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EXPONENTIAL DECAY FOR A SEMILINEAR PROBLEM WITH MEMORY

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ABSTRACT. We consider the abstract semilinear evolution equation with a memory term

$$x'(t) + Ax(t) = F\left(t, x(t), \int_0^t l(t,s) f(s, x(s))ds\right), \quad t \in I = [0,T],$$
$$x(0) = x_0, \quad x_0 \in X.$$

By the semigroup approach and using some techniques based on some inequalities we prove exponential decay in some norms for solutions of the Cauchy problem. This is established for some functions f and F with unbounded time dependent terms.

1. INTRODUCTION

In this paper we are concerned with the following initial value problem

$$(1) \quad \begin{cases} x'(t) + Ax(t) = F\left(t, x(t), \int_0^t l(t,s) f(s, x(s))ds\right), & t \in I = [0,T], \\ x(0) = x_0 \in X, \end{cases}$$

where $x \in X$, $-A$ is the infinitesimal generator of a linear semigroup e^{-tA} in a Banach space $(X, \|.\|)$ and $l(t,s)$ is a given kernel. The prime denotes time differentiation and x_0 is a given initial value. The functions $f : I \times X \to X$ and $F : I \times X \times X \to X$ are assumed to satisfy

(H1) There exist continuous functions $\varphi : I \to [0,\infty)$ and $q : I \to [0,\infty)$ such that

$$\|f(t,u)\| \le \varphi(t)\Theta(\|u\|), \ \ u \in X, \, t \in I$$

where $\Theta : [0,\infty) \to [0,\infty)$ is a continuous nondecreasing function such that

$$\Theta(\sigma(t))^2 \le q(t)\Theta(\sigma(t)^2), \ \forall \sigma, t \ge 0.$$

This last condition is similar to condition (q) in [6].

(H2) There exists a continuous function $\psi : I \to [0,\infty)$ such that

$$\|F(t,u,v)\| \le \psi(t) \left(\|u\| + \|v\|\right), \ \ u, v \in X, \, t \in I.$$

By mild solution of (1) we will mean a continuous function satisfying the associated integral equation

$$x(t) = e^{-tA}x(0) + \int_0^t e^{-(t-s)A} F\left(s, x(s), \int_0^s l(s-\tau) f(\tau, x(\tau)) d\tau\right) ds, \ t > 0.$$

In [9], Ntouyas and Tsamatos, using the Leray-Schauder alternative proved the existence of mild solutions for a similar problem with delays and nonlocal conditions provided that the semigroup is compact and the function F satisfies some Caratheodory-type condition instead of the classical requirements that F be locally Lipschitz, monotone or completely continuous.

In our case we shall consider the weakly singular kernel

$$l(t,s) = l(t-s) = (t-s)^{-\beta} e^{-\gamma(t-s)}, \ \ \beta \in (0,1), \, \gamma > 0. \tag{2}$$

This kernel multiplied by $1/\Gamma(1-\beta)$ represents a fractional derivative damping term modified to have exponential decay (see [1]). We intend to prove exponential decay of mild solutions for problem (1). To prove global existence we may mimic the argument in [9] which is based on a fixed point result (see [2]) and consists mainly in

(a) obtaining a priori bounds for mild solutions of the problem

$$\begin{cases} x'(t) + \lambda A x(t) = \lambda F\left(t, x(t), \int_0^t l(t-s) f(s, x(s)) ds\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in I, \, \lambda \in (0,1), \\ x(0) = x_0 \end{cases} \tag{3}$$

and

(b) showing that the operator $S : B = C(I, X) \to B$ defined by

$$(Sy)(t) = e^{-tA}x_0 + \int_0^t e^{-(t-s)A}F\left(s, y(s), \int_0^s l(s-\tau)f(\tau, y(\tau))d\tau\right) ds, \ \ t \in I$$

is a completely continuous operator.

Part (a), however, must be modified to apply to our kernel. Indeed, observe that the kernel defined in (2) is not bounded in a neighborhood of 0, while it is assumed in [9] that

$$\max\{|l(t,s)| : 0 \leq s \leq t \leq T\} < \infty.$$

Lemma 1 [6] *If* $0 \leq \alpha \leq 1$ *and* $\omega > 0$, *then*

$$\int_0^t (t-s)^{-\alpha} e^{\omega s} ds < \Gamma(1-\alpha)e^{\omega t}/\omega^{1-\alpha}, \ t > 0.$$

Theorem 2 *Let* f *and* F *be two functions satisfying* **(H1)** *and* **(H2)**. *Suppose further that*

$$\int_0^T B(s)ds < \int_c^\infty \frac{ds}{2s + \Theta(s)}$$

where $B(t) := \max\{\omega + [1 + M^2\psi^2(t)]/2, 2C^2\varphi^2(t)q(t)\}$, $c := 2M^2 \|x_0\|^2$ *and* $C := \Gamma^{1/2}(1-2\beta)(2\gamma)^{\frac{1}{2}-\beta}$. *Then the problem* (1) *admits at least one mild solution on* $[0, T]$.

For the reader's convenience we present below a sketch of the necessary modifications.

If e^{-tA} is a strongly continuous semigroup satisfying

$$\left\|e^{-tA}\right\| \leq Me^{\omega t}, \ \ t \geq 0, \ M \geq 1, \ \omega > 0$$

and $x(t)$ is a mild solution of (1) then

$$\begin{aligned} \|x(t)\| &\leq Me^{\omega t}\|x_0\| \\ &+ Me^{\omega t}\textstyle\int_0^t e^{-\omega s}\psi(s)\left\{\|x(s)\| + \int_0^s (s-\tau)^{-\beta}e^{-\gamma(s-\tau)}\varphi(\tau)\Theta(\|x(\tau)\|)d\tau\right\} ds, \end{aligned}$$

or

$$
\begin{aligned}
(4) \qquad e^{-\omega t}\|x(t)\| \le M\|x_0\| + M\int_0^s e^{-\omega s}\psi(s)\|x(s)\| \\
+\int_0^s (s-\tau)^{-\beta}e^{-\gamma(s-\tau)}\varphi(\tau)\Theta(\|x(\tau)\|)d\tau ds.
\end{aligned}
$$

We denote the right hand side of (4) by $u(t)$, then

$$
u'(t) = Me^{-\omega t}\psi(t)\left\{\|x(t)\| + \int_0^t (t-\tau)^{-\beta}e^{-\gamma(t-\tau)}\varphi(\tau)\Theta(\|x(\tau)\|)d\tau\right\}.
$$

By the definition of u and (4) it is clear that $\|x(t)\| \le e^{\omega t}u(t)$ for all $t \in I$ and therefore

$$
(5) \quad u'(t) \le Me^{-\omega t}\psi(t)\left\{e^{\omega t}u(t) + \int_0^t (t-\tau)^{-\beta}e^{-\gamma(t-\tau)}\varphi(\tau)\Theta(e^{\omega\tau}u(\tau))d\tau\right\}.
$$

Let us first suppose that $\beta < 1/2$. The case $\beta \ge 1/2$ can be treated along the argument to follow in the proof of part (ii) of our Theorem 5. Using Lemma 1 we may estimate the integral term in (5) as follows

$$
\int_0^t (t-\tau)^{-\beta}e^{-\gamma(t-\tau)}\varphi(\tau)\Theta(e^{\omega\tau}u(\tau))d\tau \le C\left(\int_0^t \varphi^2(\tau)q(\tau)\Theta(e^{2\omega\tau}u^2(\tau))d\tau\right)^{\frac{1}{2}}
$$

Hence

$$
(6) \quad e^{\omega t}u'(t) \le M\psi(t)\left\{e^{\omega t}u(t) + C\left(\int_0^t \varphi^2(\tau)q(\tau)\Theta(e^{2\omega\tau}u^2(\tau))d\tau\right)^{\frac{1}{2}}\right\}.
$$

Setting

$$
v(t) = e^{\omega t}u(t) + C\left(\int_0^t \varphi^2(\tau)q(\tau)\Theta(e^{2\omega\tau}u^2(\tau))d\tau\right)^{\frac{1}{2}},
$$

we see that

$$
(7) \qquad v^2(t) \le 2e^{2\omega t}u^2(t) + 2C^2\int_0^t \varphi^2(\tau)q(\tau)\Theta(e^{2\omega\tau}u^2(\tau))d\tau.
$$

Once again let $\chi(t)$ denote the right hand side of (7). Then clearly

$$
\begin{aligned}
\chi'(t) &= 4\omega e^{2\omega t}u^2(t) + 4e^{2\omega t}u(t)u'(t) + 2C^2\varphi^2(t)q(t)\Theta(e^{2\omega t}u^2(t)) \\
&\le [2\omega + 1 + M^2\psi^2(t)]\chi(t) + 2C^2\varphi^2(t)q(t)\Theta(\chi(t)).
\end{aligned}
$$

Notice that we have used (6) and (7). Finally, we may write

$$
\chi'(t) \le B(t)\{2\chi(t) + \Theta(\chi(t))\}, \quad t \in I.
$$

So

$$\int_{\chi(0)}^{\chi(t)} \frac{ds}{2s+\Theta(s)} \leq \int_0^T B(s)ds,$$

with $\chi(0) = 2u^2(0) = 2M^2 \|x_0\|^2 \equiv c$. Therefore, supposing that

$$\int_0^T B(s)ds < \int_c^{\infty} \frac{ds}{2s+\Theta(s)},$$

we see that $\chi(t)$ is bounded and consequently $x(t)$ is also bounded.

In the next section we state and prove our main results. The proofs are based on the semigroup approach and some ideas in [5], [6] and [7].

2. EXPONENTIAL DECAY

We consider the space $X = L^p(\Omega)$, $p \in (-1, \infty)$ with Ω a bounded domain of $\mathbf{R}^n$. $-A$ is a sectorial operator (see [3]) with $\mathbf{Re}\sigma(A) > b > 0$ where $\mathbf{Re}\sigma(A)$ denotes the real part of the spectrum of A. The fractional operator A^α, $0 \leq \alpha \leq 1$, is defined in the usual way on $D(A^\alpha) = X^\alpha$ and $\|x\|_\alpha = \|A^\alpha x\|$, $x \in X^\alpha$. It is known that $(X^\alpha, \|.\|_\alpha)$ is a Banach space. e^{-tA} denotes the semigroup generated by the operator A.

We will need the following Lemmas:

Lemma 3 [3] *If* $0 \leq \alpha \leq 1$, *then* $D(A^\alpha) \subset C^\nu(\bar{\Omega})$ *for* $0 \leq \nu < 2\alpha - n/p$.

Lemma 4 [3] *If* $0 \leq \alpha \leq 1$, *then*

$$\left\| A^\alpha e^{-tA} \right\|_p \leq C_1 t^{-\alpha} e^{-bt},\ t > 0$$

for some positive constant C_1.

We have the following first result

Theorem 5 *Let* $l(t,s)$ *be as in* (2) *and* $\Theta(r) = r^m$, $m \geq 1$. *suppose that* f *and* F *satisfy* **(H1)** *and* **(H2)** *respectively. Let* $y = (1-\alpha)/\alpha$, $0 < \alpha < 1$, $z = (1-\beta)/\beta$, $0 < \beta < 1$ *and* $\xi = \min\{y, z\}$. *If* φ, $\psi \in L^{q^*}(0,\infty)$ *with*

$$q^* = \begin{cases} (2\xi+1)/\xi, & \xi \leq 1, \\ 2, & \xi > 1, \end{cases}$$

then any uniformly bounded mild solution of (1) *decays exponentially as* $t \to +\infty$ *and we have*

$$\|A^{\alpha}x(t)\|_p \le Ce^{-(b-\varepsilon)t}\|x_0\|_p, \ \forall\, t \ge a > 0,$$

for some real number ε *such that* $0 < \varepsilon < b$.

Proof. Let $x(t)$ be a uniformly bounded mild solution of (1), that is a solution of

$$x(t) = e^{-tA}x(0) + \int_0^t e^{-(t-s)A}F\left(s, x(s), \int_0^s l(s-\tau)f(\tau, x(\tau))d\tau\right)ds, \tag{8}$$

where $t > 0$.

Applying the operator A^{α}, $0 < \alpha < 1$ on both sides of (8) and using Lemma 4 we obtain

$$\begin{aligned} \|A^{\alpha}x(t)\|_p &\le C_1 t^{-\alpha}e^{-bt}\|x_0\|_p \\ &+C_1\textstyle\int_0^t (t-s)^{-\alpha}e^{-b(t-s)}\left\|F\left(s, x(s), \int_0^s l(s-\tau)f(\tau, x(\tau)d\tau)\right\|_p ds. \end{aligned} \tag{9}$$

From **(H1)**, **(H2)** and (9) we infer that

$$\begin{aligned} e^{bt}\|A^{\alpha}x(t)\|_p &\le C_1 t^{-\alpha}\|x_0\|_p + C_1\textstyle\int_0^t (t-s)^{-\alpha}e^{bs}\psi(s)\|x(s)\|_p ds \\ &+C_1\textstyle\int_0^t (t-s)^{-\alpha}e^{bs}\psi(s)\left(\int_0^s (s-\tau)^{-\beta}e^{-\gamma(s-\tau)}\varphi(\tau)\|x(\tau)\|_p^m d\tau\right)ds. \end{aligned} \tag{10}$$

From the inclusion $D(A^{\alpha}) \subset L^p(\Omega)$ we have the inequality

$$\|x(t)\|_p \le C_2\|A^{\alpha}x(t)\|_p.$$

Let θ be a real number in $(0,1)$, then

$$\|x(t)\|_p = \|x(t)\|_p^{\theta}\|x(t)\|_p^{1-\theta} \le C_3\|A^{\alpha}x(t)\|_p^{\theta}\|x(t)\|_p^{1-\theta}. \tag{11}$$

Assume that $m > 1$ (if $m = 1$, it will be apparent that the estimate (11) is not necessary and we arrive at the same conclusion). Choosing θ in (11) such that $m\theta = 1$ i.e. $\theta = 1/m$ and taking into account the uniform boundedness assumption in $L^p(\Omega)$ we deduce from (10) that for any $a > 0$

$$\begin{aligned} e^{bt}\|A^{\alpha}x(t)\|_p &\le C_4\|x_0\|_p + C_5\textstyle\int_0^t (t-s)^{-\alpha}e^{bs}\psi(s)\|A^{\alpha}x(s)\|_p ds \\ &+C_6\textstyle\int_0^t (t-s)^{-\alpha}e^{(b-\gamma)s}\psi(s)\left(\int_0^s (s-\tau)^{-\beta}e^{\gamma\tau}\varphi(\tau)\|A^{\alpha}x(\tau)\|_p d\tau\right)ds \\ &= C_4\|x_0\|_p + C_5 I_1 + C_6 I_2, \ \forall t \ge a > 0. \end{aligned} \tag{12}$$

Let us denote the left hand side of (12) by $u(t)$. We shall treat I_1 and I_2 separately.

(i) If $\xi > 1$, then $0 < \alpha < 1/2$ and $0 < \beta < 1/2$. By Lemma 1, the Cauchy-Schwarz inequality, and since $\alpha < 1/2$

$$(13)\qquad \begin{aligned} I_1 &\le \int_0^t (t-s)^{-\alpha}\psi(s)u(s)ds \\ &\le \int_0^t (t-s)^{-\alpha}e^{\varepsilon s}\psi(s)e^{-\varepsilon s}u(s)ds \\ &\le C_7 e^{\varepsilon t}\left(\int_0^t \psi(s)^2 e^{-2\varepsilon s}u^2(s)ds\right)^{1/2}. \end{aligned}$$

Note that we have multiplied by $e^{\varepsilon s}.e^{-\varepsilon s}$. As in the theorem ε is a real number such that $0 < \varepsilon < b$. The Cauchy-Schwarz inequality allows us to write

$$\begin{aligned} I_3 &\equiv \int_0^s (s-\tau)^{-\beta}e^{\gamma\tau}\varphi(\tau)\left\|A^\alpha x(\tau)\right\|_p d\tau \le \int_0^s (s-\tau)^{-\beta}e^{(\gamma-b)\tau}\varphi(\tau)u(\tau)d\tau \\ &\le \int_0^s (s-\tau)^{-\beta}e^{(\gamma-b)\tau}\varphi(\tau)e^{\varepsilon\tau}e^{-\varepsilon\tau}u(\tau)d\tau \\ &\le \left(\int_0^s (s-\tau)^{-2\beta}e^{2(\gamma-b+\varepsilon)\tau}d\tau\right)^{\frac{1}{2}}\left(\int_0^s \varphi^2(\tau)e^{-2\varepsilon\tau}u^2(\tau)d\tau\right)^{\frac{1}{2}}. \end{aligned}$$

Assume that $\gamma \ge b$. In the case $\gamma < b$ we choose an ε such that $b-\gamma < \varepsilon < b$. By Lemma 1, as $\beta < 1/2$, it follows that

$$I_2 \le C_8 \int_0^t (t-s)^{-\alpha}e^{(b-\gamma)s}\psi(s)e^{(\gamma-b+\varepsilon)s}\left(\int_0^s \varphi^2(\tau)e^{-2\varepsilon\tau}u^2(\tau)d\tau\right)^{\frac{1}{2}} ds$$

or

$$I_2 \le C_8 \int_0^t (t-s)^{-\alpha}e^{\varepsilon s}\psi(s)\left(\int_0^s \varphi^2(\tau)e^{-2\varepsilon\tau}u^2(\tau)d\tau\right)^{\frac{1}{2}} ds.$$

Once again, the use of Lemma 1 with $\alpha < 1/2$ implies

$$(14)\qquad I_2 \le C_9 e^{\varepsilon t}\left(\int_0^t \psi^2(s)\int_0^s \varphi^2(\tau)e^{-2\varepsilon\tau}u^2(\tau)d\tau ds\right)^{\frac{1}{2}}.$$

The relations (12), (13) and (14) now yield for $t \ge a > 0$ after taking the square of both sides of (12)

$$(15)\qquad \begin{aligned} u^2(t) \le\ & C_{10}\Big\{\|x_0\|_p^2 + e^{2\varepsilon t}\int_0^t \psi(s)^2 e^{-2\varepsilon s}u^2(s)ds \\ & + e^{2\varepsilon t}\int_0^t \psi^2(s)\int_0^s \varphi^2(\tau)e^{-2\varepsilon\tau}u^2(\tau)d\tau ds\Big\}. \end{aligned}$$

Let us set $v(t) = e^{-2\varepsilon t}u^2(t)$, then (15) may be written as

$$(16)\quad v(t) \le C_{10}\left\{\|x_0\|_p^2 + \int_0^t \psi^2(s)v(s)ds + \int_0^t \psi^2(s)\int_0^s \varphi^2(\tau)v(\tau)d\tau ds\right\},$$

for all $t \geq a$. Denoting the right hand side of (16) by $v_1(t)$, we get

$$v_1'(t) = C_{10}\left\{\psi^2(t)v(t) + \psi^2(t)\int_0^t \varphi^2(\tau)v(\tau)d\tau\right\} \tag{17}$$

and letting $v_2(t) = \int_0^t \varphi^2(\tau)v(\tau)d\tau$, we find $v_2'(t) = \varphi^2(t)v(t) \leq \varphi^2(t)v_1(t)$.

Next, observing that v_1 is increasing and making use of [6, Lemma 1], we get

$$\frac{v_2(t)}{v_1(t)} \leq \int_0^t \frac{v_2'(s)}{v_1(s)}ds \leq \int_0^t \varphi^2(s)ds.$$

We deduce from (17) that

$$\frac{v_1'(t)}{v_1(t)} \leq C_{10}\psi^2(t)\left\{1 + \frac{v_2(t)}{v_1(t)}\right\} \leq C_{10}\psi^2(t)\left\{1 + \int_0^t \varphi^2(s)ds\right\}.$$

From which we infer

$$v_1(t) \leq v_1(0)\exp\left\{C_{10}\int_0^t \psi^2(s)\left[1 + \int_0^t \varphi^2(\tau)d\tau\right]ds\right\}.$$

Since φ, $\psi \in L^2(0,\infty)$ we conclude that

$$\|A^\alpha x(t)\|_p^2 \leq e^{-2bt}u^2(t) \leq C_{11}e^{-2(b-\varepsilon)t}\|x_0\|_p^2,\ t \geq a > 0.$$

(ii) If $\xi \leq 1$, then putting $q = (2\xi+1)/(\xi+1)$ it is easy to see that

$$\min\{1-\alpha q, 1-\beta q\} \geq \min\left\{\frac{1}{2(\xi+1)}, \frac{\xi^2}{(\xi+1)^2}\right\} > 0.$$

This relation will be needed to justify the application of Lemma 1 in the estimates of I_1 and I_2 below. Indeed, by Lemma 1 and the Hölder inequality we have

$$\begin{aligned} I_1 &\leq \left(\int_0^t (t-s)^{-\alpha q}e^{\varepsilon q s}ds\right)^{\frac{1}{q}}\left(\int_0^t \psi^{q^*}(s)e^{-\varepsilon q^* s}u^{q^*}(s)ds\right)^{\frac{1}{q^*}}, \\ &\leq K_1 e^{\varepsilon t}\left(\int_0^t \psi^{q^*}(s)e^{-\varepsilon q^* s}u^{q^*}(s)ds\right)^{\frac{1}{q^*}}, \end{aligned} \tag{18}$$

where $\frac{1}{q} + \frac{1}{q^*} = 1$; and as

$$\int_0^s (s-\tau)^{-\beta}e^{(\gamma-b)\tau}\varphi(\tau)u(\tau)d\tau \leq \\ \left(\int_0^s (s-\tau)^{-\beta q}e^{q(\gamma-b+\varepsilon)\tau}d\tau\right)^{\frac{1}{q}}\left(\int_0^s \varphi^{q^*}(\tau)e^{-\varepsilon q\tau}u^{q^*}(\tau)d\tau\right)^{\frac{1}{q^*}},$$

we see that

$$
\begin{aligned}
I_2 &\le K_2 \int_0^t (t-s)^{-\alpha} e^{\varepsilon s}\psi(s)\left(\int_0^t \varphi^{q^*}(\tau)e^{-\varepsilon q^*\tau}u^{q^*}(\tau)d\tau\right)^{\frac{1}{q^*}} \\
&\le K_2\left(\int_0^t (t-s)^{-\alpha q}e^{\varepsilon q s}ds\right)^{\frac{1}{q}}\left(\int_0^t \psi^{q^*}(s)\int_0^s \varphi^{q^*}(\tau)e^{-\varepsilon q^*\tau}u^{q^*}(\tau)d\tau ds\right)^{\frac{1}{q^*}} \\
&\le K_3 e^{\varepsilon t}\left(\int_0^t \psi^{q^*}(s)\int_0^t \varphi^{q^*}(\tau)e^{-\varepsilon q^*\tau}u^{q^*}(\tau)d\tau ds\right)^{\frac{1}{q^*}}.
\end{aligned}
\tag{19}
$$

The inequalities (12), (18) and (19) imply

$$
\begin{aligned}
e^{-\varepsilon q t}u^q(t) \le K_4 \Big\{ &\|x_0\|_p^q + \int_0^t \psi^q(s)e^{-\varepsilon q s}u^q(s)ds \\
&+ \int_0^t \psi^q(s)\int_0^s \varphi^q(\tau)e^{-\varepsilon q\tau}u^q(\tau)d\tau ds\Big\}.
\end{aligned}
$$

The rest of the proof is similar to that of part (i).

Remark. From the above proof it is clear that when $m = 1$ we do not need the uniform boundedness assumption. Below we shall meet another situation where we do not impose this condition.

In the next result we will consider the following class of nonlinearities

(H3) There exist a continuous function $\varphi : [0,\infty) \to [0,\infty)$ and $\mu \ge 0$ such that

$$\|f(t,u)\|_p \le t^\mu \varphi(t)\|u\|_p^m, \quad t \ge 0,\ u \in L^p,\ m > 1.$$

A similar class (with $\|A^\alpha u\|_p^m$ instead of $\|u\|_p^m$) has been considered in [7, p187] for the problem

$$
\begin{cases}
x'(t) + Ax(t) = f(t, x(t)), & t \in I = [0,T], \\
x(0) = x_0.
\end{cases}
$$

(H4) There exist continuous functions $\psi_1,\ \psi_2 : [0,\infty) \to [0,\infty)$ and σ_1, $\sigma_2 \ge 0$ such that

$$\|F(t,u,v)\|_p \le t^{\sigma_1}\psi_1(t)\|u\|_p^m + t^{\sigma_2}\psi_2(t)\|v\|_p,\ t \ge 0;\ u,\ v \in L^p,\ m > 1.$$

Lemma 6 ([4] or [5]). *If $\delta, \nu, \tau > 0$ and $z > 0$, then*

$$z^{1-\nu}\int_0^z (z-\zeta)^{\nu-1}\zeta^{\delta-1}e^{-\tau\zeta}d\zeta \leq K(\nu,\delta,\tau)$$

where $K(\nu,\delta,\tau) = \max\{1, 2^{1-\nu}\}\Gamma(\delta)(1+\delta/\nu)\tau^{-\delta}$.

Definition 7. We say that the initial data x_0 and the data ψ_1, g and h satisfy condition $L[C,g,h]$ whenever

$$L[C,g,h] < 1$$

where

$$\begin{aligned} L[C,g,h] &= 3^{m(q^*-1)}C_1^{q^*m}C_2^{q^*}(m-1)\left\|x_0\right\|_p^{q^*(m-1)} \\ &\quad \left\{C_3^{q^*/q}\int_0^\infty \psi_1^{q^*}(s)ds + C\int_0^\infty g^{q^*}(s)\int_0^s h^{q^*}(\tau)d\tau ds\right\}. \end{aligned}$$

Here C_1 and C_2 are the best constants in Lemma 4 and in the inequality $\left\|x(t)\right\|_p \leq M\left\|A^\alpha x(t)\right\|_p$, respectively. The constant C_3 is equal to $K(1-\alpha q, 1+q(\sigma_1 - m\alpha), bq(m-1))$; q and q^* are conjugate exponents determined as in Theorem 5.

For the reader's convenience we shall denote by C_i, $i = 4, ..., 8$ the following constants

$$C_4 = K(1-\beta q, 1+q(\mu - m\alpha), q(bm-\gamma)),$$

$$C_5 = K(1-\alpha q, 1+q(\sigma_2-\beta), q(\gamma - b)),$$

$$C_6 = K(1-\alpha q, 2+q(\sigma_2-\beta+\mu-m\alpha), qb(m-1)),$$

$$C_7 = K(1-\beta q, 1+q(\mu-m\alpha), q(a+bm-\gamma)),$$

$$C_8 = K(1-\alpha q, 1+q(\sigma_2-\varepsilon), qb(m-1)),$$

where K is the constant appearing in Lemma 6, B is the Euler Beta function and $\varepsilon > 0$ is to be determined later.

Now we are in position to state and prove our result.

Theorem 8 *Assume the same hypothesis as in Theorem 5 with* **(H3)** *and* **(H4)**.

(a) If $b < \gamma < bm$, *we suppose that* $1+q(\sigma_1 - m\alpha) > 0$, $1+q(\mu - m\alpha) > 0$, $1+q(\sigma_2 - \beta) > 0$ *and the data satisfy condition* $L\left[(C_4C_5)^{q^*/q}, \psi_2(t), \varphi(t)\right]$.

(b) If $\gamma = bm$, *we suppose that* $1+q(\sigma_1 - m\alpha) > 0$, $1+q(\mu - m\alpha) > 0$ *and the data satisfy condition* $L\left[C_6^{q^*/q} B^{q^*/q}(1-\beta q, 1+q(\mu - m\alpha)), \psi_2(t), \varphi(t)\right]$.

(c) If $\gamma > bm$, *we suppose either that:* $1+q(\sigma_1 - m\alpha) > 0$,

$$1+q(\mu - m\alpha) > 0, 1+q(\sigma_2 - \beta) > 0$$

and the data satisfy condition $L\left[(C_5C_7)^{q^*/q}, \psi_2(t), e^{at}\varphi(t)\right]$ *,or* $1+q(\sigma_1 - m\alpha) > 0$ *and the data satisfy condition*

$$L\left[\left(C_8 \frac{\Gamma(1-\beta q)}{p(\gamma - bm)^{1-\beta q}}\right)^{q^*/q}, t^{\varepsilon}\psi_2(t), t^{\mu - m\alpha}\varphi(t)\right]$$

for some $\varepsilon > 0$ *satisfying* $1+q(\sigma_2 - \varepsilon) > 0$.

Then we have

$$\|A^\alpha x(t)\|_p \leq Ct^{-\alpha}e^{-bt}\|x_0\|_p,\ t > 0$$

for some positive constant C.

Proof. Let $x(t)$ be a mild solution of (1). Then on account of **(H3)**, **(H4)** and Lemma 4 we have

$$\|A^\alpha x(t)\|_p \leq C_1 t^{-\alpha}e^{-bt}\|x_0\|_p + C_1 \int_0^t (t-s)^{-\alpha}e^{-b(t-s)}s^{\sigma_1}\psi_1(s)\|x(s)\|_p^m ds$$
$$+C_1 \int_0^t (t-s)^{-\alpha}e^{-b(t-s)}s^{\sigma_2}\psi_2(s)\left(\int_0^s (s-\tau)^{-\beta}e^{-\gamma(s-\tau)}\tau^{\mu}\varphi(\tau)\|x(\tau)\|_p^m d\tau\right) ds$$

or

$$\begin{aligned} t^\alpha e^{bt}\|A^\alpha x(t)\|_p &\leq C_1\|x_0\|_p + C_1C_2t^\alpha \int_0^t (t-s)^{-\alpha}e^{bs}s^{\sigma_1}\psi_1(s)\|A^\alpha x(s)\|_p^m ds \\ &+C_1C_2t^\alpha \int_0^t (t-s)^{-\alpha}e^{(b-\gamma)s}s^{\sigma_2}\psi_2(s)\times \\ &\left(\int_0^s (s-\tau)^{-\beta}e^{\gamma\tau}\tau^{\mu}\varphi(\tau)\|A^\alpha x(\tau)\|_p^m d\tau\right) ds, \end{aligned} \tag{20}$$

where the constants C_1 and C_2 are as in the Definition 7.

We denote the right hand side of (20) by $u(t)$, then clearly

$$\begin{aligned} u(t) &\leq C_1\|x_0\|_p + C_1C_2t^\alpha \int_0^t (t-s)^{-\alpha}e^{b(1-m)s}s^{\sigma_1 - m\alpha}\psi_1(s)u^m(s)ds + C_1C_2t^\alpha \\ &\int_0^t (t-s)^{-\alpha}e^{(b-\gamma)s}s^{\sigma_2}\psi_2(s)\left(\int_0^s (s-\tau)^{-\beta}e^{(\gamma - bm)\tau}\tau^{\mu - m\alpha}\varphi(\tau)u^m(\tau)d\tau\right) ds \\ &\leq C_1\|x_0\|_p + C_1C_2t^\alpha J_1(t) + C_1C_2t^\alpha \int_0^t (t-s)^{-\alpha}e^{(b-\gamma)s}s^{\sigma_2}\psi_2(s)J_2(s)ds, \end{aligned} \tag{21}$$

where

$$J_1(t)=\int_0^t (t-s)^{-\alpha}e^{b(1-m)s}s^{\sigma_1-m\alpha}\psi_1(s)u^m(s)ds$$

and

$$J_2(s)=\int_0^s (s-\tau)^{-\beta}e^{(\gamma-bm)\tau}\tau^{\mu-m\alpha}\varphi(\tau)u^m(\tau)d\tau.$$

The relation (21) will be our reference inequality. Here after, we estimate $J_1(t)$ and $J_2(s)$.

(a) Let $b<\gamma<bm$. By the Hölder inequality we find

$$J_1(t)\le\left(\int_0^t (t-s)^{-\alpha q}e^{bq(1-m)s}s^{q(\sigma_1-m\alpha)}ds\right)^{\frac{1}{q}}\left(\int_0^t \psi_1^{q^*}(s)u^{mq^*}(s)ds\right)^{\frac{1}{q^*}}$$

and

$$J_2(s)\le\left(\int_0^s (s-\tau)^{-\beta q}e^{q(\gamma-bm)\tau}\tau^{q(\mu-m\alpha)}d\tau\right)^{\frac{1}{q}}\left(\int_0^s \varphi^{q^*}(\tau)u^{mq^*}(\tau)d\tau\right)^{\frac{1}{q^*}}.$$

Note that as in Theorem 5 (proof of (ii)), $1-\alpha q>0$ and $1-\beta q>0$. This fact together with the conditions stated in part (a) of the theorem allow the use of Lemma 6. This leads to

$$\begin{gathered}u(t)\le C_1\|x_0\|_p+C_1C_2K_1^{1/q}\left(\int_0^t \psi_1^{q^*}(s)u^{mq^*}(s)ds\right)^{\frac{1}{q^*}}\\+C_1C_2t^\alpha\int_0^t(t-s)^{-\alpha}e^{(b-\gamma)s}s^{\sigma_2-\beta}\psi_2(s)K_2^{1/q}\left(\int_0^s \varphi^{q^*}(\tau)u^{mq^*}(\tau)d\tau\right)^{\frac{1}{qv}}ds.\end{gathered}$$

Once again, the Hölder inequality yields

$$\begin{gathered}u(t)\le C_1\left\|x_0\right\|_p+C_1C_2K_1^{1/q}\left(\int_0^t \psi_1^{q^*}(s)u^{mq^*}(s)ds\right)^{\frac{1}{q^*}}+C_1C_2t^\alpha K_2^{1/q}\\\left(\int_0^t(t-s)^{-\alpha q}e^{q(b-\gamma)s}s^{q(\sigma_2-\beta)}ds\right)^{\frac{1}{q}}\left(\int_0^t \psi_2^{q^*}(s)\int_0^s \varphi^{q^*}(\tau)u^{mq^*}(\tau)d\tau ds\right)^{\frac{1}{q^*}},\end{gathered}$$

and by Lemma 6

$$\begin{gathered}u(t)\le C_1\left\|x_0\right\|_p+C_1C_2K_1^{1/q}\left(\int_0^t \psi_1^{q^*}(s)u^{mq^*}(s)ds\right)^{\frac{1}{q^*}}\\+C_1C_2K_2^{1/q}K_3^{1/q}\left(\int_0^t \psi_2^{q^*}(s)\int_0^s \varphi^{q^*}(\tau)u^{mq^*}(\tau)d\tau ds\right)^{\frac{1}{q^*}},\end{gathered}\tag{22}$$

where

$$K_1=K(1-\alpha q,1+q(\sigma_1-m\alpha),bq(m-1)),$$
$$K_2=K(1-\beta q,1+q(\mu-m\alpha),q(bm-\gamma)),$$

and

$$K_3 = K(1-\alpha q, 1+q(\sigma_2-\beta), q(\gamma-b)).$$

As in [6, p354], using the inequality

$$(a_1+a_2+...+a_j)^k \le j^{k-1}(a_1^k+a_2^k+...+a_j^k),\ j\ge 1$$

we infer from (22) that

$$\begin{aligned} u^{q^*}(t) \le 3^{q^*-1}C_1^{q^*}\Big\{&\|x_0\|_p^{q^*} + C_2^{q^*}K_1^{q^*/q}\textstyle\int_0^t \psi_1^{q^*}(s)u^{mq^*}(s)ds \\ &+C_2^{q^*}K_2^{q^*/q}K_3^{q/p}\textstyle\int_0^t \psi_2^{q^*}(s)\int_0^s \varphi^{q^*}(\tau)u^{mq^*}(\tau)d\tau ds\Big\}. \end{aligned} \tag{23}$$

If $v(t)$ denotes the right hand side of (23), then

$$\begin{aligned} v'(t) \le 3^{q^*-1}C_1^{q^*}\Big\{&C_2^{q^*}K_1^{q^*/q}\psi_1^{q^*}(t)v^m(t) \\ &+C_2^{q^*}K_2^{q^*/q}K_3^{q^*/q}\psi_2^{q^*}(t)\textstyle\int_0^t \varphi^{q^*}(\tau)v^m(\tau)d\tau\Big\}. \end{aligned}$$

Let $w(t)=\int_0^t \varphi^{q^*}(\tau)v^m(\tau)d\tau$, then $w'(t)=\varphi^{q^*}(t)v^m(t)$ and

$$\frac{w(t)}{v^m(t)} \le \int_0^t \frac{w'(s)}{v^m(s)}ds = \int_0^t \varphi^{q^*}(s)ds.$$

Thus

$$\frac{v'(t)}{v^m(t)} \le 3^{q^*-1}C_1^{q^*}C_2^{q^*}\left\{K_1^{q^*/q}\psi_1^{q^*}(t) + K_2^{q^*/q}K_3^{q^*/q}\psi_2^{q^*}(t)\int_0^t \varphi^{q^*}(\tau)d\tau\right\}. \tag{24}$$

Integrating (24) over $[0,t]$, we get

$$\begin{aligned} &\textstyle\int_0^t \frac{v'(s)}{v^m(s)}ds \\ &\le 3^{q^*-1}C_1^{q^*}C_2^{q^*}\Big\{K_1^{q^*/q}\textstyle\int_0^t \psi_1^{q^*}(s)ds + K_2^{q^*/q}K_3^{q^*/q}\int_0^t \psi_2^{q^*}(s)\int_0^s \varphi^{q^*}(\tau)d\tau ds\Big\}. \end{aligned}$$

Therefore,

$$v(t) \le \left[v_0^{1-m} - (m-1)G(t)\right]^{-\frac{1}{m-1}},$$

where $G(t)$ is the right hand side of the latter inequality and

$$v_0 = 3^{q-1}C_1^{q^*}\|x_0\|_p^{q^*}.$$

In view of the condition $L\left[(C_4C_5)^{q^*/q}, \psi_2(t), \varphi(t)\right]$ and the definitions of $v(t)$ and $u(t)$ it is clear that

$$\left(t^\alpha e^{bt}\|A^\alpha x(t)\|_p\right)^{q^*} \le u^{q^*}(t) \le v(t) \le C\|x_0\|_p^{q^*},$$

or

$$\|A^{\alpha}x(t)\|_p \leq Ct^{-\alpha}e^{-bt}\|x_0\|_p .$$

(b) Let $\gamma = bm$, then the integral term

$$I \equiv \int_0^t (t-s)^{-\alpha} e^{(b-\gamma)s} s^{\sigma_2} \psi_2(s) J_2(s) ds$$

in (21) reduces to

$$\begin{aligned}&\textstyle\int_0^t (t-s)^{-\alpha} e^{b(1-m)s} s^{\sigma_2} \psi_2(s) \left(\int_0^s (s-\tau)^{-\beta} \tau^{\mu - m\alpha} \varphi(\tau) u^m(\tau) d\tau \right) ds \\ &\quad \equiv \textstyle\int_0^t (t-s)^{-\alpha} e^{b(1-m)s} s^{\sigma_2} \psi_2(s) J_3(s) ds.\end{aligned}$$

J_3 is estimated using the Euler Beta function in the following manner

$$\begin{aligned} J_3 &\leq \left(\int_0^s (s-\tau)^{-\beta q} \tau^{q(\mu - m\alpha)} d\tau \right)^{\frac{1}{q}} \left(\int_0^s \varphi^{q^*}(\tau) u^{mq^*}(\tau) d\tau \right)^{\frac{1}{q^*}} \\ &\leq B^{\frac{1}{q}}(1-\beta q, 1 + q(\mu - m\alpha)) s^{[1-\beta q + q(\mu - m\alpha)]/q} \left(\int_0^s \varphi^{q^*}(\tau) u^{mq^*}(\tau) d\tau \right)^{\frac{1}{q^*}} . \end{aligned}$$

Using Hölder's inequality, again, we obtain

$$\begin{aligned} I \leq & B^{\frac{1}{q}}(1-\beta q, 1 + q(\mu - m\alpha)) \\ & \left(\textstyle\int_0^t (t-s)^{-\alpha q} e^{qb(1-m)s} s^{q\sigma_2 + 1 - q\beta + q(\mu - m\alpha)} ds \right)^{\frac{1}{q}} \\ & \left(\textstyle\int_0^t \psi_2^{q^*}(s) \int_0^s \varphi^{q^*}(\tau) u^{mq^*}(\tau) d\tau ds \right)^{\frac{1}{q^*}} . \end{aligned}$$

Observe that $q\sigma_2 + 1 - q\beta + q(\mu - m\alpha) > -1$ since $(1-q\beta) + (1 + q(\mu - m\alpha)) > 0$ from the hypothesis. Hence

$$I \leq B^{\frac{1}{q}}(1-\beta q, 1 + q(\mu - m\alpha)) t^{-\alpha} K_4^{1/q} \left(\int_0^t \psi_2^{q^*}(s) \int_0^s \varphi^{q^*}(\tau) u^{mq^*}(\tau) d\tau ds \right)^{\frac{1}{q^*}},$$

where $K_4 = K(1-\beta q, 2 + q(\sigma_2 - \beta + \mu - m\alpha), qb(m-1))$.

The rest of the proof is the same as in part (a) and we obtain exponential decrease of the solution of (8) under the conditions in the theorem.

(c) Let $\gamma > bm$, then either

(i) we multiply by $e^{a\tau}.e^{-a\tau}$ inside the third integral in (21), or

(ii) we multiply by $s^{\varepsilon}.s^{-\varepsilon}$ inside the second integral in (21).

Let us first consider case (i). We have

$$\begin{aligned} I &\equiv \int_0^s (s-\tau)^{-\beta} e^{(\gamma-bm)\tau} \tau^{\mu-m\alpha} e^{a\tau}.e^{-a\tau}\varphi(\tau)u^m(\tau)d\tau \\ &\leq \left(\int_0^s (s-\tau)^{-q\beta} e^{q(\gamma-bm-a)\tau}\tau^{q(\mu-m\alpha)}d\tau\right)^{\frac{1}{q}} \left(\int_0^s e^{q^*a\tau}\varphi^{q^*}(\tau)u^{q^*m}(\tau)d\tau\right)^{\frac{1}{q^*}}. \end{aligned}$$

Since $a > \gamma - bm$, $1 - q\beta > 0$ and $1 + q(\mu - m\alpha)$ it follows from Lemma 6 that

$$I \leq K_5^{1/q} s^{-\beta} \left(\int_0^s e^{q^*a\tau}\varphi^{q^*}(\tau)u^{q^*m}(\tau)d\tau\right)^{\frac{1}{q^*}},$$

where $K_5 = K(1 - p\beta, 1 + p(\mu - m\alpha), p(a + bm - \gamma))$. Therefore

$$\begin{aligned} & u(t) \leq C_1 \|x_0\|_p + C_1C_2K_1^{1/q}\left(\int_0^t \psi_1^{q^*}(s)u^{mq^*}(s)ds\right)^{\frac{1}{q^*}} \\ & +C_1C_2K_5^{1/q}t^{\alpha}\int_0^t (t-s)^{-\alpha}e^{(b-\gamma)s}s^{\sigma_2}\psi_2(s)s^{-\beta}\left(\int_0^s e^{q^*a\tau}\varphi^{q^*}(\tau)u^{mq^*}(\tau)d\tau\right)^{\frac{1}{q^*}}ds, \\ & \quad \leq C_1 \|x_0\|_p + C_1C_2K_1^{1/q}\left(\int_0^t \psi_1^{q^*}(s)u^{mq^*}(s)ds\right)^{\frac{1}{q^*}} \\ & \quad +C_1C_2K_5^{1/q}t^{\alpha}\left(\int_0^t (t-s)^{-\alpha q}e^{q(b-\gamma)s}s^{q(\sigma_2-\beta)}ds\right)^{\frac{1}{q}} \\ & \quad \left(\int_0^t \psi_2^{q^*}(s)\int_0^s e^{q^*a\tau}\varphi^{q^*}(\tau)u^{mq^*}(\tau)d\tau ds\right)^{\frac{1}{q^*}}. \end{aligned}$$

As $1 + q(\sigma_2 - \beta) > 0$, we get

$$\begin{aligned} u(t) &\leq C_1 \|x_0\|_p + C_1C_2K_1^{1/q}\left(\int_0^t \psi_1^{q^*}(s)u^{mq^*}(s)ds\right)^{\frac{1}{q^*}} \\ &+C_1C_2K_5^{1/q}K_3^{1/q}\left(\int_0^t \psi_2^{q^*}(s)\int_0^s e^{q^*a\tau}\varphi^{q^*}(\tau)u^{mq^*}(\tau)d\tau ds\right)^{\frac{1}{q^*}}. \end{aligned}$$

Next we proceed as in part (a).

(ii) Multiplying by $s^{\varepsilon}.s^{-\varepsilon}$ inside the second integral of (21) we find

$$\begin{aligned} & U \equiv \\ & \int_0^t (t-s)^{-\alpha}e^{(b-\gamma)s}s^{\sigma_2-\varepsilon}s^{\varepsilon}\psi_2(s)\left(\int_0^s (s-\tau)^{-\beta}e^{(\gamma-bm)\tau}\tau^{\mu-m\alpha}\varphi(\tau)u^m(\tau)d\tau\right)ds \\ & \quad \leq \int_0^t (t-s)^{-\alpha}e^{(b-\gamma)s}s^{\sigma_2-\varepsilon}s^{\varepsilon}\psi_2(s)\left(\int_0^s (s-\tau)^{-q\beta}e^{q(\gamma-bm)\tau}d\tau\right)^{\frac{1}{q}} \\ & \quad \left(\int_0^s \tau^{q^*(\mu-m\alpha)}\varphi^{q^*}(\tau)u^{mq^*}(\tau)d\tau\right)^{\frac{1}{q^*}} \end{aligned}$$

or

$$U \leq \left[\frac{\Gamma(1-q\beta)}{[q(\gamma-bm)]^{1-q\beta}}\right]^{\frac{1}{q}}$$
$$\int_0^t (t-s)^{-\alpha} e^{b(1-m)s} s^{\sigma_2-\varepsilon} s^{\varepsilon} \psi_2(s) \left(\int_0^s \tau^{q^*(\mu-m\alpha)} \varphi^{q^*}(\tau) u^{mq^*}(\tau) d\tau\right)^{\frac{1}{q^*}} ds.$$

Then

$$U \leq \left[\frac{\Gamma(1-q\beta)}{[q(\gamma-bm)]^{1-q\beta}}\right]^{\frac{1}{q}} \left(\int_0^t (t-s)^{-q\alpha} e^{qb(1-m)s} s^{q(\sigma_2-\varepsilon)} ds\right)^{\frac{1}{q}}$$
$$\left(\int_0^t s^{\varepsilon q^*} \psi_2^{q^*}(s) \int_0^s \tau^{q^*(\mu-m\alpha)} \varphi^{q^*}(\tau) u^{mq^*}(\tau) d\tau ds\right)^{\frac{1}{q^*}},$$

therefore

$$U \leq K_6^{1/q} \left[\frac{\Gamma(1-q\beta)}{[q(\gamma-bm)]^{1-q\beta}}\right]^{\frac{1}{q}}$$
$$t^{-\alpha} \left(\int_0^t s^{\varepsilon q^*} \psi_2^{q^*}(s) \int_0^s \tau^{q^*(\mu-m\alpha)} \varphi^{q^*}(\tau) u^{mq^*}(\tau) d\tau ds\right)^{\frac{1}{q^*}}$$

where $K_6^{1/p} = K(1-\alpha q, 1+q(\sigma_2-\varepsilon), qb(m-1))$. The rest of the proof is similar to part (a)

Remark 2 (i) Note that the results in Theorem 8 hold without the uniform boundedness assumption.

(ii) In case (b) the theorem holds true without any condition on σ_2 other than the nonnegativity.

Remark 3 We did not address here the issue of the equivalence between (8) and (1). In fact, it is possible to impose some conditions on $F(t, u, v)$ (like Hölder continuity in t, Lipschitz continuity with respect to u and v) and use the results in Theorem 8 to see that mild solutions of (1) with **(H3)** and **(H4)** are actually strong solutions. Moreover, we can establish regularity properties for the solution. For these matters we refer to [3] or the paper of Hoshino and Yamada [4].

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