma_i : 1 \leq i \leq r\}$. If (C_8) and (C_9) hold, then every solution of equation (1.3) oscillates.*

Proof. If possible, let x_n be an non-oscillatory solution of equation (1.3). Let $x_n > 0$ for $n \geq n_0 \geq N_0$. The case $x_n < 0$ for $n \geq n_0$ may similarly be dealt with. Then either (2.1) holds or (2.2) holds by Lemma 2.4, where z_n is defined by (1.4). If (2.1) holds, then $\triangle^k z_n < 0$ for $n \geq n_1 \geq n_0$, $0 \leq k \leq m-1$.

By the Taylor series expansion we have for $n \geq n_1 + \tau$,

$$z_n = z_{n-\tau} + (\tau)\triangle\, z_{n-\tau} + \frac{(\tau)^{(2)}}{2!}\triangle^2\, z_{n-\tau} + \dots + \frac{(\tau)^{(m-1)}}{(m-1)!}\triangle^{m-1}\, z_x$$

where $n - \tau < x < n$ and $\tau > 0$. Since $\triangle^{m-1} z_n$ is monotonically decreasing, then

$$z_n < \frac{(\tau)^{(m-1)}}{(m-1)!}\triangle^{m-1}\, z_{n-\tau}.$$

Further, $z_n > -p\, x_{n-\tau}$ for $n \geq n_1$ implies that $x_n > -1/p\; z_{n+\tau}$. Hence for $n \geq n_1 + \rho$,

$$\begin{aligned}
0 &= \triangle^m z_n + \sum_{i=1}^{r} Q_i(n) f(x_{n-\sigma_i}) \\
&\geq \triangle^m z_n + \sum_{i=1}^{r} Q_i(n) f(\frac{-1}{p}\; z_{n+\tau-\sigma_i}) \\
&\geq \triangle^m z_n + f\left(-\frac{1}{p}\; z_{n+\tau-\sigma_1}\right)\sum_{i=1}^{r} Q_i(n) \\
&\geq \triangle^m z_n + f\left(-\frac{1}{p}\,\frac{(\tau-\sigma_1)^{(m-1)}}{(m-1)!}\;\triangle^m z_n\right)\sum_{i=1}^{r} Q_i(n)
\end{aligned}$$

that is,

$$\sum_{i=1}^{r} Q_i(n) + \frac{\triangle^m z_n}{f(U)} \leq 0,$$

where $u = -\frac{1}{p}\frac{(\tau-\sigma_1)^{(m-1)}}{(m-1)!}\triangle^m z_n$. Hence

$$\sum_{n_2}^{\infty}\left(\sum_{i=1}^{r} Q_i(n)\right) \leq \frac{p\,(m-1)!}{(\tau-\sigma_1)^{(m-1)}}\int_c^{\infty}\frac{du}{f(u)}$$

where $n_2 \geq n_1+\rho$ and $c = -\frac{(\tau-\sigma_1)^{(m-1)}}{p\,(m-1)!}\triangle^m z_{n_2}$. This contradicts (C_9) due to (C_8). Hence (2.2) holds. Consequently (2.4) is true. Since m is odd, then $z_n > 0$ for $n \geq n_1$ and hence $x_n > p_n\, x_{n-\tau} \geq x_{n-\tau}$. Thus $\liminf_{n\to\infty} x_n > 0$. This contradicts (C_9) in view of (2.4). Hence the proof of the theorem is complete. □

Theorem 2.6. *Let $0 \leq p_n \leq 1$, where $p > 0$ is a constant. Let m be odd and $\tau > \sigma_1 = \max\{\sigma_i : 1 \leq i \leq r\}$ and (C_5) hold. If*

$$\liminf_{n\to\infty}\sum_{n-\delta}^{n-1}\left(\sum_{i=1}^{r} Q_i(n)\right) > \frac{p(m-1)!}{\alpha \in (\tau-\delta-\sigma_i)^{(m-1)}} \tag{2.10}$$

where $0 < \delta < \tau-\sigma_1$, then every solution of equation (1.3) oscillates.

Proof. We may note that (2.10) implies (C_9). Proceeding as in the proof of Theorem 2.5, we obtain

$$z_n < \frac{(\tau)^{(m-1)}}{(m-1)!}\triangle^{m-1} z_{n-\tau},$$

for $n \geq n_1+\tau$ when (2.1) holds. Further $x_n > -1/p\; z_{n+\tau}$ for $n \geq n_1$. From (2.1) it follows that $z_n \to \infty$ as $n \to \infty$. Hence $f(z_n) > z_n$ for $n \geq n_2 \geq n_1+\rho$.

Hence for $n \geq n_3 \geq n_2 + \rho$,

$$\begin{aligned}
0 &= \triangle^m z_n + \sum_{i=1}^{r} Q_i(n) f(x_{n-\sigma_i}) \\
&\geq \triangle^m z_n + \sum_{i=1}^{r} Q_i(n) f\left(\frac{-1}{p}\ z_{n+\tau-\sigma_i}\right) \\
&\geq \triangle^m z_n - \frac{\alpha}{p} \sum_{i=1}^{r} Q_i(n)\ z_{n+\tau-\sigma_i} \\
&\geq \triangle^m z_n - \frac{\alpha}{p}\ \ z_{n+\tau-\sigma_i} \sum_{i=1}^{r} Q_i(n) \\
&\geq \triangle^m z_n - \frac{\alpha}{p} \frac{(\tau-\delta-\sigma_1)^{(m-1)}}{(m-1)!}\ \triangle^{m-1} z_{n+\delta} \sum_{i=1}^{r} Q_i(n)
\end{aligned}$$

which contradicts Lemma 2.3 in view of (2.10) because $\triangle^{m-1} z_n < 0$ for $n \geq n_3$. If (2.2) holds, we arrive at a contradiction as in the proof of Theorem 2.5. Thus the theorem is proved.

Theorem 2.7. *Suppose* $0 \leq p_n \leq 1$. *If* m *is even and* $\tau < \sigma = \min\{\sigma_i : 1 \leq i \leq r\}$ *and* (C_7) *and* (C_9) *hold, , then every solution of equation (1.3) oscillates.*

Proof. Let x_n be a non-oscillatory solution of equation (1.3) with $x_n > 0$ for $n \geq n_0 \geq N_0$. From Lemma 2.5, it follows that (2.2) holds, where z_n is defined by (1.4). Since m is even, then $z_n < 0$, $\triangle z_n > 0, ..., \triangle^{m-1} z_n > 0$ for $n \geq n_1 \geq n_0 + \rho$. Further,

$$z_{n-\tau} = z_n + (-\tau)\triangle\, z_n + \frac{(-\tau)^{(2)}}{2!}\triangle^2\, z_n + ... + \frac{(-\tau)^{(m-1)}}{(m-1)!}\triangle^{m-1}\, z_x$$

where $n - \tau < x < n$ and $\tau > 0$, implies that

$$z_n < \frac{(-\tau)^{(m-1)}}{(m-1)!}\triangle^{m-1}\, z_n$$

for $n \geq n_1$. Since $z_n > -z_{n+\tau}$ for $n \geq n_1$, then

$$\begin{aligned} 0 &= \triangle^m z_n + \sum_{i=1}^{m} Q_i(n) f(x_{n-\sigma_i}) \\ &\geq \triangle^m z_n + \sum_{i=1}^{m} Q_i(n) f(-z_{n+\tau-\sigma_i}) \\ &\geq \triangle^m z_n + f(-z_{n+\tau-\sigma_i}) \sum_{i=1}^{r} Q_i(n) \\ &\geq \triangle^m z_n + f\left(\frac{(\sigma-\tau)^{(m-1)}}{(m-1)!} \triangle^m z_n \right) \sum_{i=1}^{r} Q_i(n) \end{aligned}$$

for $n \geq n_2 \geq n_1 + \rho$. Since $\triangle^{m-1} z_n \to \infty$ as $n \to \infty$, then summing the above in equality from n_2 to ∞ yields a contradiction to (C_9) due to (C_7). A similar contradiction is obtained if $x_n < 0$ for $n \geq n_0$. Hence the proof of the theorem is complete. □

Example 2.3. Consider the difference equation

$$(2.11) \qquad \triangle^4 \left[x_n - \frac{1}{2}\, x_{n-1} \right] + \frac{3^4}{2^5} \left[x_{n-2} \right] + 3^4\, 2^{2n-11} \left[x_{n-3} \right]^3 = 0, \;\; n \geq 4$$

All the hypothesis of Theorem 2.7 are satisfied. Note that equation (2.11) has an oscillatory solution $\{x_n\} = \{(-1)^n / 2^n\}$.

The proofs of the following Theorems 2.8 and 2.9 are quite similar to that of Theorems 2.6 and 2.7 and hence omitted.

Theorem 2.8. *Let* $0 \leq p_n \leq 1$, *m be even and* $\tau < \sigma = \min\{\sigma_i : 1 \leq i \leq r\}$. *If* (C_6) *hold and*

$$\liminf_{n\to\infty} \sum_{n-\sigma}^{n-1} \left(\sum_{i=1}^{r} Q_i(n) \right) > \frac{(m-1)!}{\beta \in (\sigma-\tau-c)^{(m-1)}}$$

where $0 < c < \sigma - \tau$, *then every solution of equation (1.3) oscillates.*

Theorem 2.9. *Let* $1 \leq p_n \leq 1$, *m be even and* $\tau < \sigma = \min\{\sigma_i : 1 \leq i \leq r\}$. *If* (C_7) *and* (C_9) *hold, then every solution of equation (1.3) oscillates.*

3. Remarks

It is interesting to notice that the range of p_n, the nature of m and super linearly / sub linearity of f are closely related in the results concerning equation (1.3). We have no results for superlinear f when $0 \leq p_n \leq 1$ or $-p \leq p_n \leq 0$ irrespective of m odd or even, where p is any positive constant. No result for equation (1.3) is known if p_n changes sign with or without $-1 \leq p_n \leq 1$. The conditions imposed on $Q_i(n)$, $1 \leq i \leq r$, in all the previous theorems are sufficient. For $n = 1$ or $n \geq 2$, similar results may be obtained for equation (1.3), when $Q_i(n)$, $1 \leq i \leq r$. It seems that no result is known for $Q_i(n)$ changes sign.

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