Arab J. Math. Sc.
Volume 9, Number 1, June 2003
pp. 29-105

Nonclassical properties for some intermediate states

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ABSTRACT. In this communication we consider in some detail new classes of states. These states are intermediate states either between the pure number (Fock) state, and the (non-pure) chaotic state (thermal state), such as geometric state, or between the coherent state and number state such as binomial state. We extend our discussion to include some other states such as even (odd) coherent states, even (odd) binomial states, phased generalized binomial state...etc. In our study of these states we pay attention to a discussion of the nonclassical properties, besides the statistical properties, for example correlation functions, squeezing, and quasiprobablity distribution functions (P-representation, W-Wigner, and Q-function). Furthermore we consider the field distribution and the photon number distribution, as well as the phase properties. Finally, we give the scheme for the production of these states in addition to some applications related to some quantum optical systems.

1. INTRODUCTION

Since the earlier days of quantum optics, it is well known that the Fock (number) state and the coherent state represent two of the most fundamental states of a single boson mode [1, 2]. The number state $|n\rangle$ is determined by its photon number and the phase is entirely random. In this state the amplitude of the field has a zero expectation value. For the coherent state $|\alpha\rangle$ one can generate it from the action of the Glauber displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a})$ on the vaccum state $|0\rangle$, such that

(1.1) $$|\alpha\rangle = \exp(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a})|0\rangle = \exp(-\frac{1}{2}|\alpha|^{2})\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}|0\rangle$$

with $\hat{a}$ ($\hat{a}^{\dagger}$) standing for the annihilation (creation) boson operator, with the properties $[\hat{a}, \hat{a}^{\dagger}] = 1$, and α complex. For this state the phase is determined and the amplitude of the field has a non-zero value. In fact the coherent state is a linear combination of all $|n\rangle$ states with coefficients chosen such that the photon counting distribution is Poissonian. To generate such a state one can use the fact that a classical charge distribution radiates a field in a coherent state, while a single atom in its first excited state in the absence of external interactions radiates a field in the $|n = 1\rangle$ state. It is worthwhile to refer to another state that is the choatic (thermal) state [3], whose density operaror ρ_{th} is given by

(1.2) $$\rho_{th} = \sum_{n=0}^{\infty}\frac{\overline{n}^{n}}{(1+\overline{n})^{n+1}}|n\rangle\langle n|,$$

where $\frac{(\overline{n})^{n}}{(1+\overline{n})^{n+1}}$ is the Bose-Einstein distribution function.

Recently one finds a great deal of interest in producing and generating new states in addition to the previous states. Most of these states are intermediate states which can be generated from the above states. In fact they interpolate between distinctive states, reducing them in different limits of the parameters involved. In consequence, a unifying role is played by an intermediate state describing the physical properties of its limiting states. The earliest example in the literature is the binomial state [4], which interpolates between the coherent state and the number (Fock) state. Another is the negative binomial state [5], which bridges the coherent state and the quasi-thermal state (i.e. Susskind-Glogower phase state [6]). The notions of the even binomial states [7] (between the even-coherent and the even-number state) and the q-deformed binomial states [8] (between the q-coherent and the q-number state) have been introduced. The negative binomial states were also generalized to the even (odd) negative binomial states [9], which interpolate between the even (odd) coherent state and the even (odd) quasi-thermal state. The logarithmic state, as a special case of the negative binomial states with the $n = 0$ term removed, was also investigated [10]. There are also many other intermediate states, among which one can cite:

(i) the generalized geometric state [11], between the number state and the (nonpure) chaotic (thermal) states.

(ii) the intermediate number phase state [12], between the number state and the Pegg-Barnett phase state [13].

(iii) the intermadiate number squeezed state [14], between the number state and the squeezed coherent state; (iv) the even and odd intermediate number squeezed state [15], between the even (odd) number state and even (odd) squeezed state. Most of the theoetical studies concerning these states have focused on their construction and the possible occurrence of various nonclassical effects exhibited by them. All the above mentioned states are grouped under the category of nonclassical states of light, and the main purpose of the present work is to review some nonclassical properties of some of these states. However, to reach this goal we shall make our starting point the even and odd coherent states.

2. EVEN AND ODD COHERENT STATES

2.1. **Generation of the state:** During their study of a singular nonstationary one-dimensional oscillator Dodonov et al generated and introduced to the physics world the even and odd coherent states [16]. They have shown that these states separately form complete sets in the Hilbert spaces of even and odd functions. These functions can be written as follows

$$|\alpha>_{\pm}=\frac{1}{2}\lambda_{\pm}(|\alpha>\pm|-\alpha>), \tag{2.1}$$

or, in terms of number states,

$$|\alpha>_{+}=\lambda_{+}\exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{\sqrt{(2n)!}}|2n>, \tag{2.2}$$

and

$$|\alpha>_{-}=\lambda_{-}\exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}}|2n+1>, \tag{2.3}$$

where λ_+ and λ_- are the normalization constants for both even and odd coherent states respectively, given by the formulae

$$(2.4) \qquad \lambda_+ = \exp(\frac{1}{2}|\alpha|^2)[\cosh|\alpha|^2]^{-\frac{1}{2}}, \qquad \lambda_- = \exp(\frac{1}{2}|\alpha|^2)[\sinh|\alpha|^2]^{-\frac{1}{2}}$$

Several methods to construct these states are given in the literature, one of them is to use the inversion operator $\hat{I}$, which has the properties $\hat{I}\hat{a}\hat{I} = -\hat{a}$; and $\hat{I}\hat{a}^\dagger\hat{I} = -\hat{a}^\dagger$ where $\hat{a}$ and $\hat{a}^\dagger$ are the boson annhilition and creation operators, respectively. However, the displacement operator $\hat{D}(\alpha)$ is not invariant with respect to the coordinate such that $\hat{I}\hat{D}(\alpha)\hat{I} = \hat{D}(-\alpha)$, therefore one can construct two operators, each generating irreducible representations of the group consisting of two elements, the unit operator and the inversion operator $\hat{I}$.

The first operator generates the symmetric representation of the group and takes the form

$$(2.5) \qquad \cosh(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) = \frac{1}{2}[\hat{D}(\alpha) + \hat{D}(-\alpha)] = \hat{D}_+(\alpha) \ ,$$

while the second operator generates the anti-symmetric representation of the group, thus

$$(2.6) \qquad \sinh(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) = \frac{1}{2}[\hat{D}(\alpha) - \hat{D}(-\alpha)] = \hat{D}_-(\alpha),$$

and hence $|\alpha >_\pm = \lambda_\pm \hat{D}_\pm(\alpha)|0>$.

It is easy to see that the parity of these functions with respect to α is the same as with respect to the coordinate, therefore these functions completely describe the even and odd coherent states. Here it would be interesting to refer to the solution of the problem of the nonstationary oscillator with the wall at the origin of the coordinates, which has been obtained in terms of odd coherent states, for more details see [16].

Recently it has been shown that quantum interference between coherent states leads to generation of states whose properties are as far as one can imagine from classical states. For example one can see the superposition of two coherent states ($|\alpha\rangle_+$) can arise as a consequence of propagation of coherent light through an amplitude-dispersive medium. Furthermore it has been shown that the even coherent states exhibit ordinary (second order) squeezing as well as fourth order squeezing. Also we find that the idea of superposition of coherent states has been extended to include the quadrature variances of a continuous one-dimensional superposition of coherent states, which shows a significant reduction of fluctuations in one of the quadratures. The nonclassical properties of even and odd coherent states have been studied in details , see for example [17, 18, 19, 20].

2.2. **Photon number.** The photon number distribution for both even and odd states can be obtained from equation (2.1). For the even case we have[17]

$$P^{(e)}(n) = \begin{cases} \frac{1}{\cosh|\alpha|^2}\frac{|\alpha|^{2n}}{n!} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{2.7}$$

while the probability $P^{(o)}(n)$ of finding n photons for the odd state is

$$P^{(o)}(n) = \begin{cases} \frac{1}{\sinh|\alpha|^2}\frac{|\alpha|^{2n}}{n!} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \tag{2.8}$$

It has been shown that the photon number probability distributions are oscillatory and also resemble those associated with the Poisson distribution of the ordinary coherent state, (figure 1 of [17] may be referred to).

2.3. **Quasidistribution function.** The representation of quantum fields in phase space in terms of quasiprobabilities is widely used in the field of quantum optics, with particular emphasis on the W-Wigner [21, 22], Q-function [23], and the Glauber Sudershan P-representation [2, 24]. Therefore in this subsection we shall be concerned with the W-Wigner and Q-function for both even and odd coherent states. To find the W-Wigner and Q-functions we have to calculate the characteristic function $C(\xi, s)$, which is associated with the symmetrical order of the bosonic (photon) operators and is given by

$$C(\xi, s) = Tr[\hat{\rho}\exp(\xi\hat{a}^{\dagger} - \xi^{*}\hat{a} + \frac{s}{2}|\xi|^2), \tag{2.9}$$

where $\hat{\rho}$ is the density matrix for the desired state and s is a parameter that defines the relevant quasiprobability distribution function which takes the values $-1, 0, 1$ corresponding to Husimi Q-function, W-Wigner, and P-representation, respectively. The quasi-distribution functions $I(\beta, s)$ are defined through the Fourier transform of the characteristic function in the form

$$I(\beta, s) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} d^2\xi \exp(\xi\beta^* - \xi^*\beta)C(\xi, s), \tag{2.10}$$

where $P(\beta) = I(\beta, 1), W(\beta) = I(\beta, 0)$ and $Q(\beta) = I(\beta, -1)$.

For any state $|\psi_g\rangle$ expanded in terms of the Fock states $|n\rangle$ as follows

$$|\psi_g\rangle = \sum_n x_n|n\rangle, \qquad \text{with} \qquad \hat{\rho} = |\psi_g\rangle\langle\psi_g| \tag{2.11}$$

the generalized quasidistribution function $I(\beta, s)$ of equation (2.10) can be cast in the equivalent form [24]

$$I(\beta, s) = \frac{2}{\pi}\frac{e^{-(\frac{2}{1-s})|\beta|^2}}{1-s}\times \sum_{m,n} c_m^* c_n \frac{\beta^m\beta^{*n}}{|\beta|^{2n}}\sqrt{\frac{n!}{m!}}\left(\frac{s+1}{s-1}\right)^n\left(\frac{2}{1-s}\right)^{m-n} L_n^{m-n}\left(\frac{4|\beta|^2}{1-s}\right). \tag{2.12}$$

Hence the Wigner function becomes

$$W(\beta) = I(\beta, 0) = \frac{2}{\pi}e^{-2|\beta|^2}\sum_{m,n}(-1)^n c_m^* c_n (2\beta)^{m-n}\sqrt{\frac{n!}{m!}}L_n^{m-n}\left(4|\beta|^2\right) \tag{2.13}$$

while the Husimi Q-function takes the form

$$Q(\beta) = I(\beta, -1) = \frac{1}{\pi}\exp(-|\beta|^2)\sum_{m,n} c_m^* c_n \frac{\beta^m\beta^{*n}}{\sqrt{m!n!}} \tag{2.14}$$

where $L_n^\alpha(x) = \sum_{m=0}^n \frac{(-x)^m}{m!}\begin{pmatrix} n+\alpha \\ n-m \end{pmatrix}$ is the associated Laguerre polynomial. For P-representation one has to take the proper limit as $s \to 1$.

For the even coherent states case we can write the expression for W-Wigner as follows

$$W^{(e)}(\beta) = \frac{\exp(|\alpha|^2 - 2|\beta|^2)}{2\pi\cosh|\alpha|^2}\times [e^{-2|\alpha|^2}\cos 2(\alpha\beta^* + \alpha^*\beta) + e^{2|\alpha|^2}\cos 2(\alpha\beta^* - \alpha^*\beta)]. \tag{2.15}$$

For the odd coherent states case we have

$$(2.16)\quad W^{(o)}(\beta) = \frac{\exp(|\alpha|^2 - 2|\beta|^2)}{2\pi \sinh |\alpha|^2} \times \\ [e^{-2|\alpha|^2} \cos 2(\alpha\beta^* + \alpha^*\beta) - e^{2|\alpha|^2} \cos 2(\alpha\beta^* - \alpha^*\beta)].$$

Figures for these functions show negative values which is a sign of nonclassical effects. Similarly we can write the Q-function in the case of the even coherent states in the form

$$(2.17)\qquad Q^{(e)}(\beta) = \frac{\exp(-|\beta|^2)}{4\pi \cosh |\alpha|^2}[\cos(\alpha\beta^* + \alpha^*\beta) + e^{2|\alpha|^2} \cos(\alpha\beta^* - \alpha^*\beta)],$$

while for the odd case we get

$$(2.18)\qquad Q^{(o)}(\beta) = \frac{\exp(-|\beta|^2)}{4\pi \sinh |\alpha|^2}[\cos(\alpha\beta^* + \alpha^*\beta) - e^{2|\alpha|^2} \cos(\alpha\beta^* - \alpha^*\beta)],$$

After this quick review of the even and odd-cherent states, we look at the intermediate states; we begin with the binomial states.

3. THE BINOMIAL STATES

As a natural way to study the intermediate states, especially the states that interpolate between coherent states and number states, one may tackle the binomial state [4, 25]. The binomial state $|\eta, M>$ is a linear combination of the nnumber states $|0>, |1>, \ldots\ldots..|M>$ with coefficients chosen such that the photon counting probability distribution is binomial with mean ηM.

The binomial state is defined as

$$(3.1)\qquad |\psi_{\eta,M}\rangle = |\eta, M\rangle = \sum_{n=0}^{M} B_n^M |n\rangle,$$

where B_n^M are the binomial coefficients given by

$$(3.2)\qquad B_n^M = \left[\binom{M}{n} \eta^n (1 - |\eta|^2)^{M-n}\right]^{\frac{1}{2}}$$

The binomial state is quantum mechanical in nature and it produces light that is antibunched and has sub-Poissonian behaviour, but cannot have the minimum uncertainty product, where it contains a finite number of $|n\rangle$. To generate the binomial state different methods may be used [7]. One of these methods is to use the Hamiltonian

$$H = \mu J_+ + \mu^* J_-, \tag{3.3}$$

where $J_+(J_-)$ are the raising (lowering) operators related to the angular momentum operator, which satisfy the $SU(2)$ commutation relation $[J_+, J_-] = 2J_z$ and $[J_z, J_\pm] = \pm J_\pm$.

The evolution operator of the Hamiltonian (3.3) can be written as

$$U(t) = \exp(\zeta J_+ - \zeta^* J_-) \tag{3.4}$$

where $\zeta = (-i\mu t)$. Now if we use equation (3.4) to act on the state $|l, -l>$ where l is the co-operation number, we find

$$|\psi\rangle = U(t)|l,-l\rangle = \exp(\zeta J_+ - \zeta^* J_-)|l,-l\rangle \tag{3.5}$$

By taking $\zeta = \theta \exp(-\frac{i}{2}\varphi)$ and define $\tau = e^{-i\varphi}\tan\frac{\theta}{2}$, equation (3.5) becomes

$$|\psi\rangle = (1+|\tau|^2)^{-l} \sum_{m=-l}^{l} \begin{pmatrix} 2l \\ l+m \end{pmatrix}^{\frac{1}{2}} \tau^{l+m}|l,m\rangle \tag{3.6}$$

By setting $\tau = \eta/(1-|\eta|^2), l+m = n$ and $2l = M$ equation (3.6) takes the form of equation (3.1). Thus, as one can see, there is an advantage to consider the binomial state as a good example of the intermediate state. However we shall concentrate on the so-called even and odd binomial states [7, 26, 27]. These states interpolate between even (odd) coherent states and even (odd) number states. This will be seen in the following subsections.

3.1. **The even binomial state.** The even binomial state $|\psi_e\rangle$ is defined as follows

$$\begin{aligned} |\psi_e\rangle &= \frac{\lambda_1}{2}(|M,\eta\rangle + |M,-\eta\rangle) \\ &= \lambda_1 \sum_{n=0}^{[\frac{M}{2}]} \begin{pmatrix} M \\ 2n \end{pmatrix}^{\frac{1}{2}} \eta^{2n}(1-|\eta|^2)^{\frac{M-2n}{2}}|2n\rangle = \lambda_1 \sum_{n=0}^{[\frac{M}{2}]} B_{2n}^{M}|2n\rangle \end{aligned} \tag{3.7}$$

where $[\frac{M}{2}]$ is the largest integer less than or equal to $(M/2)$, and λ_1 is the normalization constant given by

$$|\lambda_1|^2 = 2[1+(1-2|\eta|^2)^M]^{-1} \tag{3.8}$$

As a limiting case when $\eta \to 0$ and $M \to \infty$ such that $M|\eta|^2 \to |\alpha|^2$, equation (3.7) immediately reduces to equation (2.2), corresponding to the even coherent state $|\alpha_+\rangle$.

i) Correlation function

To discuss anti-bunching we have to consider the Glauber second order (zero time) correlation function

$$g^{(2)}(0) = \frac{< \hat{a}^{\dagger 2}\hat{a}^2 >}{< \hat{a}^{\dagger}\hat{a} >^2} \tag{3.9}$$

From Equation (3.7) we calculate the expectation value of the photon number $\overline{n}$, as well as the second moment $\overline{n^2}$, from which we can rewrite equation (3.9) as follows

$$g^{(2)}(0) = (1 - \frac{1}{M})\frac{[1 + (1 - 2|\eta|^2)^M][1 + (1 - 2|\eta|^2)^{M-2}]}{[1 - (1 - 2|\eta|^2)^{M-1}]^2} \tag{3.10}$$

Equation (3.10) goes to $\coth^2 |\alpha|^2$ if $\eta \to 0$ and $M \to \infty$,which represents the value of the correlation function for the even coherent state case. Numerical investigation of equation (28) shows that the system has super-Poissonian behavior for small value of the parameter η ,but for large value of M the interval of the super-Poissonian diminshes,while the system shows sub-Poissonian for large value of η. This means that increasing the number of photons in the even binomial state changes the distribution from super-Poissonian to sub-Poissonian (see figure 1 (a,b)).

ii) Squeezing

The squeezing phenomenon represents one of the most interesting phenomena in the field of quantum optics, and is a direct quantum effect of Heisenberg's uncertainty principle. It reflects the reduced quantum fluctuations in one of the field quadratures at the expense of the other corresponding stretched quadratures. Our aim in the present subsection is to consider two cases of squeezing. The first is the normal squeezing which can be discussed through the fluctuations in the quadratures

$$\hat{X} = \frac{1}{2}(\hat{a}^\dagger + \hat{a}); \qquad \hat{Y} = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger), \tag{3.11}$$

and $\hat{a}^\dagger$satisfies the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, while $[\hat{X}, \hat{Y}] = \frac{i}{2}$.

The second case is to consider the amplitude squared squeezing, which arises naturally in the second -harmonic generation and in a number of nonlinear optical processes that is defined through

$$(3.12) \qquad d_1 = \frac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2}); \qquad d_2 = \frac{1}{2i}(\hat{a}^2 - \hat{a}^{\dagger 2})$$

which satisfy the commutation relation $[d_1, d_2] = i(1 + 2n)$.

To discuss squeezing we have first to calculate the expectation value of the operator $\hat{a}^{\dagger 2s}$ with respect to the state given by equation (3.8). For $2s < M$, we find

$$(3.13) \quad \langle \hat{a}^{\dagger 2s} \rangle = |\lambda_1|^2 \frac{\eta^{*2s}}{(1 - |\eta|^2)^s} \times \sum_{n=0}^{[\frac{M}{2}]-s} \binom{M}{2n} \left[\frac{(M-2n)!}{(M-2n-2s)!} \right]^{\frac{1}{2}} |\eta|^{4n} (1 - |\eta|^2)^{M-2n},$$

where s is a positive integer .

The field is said to be squeezed when $\overline{\Delta X}^2$ or $\overline{\Delta Y}^2 \leq \frac{1}{4}$ and $\overline{\Delta X}^2 . \overline{\Delta Y}^2 \geq \frac{1}{16}$, hold. From the numerical study one can see squeezing occur for all values of M , but it becomes more effective as M increases, however the maximum squeezing is shfited to lower η as M increases. Also it is noted that as we change the phase of the parameter η the squeezing changes between $\overline{\Delta \hat{X}}^2$ and $\overline{\Delta \hat{Y}}^2$. For the second case (amplitude squared squeezing) the field is said to be an amplitude-squared squeezed state if $\overline{\Delta d_1}^2$ or $\overline{\Delta d_2}^2 < \overline{n} + \frac{1}{2}$ and $\overline{\Delta d_1}^2$ $\overline{\Delta d_2}^2 \geq (\overline{n} + \frac{1}{2})^2$ hold. The numerical study of this phenomenon proves that amplitude squared squeezing is effective for large value of M , however the point of maximum squeezing in this case moves slightly as M increases (see figures 2 and 3 of [7]).

iii) Quasi-distribution function

As in the previous section we shall consider some statistical aspects related to the even binomial state. We are only concerned with the quasiprobability distribution function, W-Wigner and Q-functions. This is because the P representation is highly singular due to the non-classical charater of the even binomial state. From equation (2.9) with $\hat{\rho} = |\psi_e\rangle\langle\psi_e|$, together with equation

(3.7), one can calculate the characteristic function $C^{(e)}(\xi, s)$ to be

$$(3.14)\quad C^{(e)}(\xi, s) = |\lambda_1|^2(1-|\eta|^2)^M \times \sum_{n=0}^{[\frac{M}{2}]} \binom{M}{2n} \left[\frac{|\eta|^2}{1-|\eta|^2}\right]^{2n} \exp[-\frac{1}{2}(s+1)|\xi|^2] L_{2n}(|\xi|^2),$$

where $L_n(x)$ are Laguerre polynomials of order n given by

$$(3.15)\qquad L_n(x) = \sum_{r=0}^{n} (-)^n \frac{n!}{(r!)^2(n-r)!} x^r.$$

Now if we insert equation (2.13) into equation (2.14) and perform the integration we can obtain for $s = 0$ the W-Wigner functions, thus

$$(3.16)\quad W^{(e)}(\beta) = \frac{2}{\pi}|\lambda_1|^2(1-|\eta|^2)^M \times \sum_{n=0}^{[\frac{M}{2}]} \binom{M}{2n} \left[\frac{|\eta|^2}{1-|\eta|^2}\right]^{2n} \exp[-2|\beta|^2] L_{2n}(4|\beta|^2).$$

For $s = -1$ we obtain the Q-function in the form

$$(3.17)\qquad Q^{(e)}(\beta) = \frac{|\lambda_1|^2}{\pi}(1-|\eta|^2)^M \sum_{n=0}^{[\frac{M}{2}]} \binom{M}{2n} \left[\frac{|\beta\eta|^2}{1-|\eta|^2}\right]^{2n} \exp[-|\beta|^2],$$

where we have only taken the diagonal terms of the density matrix $\hat{\rho}$. In figures (5) and (6) we plot the Wigner function $W(\beta)$ of equation (3.16) when $|\eta|$ is taken very small (~ 0.2). The distribution is almost Gaussian and the shape of the field is insensitive to change in M . Increasing the value of M makes the peak slightly sharper. However, if we increase $|\eta|$ to 0.6, which means effects due to higher excitations are of considerable importance, we note a remarkable change in the shape of the function as M increases. Sharpening of the peak as well as the appearance of shallower wobbles are signs of the contribution due to higher excitations. In contrast to this we find that the $Q-$ function of equation (3.17) is insensitive to any change in either $|\eta|$ or M. For more details see figures 2-4.

3.2. **The odd binomial state.** The odd binomial state is the state which interpolates between the odd coherent state and odd number state and is defined by [27]

$$(3.18)\qquad |\psi_o\rangle = \frac{\lambda_2}{2}(|M,\eta\rangle - |M,-\eta\rangle) = \lambda_2 \sum_{n=0}^{[\frac{M-1}{2}]} B_{2n+1}^M |2n+1\rangle$$

where λ_2 is the normalization constant given by

$$(3.19)\qquad |\lambda_2|^2 = 2[1-(1-2|\eta|^2)^M]^{-1}$$

while B_{2n+1}^M is of the form

$$(3.20)\qquad B_{2n+1}^M = \left(\begin{array}{c} M \\ 2n+1 \end{array}\right)^{\frac{1}{2}} \eta^{2n+1}(1-|\eta|^2)^{\frac{M-2n-1}{2}}$$

Since the linear combination of the odd binomial state does not contain the vacuum state, the range of the parameter η will be between 0 and 1 such that $0 < |\eta| \leq 1$. By taking the limit $M \to \infty$ and $\eta \to 0$ equation (3.18) tends to equation (2.3), corresponding to the odd coherent state.

i)Sub-Poissonian behavior

To discuss the sub-Poissonian behavior of the state one needs to find the explicit expression of the Glauber second order correlation function (3.9). For the odd binomial state we find

$$(3.21)\qquad g^{(2)}(0) = (1-\frac{1}{M})\left[1 - 4\frac{(1-2|\eta|^2)^{(M-2)}(1-|\eta|^2)^2}{[1+(1-2|\eta|^2)^{M-1}]^2}\right]$$

From the above function we can deduce that $g^{(2)}(0)$ is always less than one so far as both η and M are finite. However, if we increase the value of M and decrease the value of η at the same time, such that $\eta \to 0$ as $M \to \infty$, then we find the correlation function tends to $\tanh^2|\alpha|^2$, so that a sub-Poissonian effect does exist for the odd coherent state. This emphasises that the odd binomial state has sub-Poisonnian behavior. Also we may point that, for large values of M with fixed value of η, the function approaches unity more rapidly for $\eta \approx 1$ and persists and the system shows coherence behavior [27].

ii) Squeezing

Since the normal squeezing is based on the definition of the field quadrature operators given by equation (3.11), therefore in this case we find

$$(3.22)\quad \langle \hat{a}^{2s}\rangle = |\lambda_2|^2 \frac{\eta^{2s}}{(1-|\eta|^2)^s} \times$$

$$\sum_{n=0}^{[\frac{M-1}{2}]-s} \begin{pmatrix} M \\ 2n+1 \end{pmatrix} \left[\frac{(M-2n-1)!}{(M-2n-2s-1)!}\right]^{\frac{1}{2}} |\eta|^{4n+2}(1-|\eta|^2)^{M-2n-1},$$

while the expectation value of the photon number is

$$(3.23)\qquad \langle \hat{a}^{\dagger}\hat{a}\rangle = |\eta\lambda_2|^2 (M/2)[1+(1-2|\eta|^2)^{M-1}].$$

Numerical investigation proves that the odd binomial state does not show squeezing whatever the values of the parameters η and M . However for amplitude -squared squeezing the situation is different where the squeezing get more pronounced as M increases, and the maximum point moves toward higher values of η (see [27]).

iii)Quasiprobability distribution

Following the same procedure as in the previous subsection we can calculate the quasidistribution functions for the odd binomial state. Then, from equations (2.13,14) and (3.18), and after performing the integral over the whole complex plane, we have the expressions of the W-Wigner function when $s=0$ in the form

$$(3.24)\quad W^{(o)}(\beta) = -2\frac{|\lambda_2|^2}{\pi} e^{-2|\beta|^2} \sum_{n=0}^{[(M-1)/2]} |B_{2n+1}^M|^2 L_{(2n+1)}(4|\beta|^2)$$

$$+\,2 \sum_{\substack{m,n=0\\ m>n}}^{[(M-1)/2]} \sqrt{\frac{(2m+1)!}{(2n+1)!}} |B_{2n+1}^M||B_{2n+1}^M| (2|\beta|^2)^{2(n-m)} L_{2n+1}^{2(n-m)}(4|\beta|^2)\times$$

$$\cos\{2(n-m)(\varphi+\zeta)\}$$

where B_{2n+1}^M is the binomial coefficient given by equation (3.20).

Similarly we can find the expression for the Q-function when $s = -1$ in the form

$$(3.25)\quad Q^{(o)}(\beta) = \frac{|\lambda_2|^2}{\pi} e^{-2|\beta|^2} \sum_{n=0}^{[(M-1)/2]} |B_{2n+1}^M|^2 \frac{|\beta|^{4n+2}}{(2n+1)!} + 2 \sum_{\substack{m,n=0 \\ m>n}}^{[(M-1)/2]} \frac{|\beta|^{2n+2m+2}}{\sqrt{(2n+1)!(2m+1)!}} |B_{2n+1}^M||B_{2n+1}^M| \times \cos\{2(n-m)(\varphi+\zeta)\},$$

where we have taken $\beta = |\beta|e^{i\zeta}$ and $\eta = |\eta|e^{i\phi}$.

The last two equations consist of two parts; the first part represents the diagonal terms, while the second part represents the off-diagonal terms of the density matrix $\hat{\rho}$. On the other hand to, deduce the expectation value of the density matrix $\hat{\rho}$ with respect to the coherent state β, one may use the relation $\langle\beta|\hat{\rho}|\beta\rangle = \pi Q(\beta)$. From the numercial study of the odd binomial state we find that when η has a small value ($\eta = 0.1$) and $M = 5$, the Wigner function has a hole on the summit similar to that found of the geometric state. However as we increase the value of η such that $\eta = 0.6$, keeping the value of the parameter M fixed, we have four asymmetric peaks with a chaotic behavior, where the interference between the component states results in the selective preservation of the nonclassical effects during the amplification process. Increasing the value of the parameter$M$ (~ 17), and keeping the parameter η with the same value, we find the four peaks are shifted and the chaotic behavior becomes pronounced. With respect to the Q-function, we find that when η is small (~ 0.1) and $M(\sim 5)$ the function represents the case of the Fock state. However when we increase the value of the parameter η then the probability of having single photons also increases, where we have four adjacent deformed peaks at the center, (see figures 5 and 6).

4. THE PHASED GENERALIZED BINOMIAL STATE

Our purpose in the present section is to introduce a new class of the intermediate states. The idea of introducing such states is not only to give a wider range for studying the intermediate states, but also to generalize the so-called orthogonal even coherent state which is given by [28]

$$| \phi\rangle = B^{1/2}[| \alpha\rangle_+ + | i\alpha\rangle_+], \tag{4.1}$$

where B is the normalization constant and $| \alpha\rangle_+$ is the usual even coherent state given by equation (4). It is worthwhile to refer to the origin of the orthogonal-even coherent state. This state is revealed by the study of the definition of equation (4.1), where in the complex α-plane the vector representing $(i\alpha)$ is rotated $\pi/2$ degrees from α, and therefore the even coherent state $| i\alpha\rangle_+$ is orthogonal to the state $| \alpha\rangle_+$.

The state we shall introduce is called a *phased generalized binomial state*, and is defined by [29]

$$| \chi\rangle = A^{1/2}\left[| \eta, M\rangle + e^{i\overline{\psi}} | \eta e^{i\overline{\phi}}, M\rangle\right], \tag{4.2}$$

where A is the normalization constant, and $\overline{\psi}$ and $\overline{\phi}$ are two different phases, that enable us to get different states. It is interesting to point out that the phased generalized binomial state can be regarded as a generalization to the previous states which have already been considered in the present work. Also we can say the state $|\chi\rangle$ would generalize the result of [28], while the state $| \eta, M\rangle$ can be regarded as the usual binomial state or even binomial or odd binomial state, but this depends upon the value of the phase parameters $\overline{\psi}$ and $\overline{\phi}$. The normalization constant for the state (4.2) is given by

$$A = \frac{1}{2}\left[1 + Re e^{i\overline{\psi}}\langle \eta, M | \eta e^{i\overline{\phi}}, M\rangle\right]^{-1}, \tag{4.3}$$

while for the usual binomial state, we find that A takes the form

$$A_b = \frac{1}{2}\left[1 + Re e^{i\overline{\psi}}\left(1 + (e^{i\overline{\phi}} - 1) | \eta |^2\right)^M\right]^{-1} \tag{4.4}$$

In the limiting case we find for $M \to \infty$,and $\eta \to 0$ such that $M|\eta|^2 = |\alpha|^2$ equation (4.4) tends to

$$\overline{A_b} = \frac{1}{2}\left[1 + \{\cos(\overline{\psi} + | \alpha |^2 \sin\overline{\phi})\}e^{|\alpha|^2(\cos\overline{\phi} - 1)}\right]^{-1} \tag{4.5}$$

It is clear from the definition of $| \chi\rangle$ that when $\overline{\psi} = 0$ and $\overline{\phi} = \pi l$, $l = 1, 3, 5, \ldots$ then the state (4.2) reduces to the even-binomial state (see equation (3.7)) and it reduces to the odd binomial state,when $\overline{\phi} = \overline{\psi} = \pi s$, $s = 1, 3, 5, \ldots$(see equation(3.8)).

In the forthcoming subsections we shall consider, as usual, the statistical properties of the state $|\chi\rangle$ given by equation (4.2), where we study the case when $|\ \eta, M\rangle$ is binomial, and even (odd) binomial states, provided we take $\phi = \frac{\pi}{2}$ and we call the state in this case a *phased orthogonal binomial* or *even (odd) binomial state.*

4.1. **Phased orthogonal states.**

4.1.1. *Even binomial state.* In this subsection, we consider the case when $\overline{\phi} = \frac{\pi}{2}$ in the state $|\chi\rangle$ equation (4.2). In this case the state $|\eta, M\rangle_e$ represents the usual even binomial state, where equation (4.2) reduces to

$$(4.6) \qquad |\ \chi\rangle_e = A_e^{1/2}\left[|\ \eta, M\rangle_e + e^{i\overline{\psi}}\ |\ i\eta, M\rangle_e\right]$$

It is clear that

$$(4.7) \qquad {}_e\langle \eta, M\ |\ i\eta, M\rangle_e = |\ \eta\ |^2 \sum_{n=0}^{[M/2]} \binom{M}{2n} (i\ |\ \eta\ |^2)^{2n}(1-|\ \eta\ |^2)^{M-2n}$$
$$= |\ \lambda_1\ |^2\ Re[1 + (i-1)\ |\ \eta\ |^2]^M,$$

and then the normalization constant A becomes

$$(4.8) \qquad A_e = \frac{1}{2}\{1+|\ \lambda_1\ |^2 \cos\overline{\psi} Re[1 + (i-1)\ |\ \eta\ |^2]^M\}^{-1}$$

where λ_1 is the normalization constant for the even binomial state given by equation (3.8).

As we stated above the even binomial state tends to the even coherent state when $M \to \infty, \eta \to 0$ such that $M|\eta|^2 = |\alpha|^2$. In this case we find equation (4.8) tends to the form

$$(4.9) \qquad \overline{A_e} = \frac{1}{2}\cosh|\ \alpha\ |^2\left[\cosh|\ \alpha\ |^2 + \cos\overline{\psi}\cos|\ \alpha\ |^2\right]^{-1}$$

For $\overline{\psi} = 0$ one can find equation (4.9) is exactly equation (5) of [28], and the state $|\ \chi\rangle_e$ reduces to the orthogonal-even coherent state.

Second and fourth order squeezing

Now let us employ the Hong-Mandel definition [30] to study the second and the fourth order squeezing. To do so we have to calculate the expectation values for different order operators using the state $|\ \chi\rangle_e$. It is clear that the expectation values for both $\hat{a}$ and $\hat{a}^\dagger$ vanish, and the expectation value for $\hat{a}^2$ is given by

$$(4.10)\quad \langle \hat{a}^2\rangle_e = -2iA_e \sum_{n=0}^{[\frac{M-1}{2}]} \left(B^M_{2n}\right)^* \left(B^M_{2n+2}\right) \sqrt{(2n+2)(2n+1)}\,\sin(\overline{\psi}+n\pi),$$

while the expectation value for the photon number operator $a^\dagger a$ is

$$(4.11)\qquad \langle \hat{a}^\dagger \hat{a}\rangle_e = 2A_e \sum_{n=1}^{[\frac{M}{2}]} 2n \mid B^M_{2n}\mid^2 [1+\cos(\overline{\psi}+n\pi)].$$

Similarly we can obtain the expectation value for $\hat{a}^{\dagger 2}\hat{a}^2$ as

$$(4.12)\qquad \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle_e = 2|A_e|^2 \sum_{n=1}^{[\frac{M}{2}]} 2n(2n-1) \mid B^M_{2n}\mid^2 [1+\cos(\overline{\psi}+n\pi)]$$

The expectation value for a^4 can also be given as

$$(4.13)\quad \langle \hat{a}^4\rangle_e = \frac{2|A_e|^2 \mid\eta\mid^4 e^{4i\theta}}{(1-\mid\eta\mid^2)^2} \sum_{n=0}^{[\frac{M-4}{2}]} \mid B^M_{2n}\mid^2 \times \sqrt{(M-2n)(M-2n-1)(M-2n-2)(M-2n-3)} \times [1+\cos(\overline{\psi}+n\pi)]$$

Finally the expectation value of $\hat{a}^{\dagger 3}\hat{a}$ is given by

$$(4.14)\quad \langle \hat{a}^{\dagger 3}\hat{a}\rangle_e = \frac{2|A_e|^2|\eta|^2 e^{-2i\theta}}{(1-\mid\eta\mid^2)} \sum_{n=0}^{[\frac{M-4}{2}]} 2n \mid B^M_{2n}\mid^2 \times \sqrt{(M-2n)(M-2n-1)}\,\sin(\overline{\psi}+n\pi),$$

where $\eta = |\eta|\exp(i\theta)$

Now let us define two quadrature operators for the field, $X_1 = \sqrt{2}X$ and $X_2 = \sqrt{2}Y$ where X and Y are given by equation (3.11). These quadratures satisfy the commutation relation $[X_1, X_2] = i$ and play the same role as the position and the momentum operators, $\hat{q}$ and $\hat{p}$ respectively, where $[\hat{q},\hat{p}] = i$.

For the second-order squeezing we may write the quadratures $\overline{\triangle X_1}^2$ and $\overline{\triangle X_2}^2$ in terms of $\hat{a}$,$\hat{a}^\dagger$ as follows:

$$\overline{\triangle X_{1,2}}^2 - 1/2 = \langle \hat{a}^\dagger \hat{a} \rangle_e \pm Re\langle \hat{a}^2 \rangle_e \tag{4.15}$$

Note that $\overline{\triangle X_i}$ $(i = 1, 2)$ stand for the dispersion evaluated in the orthogonal even binomial state. It is interesting to point out that if we differentiate the resulting expression with respect to θ and set the result equal to zero, we find a necessary condition for $\overline{\triangle X_1}$ (or $\overline{\triangle X_2}$) to be extremum. That is (where $\sin \overline{\psi}$ $\neq 0$) $\cos 2\theta = 0$, where the extreme values of $\theta = \pi/4, 3\pi/4$ are independent of $\mid \eta \mid$. On the other hand the extreme values with respect to $\overline{\psi}$ are $\overline{\psi} = 0, \pi$.

The fourth order moments of X_1 and X_2 in the state $\mid \chi \rangle_e$ are given by

$$\overline{\triangle X_1}^4 = \frac{3}{4}\Big[1 + 2\langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle + \frac{2}{3} Re\langle \hat{a}^4 \rangle + 4\langle \hat{a}^\dagger \hat{a} \rangle + 4Re\langle \hat{a}^2 \rangle + \frac{8}{3} Re\langle \hat{a}^{\dagger 3} \hat{a} \rangle\Big] \tag{4.16}$$

$$\overline{\triangle X_2}^4 = \frac{3}{4}\Big[1 + 2\langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle + \frac{2}{3} Re\langle \hat{a}^4 \rangle + 4\langle \hat{a}^\dagger \hat{a} \rangle - 4Re\langle \hat{a}^2 \rangle - \frac{8}{3} Re\langle \hat{a}^{\dagger 3} \hat{a} \rangle\Big] \tag{4.17}$$

where the fourth order uncertainty principle must satisfy the condition

$$\overline{\triangle X_1}^4 . \overline{\triangle X_2}^4 \geq 9/16 \tag{4.18}$$

To measure fourth order squeezing in the quadratures $\overline{\triangle X_1}^4$ and $\overline{\triangle X_2}^4$ from zero, we can rewrite equations (4.16) and (4.17) in the form

$$Q_1 = \frac{4}{3}\overline{\triangle X_1}^4 - 1, \qquad Q_2 = \frac{4}{3}\overline{\triangle X_2}^4 - 1 \tag{4.19a}$$

Thus we may conclude that X_1 and X_2 will exhibit fourth- order squeezing whenever

$$Q_1 < 0 \quad \text{or} \quad Q_2 < 0 \tag{4.19b}$$

It can be shown by calculation that fourth-order squeezing coincides with second order squeezing for both θ and $\overline{\psi}$.

To discuss the second and fourth-order moment of squeezing in the phased orthogonal-even coherent state, we have to take the limit of equations (4.15-17). After a long and tedious calculation one finds the following expressions:

$$(4.20a)\quad \overline{\triangle X_1}^2 = \frac{1}{2} + |\alpha|^2 \frac{[\sinh|\alpha|^2 - \sin|\alpha|^2 \cos\overline{\psi}]}{\cosh|\alpha|^2 + \cos\overline{\psi}\cos|\alpha|^2} + \frac{|\alpha|^2 \cos|\alpha|^2 \sin\overline{\psi}\sin 2\theta}{\cosh|\alpha|^2 + \cos\overline{\psi}\cos|\alpha|^2}$$

and

$$(4.20b)\quad \overline{\triangle X_2}^2 = \frac{1}{2} + |\alpha|^2 \frac{[\sinh|\alpha|^2 - \sin|\alpha|^2 \cos\overline{\psi}]}{\cosh|\alpha|^2 + \cos\overline{\psi}\cos|\alpha|^2} - \frac{|\alpha|^2 \cos|\alpha|^2 \sin\overline{\psi}\sin 2\theta}{\cosh|\alpha|^2 + \cos\overline{\psi}\cos|\alpha|^2}$$

$$(4.21a)\quad \overline{\triangle X_1}^4 = \frac{3}{4}\Big[1 + 4[\cosh|\alpha|^2 + \cos\overline{\psi}\cos|\alpha|^2]^{-1}[|\alpha|^4 [C\cosh|\alpha|^2 - D\cos\overline{\psi}\cos|\alpha|^2 - \frac{2}{3}\sin|\alpha|^2 \sin 2\theta\sin\overline{\psi}] + |\alpha|^2 [\sinh|\alpha|^2 - \cos\overline{\psi}\sin|\alpha|^2 + \cos|\alpha|^2 \sin\overline{\psi}\sin 2\theta]]\Big]$$

$$(4.21b)\quad \overline{\triangle X_2}^4 = \frac{3}{4}\Big[1 + 4[\cosh|\alpha|^2 + \cos\overline{\psi}\cos|\alpha|^2]^{-1}[|\alpha|^4 [C\cosh|\alpha|^2 - D\cos\overline{\psi}\cos|\alpha|^2 + \frac{2}{3}\sin|\alpha|^2 \sin 2\theta\sin\overline{\psi}] + |\alpha|^2 [\sinh|\alpha|^2 - \cos\overline{\psi}\sin|\alpha|^2 - \cos|\alpha|^2 \sin\overline{\psi}\sin 2\theta]]\Big]$$

where

$$(4.22)\quad C = \frac{1}{2}(1 + \frac{\cos 4\theta}{3}), \qquad D = \frac{1}{2}(1 - \frac{\cos 4\theta}{3})$$

From the expression for $\overline{\triangle X_1}^4, \overline{\triangle X_2}^4$, one can easily realize that there is a fourth order simultaneous squeezing in the quadrature components for both the phased orthogonal even binomial state and the phased orthogonal even coherent state (see figure 12 of [28]).

4.1.2. *Odd binomial state.* Now we turn our attention to the phased orthogonal odd binomial state $\mid \chi\rangle_o$, which is given by replacing $\mid \eta, M\rangle_o$ instead of $\mid \eta, M\rangle$ in the expression (4.2) taking into account $\overline{\phi} = \pi/2$, $\overline{\psi} = \overline{\psi} + \pi/2$. Hence the state $|\chi\rangle$ is

$$\mid \chi\rangle_o = A_o^{1/2}\left[\mid \eta, M\rangle_o + e^{(i\overline{\psi}+\pi/2)} \mid i\eta, M\rangle_o\right] \tag{4.23}$$

The normalization constant A_o is given by

$$A_o = \frac{1}{2}\left[1- \mid \lambda_2 \mid^2 \cos\overline{\psi}[1 + (i-1) \mid \eta \mid^2]^M\right]^{-1} \tag{4.24}$$

where λ_2 is the normalization constant for the odd binomial state defined by equation (3.8).

Second and fourth order squeezing

A similar discussion can be given as in the previous subsection. The extreme values for the present case are identical with those obtained for the case of the phased orthogonal even-binomial state. This is due to the fact that we take the phase $\overline{\psi}$ of equation (49) equal to $\overline{\psi}+\pi/2$. Numerical investigations reveal that the optimal case of θ and $\overline{\psi}$ does not introduce fourth order squeezing. This is in contrast to the phased orthogonal even binomial state, which has simultaneous fourth-order squeezing.

For the phased orthogonal odd coherent state, the normalization constant A_o of equation (4.24) reduces to

$$\overline{A_o} = \frac{1}{2}\sinh \mid \alpha \mid^2 [\sinh \mid \alpha \mid^2 - \cos\overline{\psi}\sin \mid \alpha \mid^2]^{-1} \tag{4.25}$$

The second order moments of squeezing are given respectively as:

$$\overline{\triangle X_1}^2 = \frac{1}{2} + \mid \alpha \mid^2 \frac{[\cosh \mid \alpha \mid^2 - \cos|\alpha|^2 \cos\overline{\psi})]}{\sinh \mid \alpha \mid^2 - \cos\overline{\psi}\sin \mid \alpha \mid^2} - \frac{\mid \alpha \mid^2 \sin \mid \alpha \mid^2 \sin\overline{\psi}\sin 2\theta}{\sinh \mid \alpha \mid^2 - \cos\overline{\psi}\sin \mid \alpha \mid^2} \tag{4.26a}$$

and

$$(4.26b)\quad \overline{\triangle X_2}^2 = \frac{1}{2} + |\alpha|^2 \frac{[\cosh|\alpha|^2 - \cos|\alpha|^2\cos\overline{\psi})]}{\sinh|\alpha|^2 - \cos\overline{\psi}\sin|\alpha|^2} + \frac{|\alpha|^2\sin|\alpha|^2\sin\overline{\psi}\sin 2\theta}{\sinh|\alpha|^2 - \cos\overline{\psi}\sin|\alpha|^2}$$

while the fourth order squeezing are

$$(4.27a)\quad \overline{\triangle X_1}^4 = \frac{3}{4}\Big[1 + 4[\sinh|\alpha|^2 - \cos\overline{\psi}\sin|\alpha|^2]^{-1}[|\alpha|^4[C\sinh|\alpha|^2 + D\cos\overline{\psi}\sin|\alpha|^2 - \frac{2}{3}\cos|\alpha|^2\sin 2\theta\sin\overline{\psi}] + |\alpha|^2[\cosh|\alpha|^2 - \cos\overline{\psi}\cos|\alpha|^2 - \sin|\alpha|^2\sin\psi\sin 2\theta]]\Big]$$

and

$$(4.27b)\quad \overline{\triangle X_2}^4 = \frac{3}{4}\Big[1 + 4[\sinh|\alpha|^2 - \cos\overline{\psi}\sinh|\alpha|^2]^{-1}[|\alpha|^4[C\sinh|\alpha|^2 + D\cos\overline{\psi}\sin|\alpha|^2 + \frac{2}{3}\cos|\alpha|^2\sin 2\theta\sin\overline{\psi}] + |\alpha|^2[\cosh|\alpha|^2 - \cos\overline{\psi}\cos|\alpha|^2 + \sin|\alpha|^2\sin\overline{\psi}\sin 2\theta]]\Big]$$

where the constants C and D are given by equations (4.22).

In comparing equations (4.20-21) with Eqs. (4.26-27), we come to the conclusion that the behaviour of $\overline{\psi}$ and θ in both cases is identical, provided we replace $(\overline{\psi} + (\pi/2))$ by $\overline{\psi}$.

4.1.3. *The distribution functions.* Here we pay attention to use the quasiprobability phase-space distributions to examine the representation of the phased generalized binomial states given by equation (4.2). By using equations (2.12) we will be able to get an explicit form for the quasiprobability distribution function for different forms of phased orthogonal states. Thus for the density matrix $\hat{\rho}$ generated by equation (4.2) we obtain the following expression for the quasiprobability function as

$$(4.28)\quad I(\beta,s)=\frac{2A}{\pi}\frac{(1-|\eta|^2)^M}{1-s}\exp\left(\frac{-2\mid\beta\mid^2}{1-s}\right)\left[\sum_{n=0}^{M}(-)^n\left(\frac{1+s}{1-s}\right)^n\binom{M}{n}\right.$$

$$\times\left(\frac{|\eta|^2}{1-|\eta|^2}\right)^n P_n^2 L_n\left(\frac{4\mid\beta\mid^2}{1-s^2}\right)+$$

$$2\sum_{\substack{m,n=0\\m>n}}^{M}(-1)^n\sqrt{\binom{M}{n}\binom{M}{m}\frac{n!}{m!}}\left(\frac{1+s}{1-s}\right)^n\left(\frac{|\eta|^2}{1-|\eta|^2}\right)^{\frac{m+n}{2}}\left(\frac{2|\beta|}{1-s}\right)^{(m-n)}$$

$$\left.\times P_n\,P_m\,L_n^{(m-n)}\left(\frac{4\mid\beta\mid^2}{1-s^2}\right)\right],$$

where

$$(4.28a)\qquad P_nP_m=4[\cos[(\theta-\delta-\frac{\overline{\phi}}{2})(m-n)]\cos(\frac{\psi+n\overline{\phi}}{2})\cos(\frac{\psi+m\overline{\phi}}{2})$$

The function $L_n^{(\gamma)}(y)$ in equation (4.28) is the associated Laguerre polynomial defined after (2.15). Note that in our calculations we have taken $\eta=|\eta|e^{i\theta}$ and $\beta=|\beta|e^{i\delta}$, where θ and δ are the phases of η and β respectively.

The Wigner function of the phased orthogonal even binomial state is

$$(4.29)\quad W(\beta)=\frac{2A_e}{\pi}(1-|\eta|^2)^M\exp(-2|\beta|^2)\left[\sum_{n=0}^{[M/2]}\binom{M}{2n}\left(\frac{|\eta|^2}{1-|\eta|^2}\right)^{2n}\right.$$

$$\times P_{2n}^2L_{2n}(4|\beta|^2)+2\sum_{\substack{m,n=0\\m>n}}^{[M/2]}\sqrt{\frac{2n!}{2m!}\binom{M}{2n}\binom{M}{2m}}$$

$$\left.\times\left(\frac{|\eta|^2}{1-|\eta|^2}\right)^{m+n}(4|\alpha|^2)^{(m-n)}P_{2n}P_{2m}L_{2n}^{2(m-n)}(4|\beta|^2)\right]$$

and

$$(4.29a)\quad P_{2n}P_{2m}=4\cos[(2\theta-2\delta-\overline{\phi})(m-n)]\cos\left(\frac{\psi}{2}+n\overline{\phi}\right)\cos\left(\frac{\psi}{2}+m\overline{\phi}\right).$$

The Q-function is

$$Q(\beta)=\frac{A_e}{\pi}(1-|\eta|^2)^M\exp(-|\beta|^2)\left[\sum_{n=0}^{[M/2]}\binom{M}{2n}\left(\frac{|\eta|^2}{1-|\eta|^2}\right)^{2n}\times\frac{(|\beta|^2)^{2n}}{2n!}P_{2n}^2+2\sum_{\substack{m,n=0\\m>n}}^{[M/2]}\sqrt{\binom{M}{2n}\binom{M}{2m}}\left(\frac{|\eta|^2}{1-|\eta|^2}\right)^{(m+n)}\times\frac{|\beta|^{2(M+n)}}{\sqrt{2n!2m!}}P_{2n}P_{2m}\right] \tag{4.30}$$

Finally we would like to mention that, as a result of the non-classical character of the present state, we find the P- representation function highly singular . Therefore consideration of the P-representation for the state $|\chi>$ is meaningless. Now we shall make use of the Wigner function (4.29) to get the probability distribution function $P(x)$ by integrating $W(\beta)$, with $\beta=x+iy$, over the imaginary variable y , where

$$P(x)=\int_{-\infty}^{\infty}W(x+iy)dy \tag{4.31}$$

Substituting equation (4.29) into equation (4.31), we get

$$P(x)=\sqrt{\frac{2}{\pi}}A_e(1-\mid\eta|^2)^M\exp(-2x^2)\left[\sum_{n=0}^{[M/2]}\binom{M}{2n}\frac{|\eta|^2}{2(1-|\eta|^2)}\right]^{2n}\times P_{2n}^2\frac{H_{2n}^2(\sqrt{2}x)}{2n!}+2\sum_{\substack{m,n=0\\m>n}}^{[M/2]}\sqrt{\binom{M}{2n}\binom{m}{2m}}\times\left(\frac{|\eta|^2}{2(1-|\eta|^2)}\right)^{m+n}P'_{2n}P'_{2m}\frac{H_{2M}(\sqrt{2}x)H_{2n}(\sqrt{2}x)}{\sqrt{2n!2m!}} \tag{4.32}$$

where $H_m(z)$ is the Hermite polynomial of order m,

$$H_m(z)=\sum_{r=0}^{[\frac{m}{2}]}\frac{(-1)^r m!(2z)^{m-2r}}{r!((m-2r)!} \tag{4.32a}$$

and

$$(4.32b) \qquad P'_{2n}P'_{2m} = 4\cos[(2\theta+\pi)(m-n)]\cos\left(\frac{\psi}{2}+n\frac{\pi}{2}\right)\cos\left(\frac{\psi}{2}+m\frac{\pi}{2}\right)$$

The characteristic function and the quasi-probability distribution functions for phased orthogonal odd binomial state are calculated as previously.

The Wigner function $W(\beta)$ has been investigated for different values of M, η and $\overline{\psi}$. For small values of η (~ 0.1), we find that the contribution of the vacuum state is the dominant one for $\overline{\psi} = 0$ and $\pi/2$. For $\overline{\psi} = \pi$ where the Fock state $|2\rangle$ is the only existing state when $M = 4$, the figures for $W(\beta)$ represent this state clearly whatever the value of η. For the intermediate $\eta = 0.5$ we find for $\overline{\psi} = 0$ and $M = 4$ that the Gaussian is deformed slightly with the appearance of shoulders. This is due to the slight effect of the state $|4\rangle$ in this case to the vacuum, while for $\overline{\psi} = 90$ and $M = 4$ the figure is assymmetric the effect of the state $|2\rangle$ deforms the Gaussian of the earlier case (see figure.2a of [29]). The nonclassical effect is apparent. Increasing M adds more states to be considered. The central peak is surrounded by wobbling circles, and changing $\overline{\psi}$ changes the symmetry (the case of $M = 16$ and $\overline{\psi} = 180$ are presented in figure.2b of [29]). When η is increased (to ~ 0.9) the effect of the Fock state with higher excitations, especially when M takes larger values, is pronounced. This is shown in figures. 3c and 3d of [29]. Increasing the value of M shows breaking up of the outer engulfing circles. Changes in $\overline{\psi}$ result in assymmetry in the figures, especially when we take $\overline{\psi} = \frac{\pi}{2}$. Therefore the shape of the quasiprobabilities is very sensitive to the choice of the phase $\overline{\psi}$.

Investigations of the formula (2.25) for the Q-function reveal that for small value of η we find the Gaussian form that characterizes the vacuum state for all values of M and $\overline{\psi} = 0$ or $\pi/2$. However when we take $\overline{\psi} = 180$, and $M \sim 4$, the case of the state $|2\rangle$ results and the figure for the Q- function is representative of this case whatever the value of η (see for example figure 4d of [4], which represents the Fock $|5\rangle$). For intermediate values of $\eta = 0.5$ we find a displacement towards the centre($\alpha = 0$) which is a sign of squeezing. This effect is pronounced for the case $\overline{\psi} = 90$ and $M(\sim 4)$. However when M is increased one finds a split into four summits, choosing $\overline{\psi}$ makes the interference between the summits greater ($\overline{\psi} = 0$) or the split clearer (as when we take $\overline{\psi} = 90$ or π). As η increases to ~ 0.9 one finds that the height of these summits depends on the choice of $\overline{\psi}$.

4.1.4. *Phase properties.* The notion of the phase in quantum optics has found renewed strong interest because of the existance of phase-dependent quantum noise. The new measurements [32] in this field opened the way to deeper understanding of the quantum nature of the phase. There are many different approaches to this problem [33]. To calculate the phase properties for the even and odd phased orthogonal binomial states we adopt the Pegg-Barnett formalism [34]. In this approach, and on the $(s+1)$ dimensional subspace, one chooses the $(s+1)$ orthonormal phase states as bases, thus defined by

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle, \tag{4.33}$$

where

$$\theta_m = \theta_0 + \frac{2\pi m}{s+1}; \quad m = 0, 1, 2. \ldots, s \tag{4.33a}$$

The phase operator is then defined as

$$\hat{\phi}_\theta = \sum_{m=0}^{2} \theta_m |\theta_m\rangle\langle\theta_m|, \tag{4.34}$$

which has the eigenstate $|\theta_m\rangle$ as its eigenstate with the eigenvalue θ_m. The probability distribution for any state $|\psi\rangle$ is given by

$$P(\theta_m) = |\langle\theta_m|\psi\rangle|^2 \tag{4.35}$$

This can be used to compute various moments and then the limit $s \to \infty$ is taken. The continuous-phase distribution function $P(\theta)$ is introduced by

$$P(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi} |\langle\theta_m|\psi\rangle|^2 \tag{4.36}$$

For any state of the form $|\psi\rangle = \sum_n c_n|n\rangle$, we find that $P(\theta)$ takes the form

$$P(\theta) = \frac{1}{2\pi}\left(1 + \sum_{m\neq n} c_m c_n^* \exp[i(n-m)\theta]\right) \tag{4.37}$$

From this phase probability function the moments can be calculated. When the phase reference angle θ_0 is put equal to zero, we find that

$$\langle\theta\rangle = 0$$

$$\langle\theta^2\rangle = \frac{\pi^2}{3} + 2\sum_{n\neq m} c_m c_n^* \frac{(-1)^{m-n}}{(m-n)^2} \tag{4.38}$$

When the function $P(\theta)$ of equation (4.37) is plotted we note the breaking up of the figure for the probability distribution funtion for the case of $\psi = 0$. This is only due to the presence of the states $|4n\rangle$ in the state, and hence we get a four- fold degenercy. This is in general trend with the case of the even states where the probability is split only on two, (see [29]).

5. GENERALIZED GEOMETRIC STATE

As we have stated earlier the geometric state presents the gradual behavior of some quantum optical systems where the state of the field changes from the pure number (Fock) state to the non-pure chaotic state [11, 35, 36]. This means a field state that interpolates between the number state $|n\rangle$ and the chaotic state with density given by equation (1.2).

5.1. **Definition.** We define the normalized generalized two parameter state $|Y, M\rangle$ as follows:

$$|Y, M\rangle = \lambda_0 \sum_{n=0}^{M} Y^{\frac{n}{2}} |n\rangle \tag{5.1}$$

where Y is a complex parameter and its phase is random in general and the normalization constant λ_0 satifies

$$|\lambda_0|^2 = \frac{1-|Y|}{1-|Y|^{M+1}}, \qquad |Y| \neq 1 \tag{5.1a}$$

The limiting cases of the definition in equation (5.1) follow:

(a) Chaotic state. For $|Y| < 1$ $(= \frac{\overline{n}}{1+\overline{n}})$ and $M \to \infty$, the density operator in this case is

(5.2a)
$$\hat{\rho}_{Y,\psi} = \lim_{M\to\infty} |Y, M\rangle\langle Y, M|$$
$$= \lim_{M\to\infty} |\lambda_0|^2 \sum_{n,n'=0}^{\infty} Y^{\frac{n}{2}} Y^{*\frac{n'}{2}} |n\rangle\langle n'|$$

If $Y = |Y|e^{2i\psi}$ and ψ is a random phase, then the average over ψ gives

(5.2b)
$$(\hat{\rho}_{Y,\psi})_{av} = \frac{1}{2\pi}\int_0^{2\pi} \hat{\rho}_{Y,\psi} d\psi$$
$$= \frac{(2\pi)^{-1}}{1+\overline{n}} \sum_{n,n'=0}^{\infty} |Y|^{\frac{n+n'}{2}} \int_0^{2\pi} e^{i(n-n')\psi} d\psi |n\rangle\langle n'|$$
$$= \sum_{n=0}^{\infty} \frac{(\overline{n})^n}{(1+\overline{n})^{n+1}} |n >< n|.$$

This is identical with the single-mode chaotic state with mean photon number $\overline{n}$.

(b) The number state. For $|Y| \to \infty$ and M finite, equation (5.1) reduces to the number state $|M>$.

(c) The vacuum state $|0\rangle$. This is either obtained by taking the limit $|Y| \to 0$ or equivalently, by taking $M = 0$.

(d) The phase state $|\theta_m\rangle$ (see equation (4.33)) of Pegg and Barenett [34]. For $y = e^{2i\theta_m}$, $|Y| = 1$ and $s = M$. This is the partially coherent phase state, and when s$\to \infty$ the coherent phase state [33] results.

5.2. **Properties.** The mean value for the mth moment of the photon number operator in the generalized geometric state is given by

(5.3)
$$\langle \hat{n}^m \rangle = |\lambda_0|^2 \sum_{n=0}^{M} n^m |Y|^n.$$

In particular, for $m = 1, 2$ we have

$$(5.3a) \quad \langle \hat{n} \rangle = |\lambda_0|^2 \sum_{n=0}^{M} n|Y|^n$$
$$= |Y|(1-|Y|)^{-1}(1-|Y|^{M+1})^{-1}[1-(M+1)|Y|^M + M|Y|^{M+1}].$$

and

$$(5.3b) \quad \langle \hat{n}^2 \rangle = |\lambda_0|^2 \sum_{n=0}^{M} n^2|Y|^n$$
$$= \frac{[|Y|(1+|Y|) - (M+1)^2|Y|^{M+1} + (2M^2+2M-1)|Y|^{M+2} - M^2|Y|^{M+3}]}{(1-|Y|)^2(1-|Y|^{M+1})}.$$

Note that from (5.3a) and (5.3b) we have in the chaotic-state limit $\langle \hat{n} \rangle \longrightarrow \overline{n}$ and $\langle \hat{n}^2 \rangle \longrightarrow \overline{n}(2\overline{n}+1)$, and in the number state $|M>$ limit $\langle \hat{n} \rangle \longrightarrow M$, and $\langle \hat{n}^2 \rangle \longrightarrow M^2$.

To calculate the normalized second-order correlation function one can use equation (3.9). In this case we have the expression

$$(5.4) \quad g^{(2)}(0) = (1-|Y|^{M+1})^{-1}[1+M|Y|^{M+1} - (1+M)|Y|^M]^{-2}$$
$$[2 - M(M+1)|Y|^{M-1} + 2(M^2-1)|Y|^M - M(M-1)|Y|^{M+1}],$$

which goes to 2 for chaotic state and goes to $(1-\frac{1}{M})$ for the number state $|M\rangle$.

For the special case of $M = 1, g^{(2)}(0) = 0$, an expected result since the state $|Y, 1>$ does not contain in its expansion the photon number state $|2>$. figures 7a and 7b show the behavior of $g^{(2)}(0)$ against $|Y| < 1$. For $M = 2$ (figure 7a), $0.69 < g^{(2)}(0) \leq 1.94$. In the range $0 < |Y| < 0.36$, there is a partial coherent property $g^{(2)}(0) > 1$. For $0.36 < |Y| < 0.9$ the antibunching effect $(g^{(2)}(0) < 1)$ is clear but it is less, compared with the photon number state [$g^{(2)}(0) = \frac{1}{2}$ for the state $|2>$]. For higher values of $M = 10$, the chaotic behavior is exhibited $(g^{(2)}(0) = 2)$ for $|Y| < 0.3$. In fact, as $M \longrightarrow 100$, $g^{(2)}(0) = 2$ for the whole range of $0 < |Y| < 0.95$. The ratio of the variance function of the photon number $(\Delta\hat{n})^2$ to the mean photon number (the Fano factor) is defined as

$$(5.6) \qquad F = \frac{(\Delta\hat{n})^2}{<\hat{n}>} = \frac{<\hat{n}^2> - <\hat{n}>^2}{<\hat{n}>}.$$

(For a number state, $F = 0$.). In the special case of $M = 1$,

(5.6a) $$F = 1 - \frac{|Y|}{1+|Y|} = \frac{1}{1+|Y|} < 1,$$

the generalized geometric distribution shows sub-Poissonian behavior (figure 8a). For $M = 2$ (figure 8b) the sub-Poissonian behavior is shown in the range $0.37 \leq |Y| < 0.9$. The cases of $M = 10, 100$ indicate the chaotic character of the state ($F > 1$). See figure 8.

5.2.1. *Squeezing.* Now, as usual, we shall examine the squeezing property for the generalized geometric state. To reach this we have to use equation (3.11). In this case we need to calculate the expectation value for both quadrature variances $\hat{X} = \sqrt{2}X_1$ and $\hat{X}_2 = \sqrt{2}Y$. Thus we find the variance $(\Delta\hat{X}_1)^2$ takes the form

(5.7a) $$2(\Delta\hat{X}_1)^2 =< \hat{a}^2 + \hat{a}^{\dagger 2} > +2\langle\hat{n}\rangle + 1 - < \hat{a} + \hat{a}^\dagger >^2$$

Similarly for $\hat{X}_2$ we have.

(5.7b) $$2(\Delta\hat{X}_2)^2 = - < \hat{a}^2 + \hat{a}^{\dagger 2} > +2\langle\hat{n}\rangle + 1 + < \hat{a} - \hat{a}^\dagger >^2$$

On the other hand we find

(5.8a) $$\langle\hat{a}^2\rangle = |\lambda_0|^2 (Y^*)^{-1} \sum_{n=2}^{M} |Y|^n \sqrt{n(n-1)} = \langle\hat{a}^{\dagger 2}\rangle^*$$

This leads to

(5.8b) $$\langle\hat{a}^2 + \hat{a}^{\dagger 2}\rangle = 2|\lambda_0|^2 |Y|^{-2} \cos\phi \sum_{n=2}^{M} |Y|^n \sqrt{n(n-1)},$$

where $Y = |Y|e^{i\phi}$, and ($\phi = 2\psi$). Similarly we can show that

$$\langle\hat{a} + \hat{a}^\dagger\rangle = 2|\lambda_0|^2 |Y|^{-\frac{1}{2}} \cos\frac{\phi}{2} \sum_{n=1}^{M} |Y|^n \sqrt{n},$$

(5.8c) $$\langle\hat{a} - \hat{a}^\dagger\rangle = 2i|\lambda_0|^2 |Y|^{-\frac{1}{2}} \sin\frac{\phi}{2} \sum_{n=1}^{M} |Y|^n \sqrt{n}$$

The numerical results for the variance expressions,

(5.9) $$S_1 = 2(\Delta\hat{X})^2 - 1, \qquad S_2 = 2(\Delta\hat{Y})^2 - 1$$

where $S_{1,2} < 0$ signify squeezing, are presented in figure 3 of [11]. In the case $M = 1$ the component S_1 shows squeezing for $\phi = 0$ up to $|Y| \approx 0.98$, but for $\phi = \frac{\pi}{4}$ a lesser squeezing occurs for shorter range of $|Y|$ (namely, $0 < |Y| \leq 0.7$). There is no squeezing at all for $\phi = \frac{\pi}{2}$. In conformity with the case $M = 1$, the component S_2 does not exhibit squeezing for the same values of ϕ. For $M = 10, S_2$ shows some squeezing for $\phi = 0, \frac{\pi}{4}$. The same is true for the case $M = 50$, but the magnitude of squeezing is less. In both cases of $M = 10$ and $M = 50$ there is no squeezing in S_1 as expected. Note that for $|Y| \geq 1$, both $S_{1,2}$ are positive for all ϕ hence there is no squeezing. This is consistent with the fact that as, $|Y| \to \infty$, the generalized geometric state tends to a Fock (number) state which does not exhibit squeezing in $S_{1,2}$, see figure 9.

5.3. **Quasiprobability distribution function.** As we have done in the previous sections of the present work, we shall consider the quasiprobability distribution. From the characteristic function equation (2.10) we shall be able to find these functions, therefore if we take the parameter $s = 1$ corresponding to normally ordered characteristics function, and we take the density matrix $\hat{\rho} = |Y, M\rangle\langle Y, M|$, where $|Y, M\rangle$ is the state given by equation (5.1), then after some calculations we have

$$C_p(\xi) = |\lambda_0|^2 \sum_{n',n=0}^{M} \sqrt{\frac{n!}{n'!}} Y^{\frac{n}{2}} Y^{*\frac{n'}{2}} \xi^{(n'-n)} L_n^{(n'-n)}(|\xi|^2) \tag{5.10}$$

where $L_n^{(m)}(z)$ is the associated Laguerre polynomial given earlier.

Although the P representation affords a convenient way of evaluting the ensemble averages of normally ordered operators, the function is highly singular as a results of the nonclassical charater of the state (5.1). Therefore we shall consider the Wigner function which can be calculated if one uses equation (2.13). In this case we have

$$W(\beta) = \frac{2}{\pi} |\lambda_0|^2 e^{-2|\beta|^2} \times \sum_{n',n=0}^{M} (-1)^n \sqrt{\frac{n!}{n'!}} (2\beta)^{(n'-n)} Y^{\frac{n}{2}} Y^{*\frac{n'}{2}} L_n^{(n'-n)}(4|\beta|^2) \tag{5.11}$$

where we used the generating function [36]

(5.11a) $$(1+k)^{\gamma}\exp[-kz] = \sum_{n=0}^{\infty} k^{\gamma} L_n^{(\gamma-n)}(z)]$$

As a special case if we take the phase $\psi = 0$, then equation (5.11) reduces to

(5.12) $$W(\beta) = \frac{2}{\pi}|\lambda_0|^2 e^{-2|\beta|^2} \sum_{n=0}^{M} (-1)^n |Y|^n L_n(4|\beta|^2)$$

In the chaotic state where $M \to \infty, |Y| < 1$ we get

(5.12a) $$[W(\beta)]_{ch} = \frac{1}{\pi}\exp[-(\overline{n}+\frac{1}{2})^{-1}|\beta|^2]$$

Figures 10-12 show the Wigner function $W(\beta)$ as a function of $\beta \equiv \mathrm{Re}(\beta) + i\,\mathrm{Im}(\beta)$ for different values of M and $|Y|$. See also figures 4-6 in [35]. From equation (5.12) it is clear that $W(\beta)$ is a symmetric function in both $\mathrm{Re}(\beta)$ and $\mathrm{Im}(\beta)$. For $|Y| = 0.1, W(\beta)$ is insensitive to M and its Gaussian-like form has its peack at $\mathrm{Re}(\beta) = \mathrm{Im}(\beta) = 0$. As β increases and for $M = 1, 2, W(\beta)$ exhibits a hole at its center, and eventully $W(\beta)$ behaves as a Gaussian similar to that of the chaotic state. In all cases $W(\beta)$ is positive for $|Y| < 1$. As for $|Y| > 1$ and for increasing M the presence of the Laguerre polynomial term in equation (5.12) is more effective and hence $W(\beta)$ becomes negative around its center. In fact, equation (5.12) reduces to that for a number (Fock) state $|M\rangle$, namely, in the limit $|Y| \gg 1$ and for fixed M,

(5.12b) $$[W(\beta)]_{Fock} = \frac{2}{\pi}(-)^M L_M(4|\beta|^2)\exp[-2|\beta|^2]$$

which tends to zero for $|\beta| \gg 1$.

Finally we calculate the Q function, which can be used to express the ensemble averages of antinormally ordered operators as a simple intergal. After carrying out the integration in equation (2.10) with $s = -1$

(5.13) $$Q(\beta) = \frac{|\lambda_0|^2}{\pi}\frac{(-)^M}{M!} L_M^{-M-1}(|\beta|^2|Y|)\exp[-|\beta|^2].$$

When M tends to infinity and $|Y| < 1$, we have the corresponding formula of the Q -function in the chaotic state,

(5.13a) $$[Q(\beta)]_{ch} = \frac{1}{\pi}(\overline{n}+1)^{-1}\exp[-(\overline{n}+1)^{-1}|\beta|^2]$$

Now let us discuss the probability distribution function for the generalized geometric state $P(x)$ associated with the quadrature x . To do so let us use equation (5.11) to calculate the integral given by equation (4.31). With the aid of equation (5.11a) we obtain the following result

$$(5.14)\quad P(x) = (\frac{2}{\pi})^{\frac{1}{2}}|\lambda_0|^2 e^{-2x^2} \times \sum_{n,n'=0}^{M} (n'!n!)^{-\frac{1}{2}}(2)^{-\frac{1}{2}(n+n')} Y^{\frac{n}{2}} Y^{*\frac{n'}{2}} H_{n'}(\sqrt{2x}) H_n(\sqrt{2x})$$

The function $H_n(x)$ in equation (5.14) is the Hermite polynomial. The behavior of $P(x)$ is discussed in [35] in the average state, i.e., terms with $n = n'$ in the summation in equation (5.14). For very small $|Y| = 0.1$, $P(x)$ is insensitive to M (the same for the Wigner function $W(\beta)$) and $P(x)$ is of Gaussian-like form. For increasing $|Y| = 0.8$, the hole exhibited in $W(\beta)$ for $M = 1$ is now reflected in the non-monotonic decay of $P(x)$. The emergence of a peak in $W(\alpha)$ for $N = 2$ at its center results in the monotonic decay of $P(x)$. For $M \gg 1$, the Gaussian behavior of $P(x)$ is reached as expected for the chaotic state field. Now, for $|Y| > 1$, the negative values of $W(\beta)$ at its center,due to the Laguerre polynomials, results in the relatively reduced initial value of $P(x = 0)$. As M increase, $P(x)$ has an oscillating behavior of growing envelope for some range of x before it vanishes. For greater $|Y|$, the behavior of $P(x)$ coincides with that for a Fock state $|M>$, namely

$$(5.14a)\qquad P_{Fock}(x) = (\frac{2}{\pi})^{\frac{1}{2}}(2^M M!)^{-1} H_M^2(\sqrt{2x}) \exp(-2x^2)$$

To complete this subsection we shall consider the non-average Wigner function. It will be convenient for us to use the series representations of these quasiprobability distributions in the form

$$(5.15)\qquad F(\beta, p) = \frac{2}{\pi(1-p)} \sum_{n=0}^{\infty} \left(\frac{p+1}{p-1}\right)^n |< Y, M|\hat{D}(\beta)|n >|^2,$$

where $\hat{D}(\beta)$ is the Glauber displacement operator. Evaluating the matrix elements in the above equation we find that $F(\beta, p)$ for the generalized geometric state can be expressed as an expansion over the associated Laguerre polynomials as

$$(5.16)\quad F(\beta,p)=\frac{2\exp[-2|\beta|^2/(1-p)]}{\pi(1-p)}\frac{1-|Y|^2}{1-|Y|^{2(M+1)}}\times$$

$$\sum_{m,n=0}^{\infty} Y^{*m}Y^{n}\beta^{m-n}\left(\frac{n!}{m!}\right)^{\frac{1}{2}}(\frac{2}{1-p})^{m-n}\left(\frac{p+1}{p-1}\right)^{n}L_n^{(m-n)}\left(\frac{4|\beta|^2}{1-p^2}\right),$$

The quasiprobability distributions $F(\beta,p)$ will be the $W-$ Wigner function if $p=0$,the $Q-$function if $p=-1$ and the $P-$ function if $p=1$. The Wigner function for the generalized geometric function is now given by

$$(5.17)\quad W(\beta)=\frac{2}{\pi}\exp[-2|\beta|^2]\frac{1-|Y|^2}{1-|Y|^{2(M+1)}}\{\sum_{n=0}^{\infty}(-)^n|Y|^{2n}L_n(4|\beta|^2)$$

$$+2\,\mathrm{Re}\sum_{\substack{n,m\\ m>n}}(-1)^n\left(\frac{n!}{m!}\right)^{\frac{1}{2}}Y^{*m}Y^{n}(2\beta)^{m-n}L_n^{(m-n)}(4|\beta|^2)\}$$

where $L_\alpha^\beta(x)$ is the associated Laguerre polynomial.

The Wigner function is particularly important as it gives the correct probability for a chosen observable by integration over the conjugate observable. It is a complete representation of the state of the system since there is a one-to-one correspondence between the Wigner function and the wavefunction of the state. Thus, if we can map the Wigner function, then we have a complete representation of the state. Investigations of the Wigner quasiprobability function of equation (5.17) reveal that it has an almost Gaussian shape for small values of $|Y|$ (<0.3) and is largely insensitive to any change in the values of M. This is due to the dominance of the effect of the vacuum state over the effects of the higher excitations. As $|Y|$ increases, however, distinct regions appear for which the Wigner function displays (non-classical) negative regions. The structure changes dramatically once $|Y|$ exceeds unity. For these values the vacuum state no longer has the highest probability in the number state expansion and we see the appearence of crescent-like structures which occur at radii characteristic of the photon number states is noticed. For some different values of M such that $M=2$ and the $M=12$ one can see the characteristic number state rings and the anisotropy characteristic of a preferred phase.

5.4. **phase properties of the generalized geometric states.** The Hermitian optical phase operator is defined in finite-dimensional state space ψ_s by equation (4.34). The use of the phase operator involves evaluating expectation values and moments as function of s before letting s tend to infinity. The importance of this limiting procedure has been discussed in detail (see for example [38]). Since equation (4.36) can be used as a probability density for calculating the moments of the Hermitian optical phase operator, therefore if we take the argument of Y to be zero so that Y is real and positive, then one can find that the phase probability density is

$$P(\theta) = \frac{1}{2\pi} \frac{1-|Y|^2}{1-|Y|^{2(M+1)}} \frac{1-2Y^{M+1}\cos[(M+1)\theta]+Y^{2(M+1)}}{1-2Y\cos\theta+Y^2} \tag{5.18}$$

We also choose the value of θ_0 in equation (4.33) to be $-\pi$ so as to ensure the sensible result that the expectation value of the phase operator is the argument of Y. With these choices equation (4.34) gives

(5.19)

$$< \hat{\phi}_\theta >= 0, \quad \Delta\hat{\phi}_\theta^2 = \frac{\pi^2}{3} + \frac{4}{1-Y^{2(M+1)}} \sum_{m=1}^{M} \frac{\cos m\pi}{m^2} Y^m \left(1 - \frac{Y^{2(M+1)}}{Y^{2m}}\right).$$

It is worth noting that, in the limit as M tends to infinity, this variance becomes

$$\Delta\hat{\phi}_\theta^2(M\to\infty) = \frac{\pi^2}{3} + 4dilog(1+Y), \tag{5.19a}$$

where $dilog(.)$ is the dilogarithm function. The form of this variance is similar to that found for the variance of the phase sum associated with the two-mode squeezed vacuum state. This is a consequence of the similarity between the form of the coefficients in the number state expansions for the coherent phase state and the two-mode squeezed vacuum.

Investigation of $P(\theta)$ of equation (5.18) for different values of M shows that for the values $0 \le |Y| \le 0.99$, $1.01 \le |Y| \le 20$ and $-\pi \le \theta \le \pi$, $P(\theta)$ is almost constant (about $\frac{1}{2\pi}$) for $|Y| \le 0.2$ whatever the value of M, which means that the vacuum is the dominant state in this case (compare with the discussion of the Wigner function for small values of $|Y|$). However, as $|Y|$ increases, a peak is produced at around $\theta = 0$. This peak sharpens as M increases, tending towards a delta function as $M \to \infty, Y \to 1$ (the phase states). As $|Y|$ increases and takes values greater than unity, this peak gradually disappears.

Finally it settles to a constant value (about $\frac{1}{2\pi}$) as $Y > 10$ (it tends to the Fock state $|M>$ as $|Y| \to \infty$, and hence phase information is lost). Discussion of the fluctuations in $< \Delta\hat{\phi}_\theta^2 >$ and $< \Delta\hat{n}^2 >$ for different values of M show that [37], as M increases, the fluctuations in n are increased near $|Y| = 1$. It is to be noted that both $< \Delta\hat{n}^2 >$or $< \Delta\hat{\phi}_\theta^2 >$ go to zero as $|Y| \to 0$ or $|Y| = 1$ and $M \to \infty$. This is expected, since $|Y| = 0$ is the vacuum state and $|Y| \to 1$ with $M \to \infty$ is essentially the phase state $|\theta_m >$ and, as has been reported, the expectation value of the commutator $[\hat{n}, \hat{\phi}]$ vanishes for both these states (see [37] for details).

5.5. **Production schemes.** In this subsection we present two schemes for production of such states. first one depends on a generalized Jaynes Cummings model while the second relies on the $SU(1,1)$ algebra. Generalizations to the JC model that include nonlinear interactions (in boson and spin variables) have been proposed recently. An interaction Hamiltonian for one of these generalizations that describes multiphoton processes in finite-level atomic systems is of the form

$$H_{int} = \sum_{j=1}^{2r} \frac{g_j}{j!}\{(\hat{a}S_+)^j + (\hat{a}^\dagger S_-)^j\}, \tag{5.20}$$

where S_3, S_+, and S_- are the inversion, raising, and lowering operators which describe the atomic systems having $(2S+1)$ states. They satisfy the commutation relations $[S_3, S_\pm] = \pm S_\pm$ and S_3 has the eigenvalues m such that $S_3|m >= m|m>$ where $-S \leq m \leq S$. The field operators $\hat{a}$ and $\hat{a}^\dagger$ satisfy the usual commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. The coupling constants g_j couple the atomic system to the field; finally $r \leq S$. This Hamiltonian produces the JC model for $r = \frac{1}{2}$, $S = \frac{1}{2}$. When $r = \frac{1}{2}$, and $S = \frac{3}{2}$ it gives the model discussed by Senitzky [39]. Several values for S were investigated by Buck and Sukumar [40]. The case for general S and $r = \frac{1}{2}$ is the well-known Dicke model and the Tavis-Cummings model [41]of cooperative two-level atoms. Taking $r = 1$, and $S = 1$, equation (5.20) describes a three-level atom in interaction with a single mode in which transitions between neighboring levels are effected by single-photon processes, while the transition between the upper and lower levels is effected through a two-photon process; which is a special case of a Hamiltonian considered for the three-level atom system. Thus we may say the that interation model (5.20) then represents a $(2r+1)$-level atom interacting with one mode of the radiation field where one-photon transitions occur between neighboring levels , two-photon transitions occur between levels as indicated i and $i+2$, $i < 2r+1$and so on, where $2r$-photon transitions occur between the two extreme levels of the atom. The model (5.20) can be used to describe some processes such as multiphonon transitions in a two-level atomic system, multiphonon lasers, and Ramann and hyper-Ramann processes.

The operators $S_{\pm}^{j}$ when applied to the state vector $|m\rangle$, for $r = S$, give

$$S_{+}^{j}|m>= \{\frac{(S-m)!(S+m+j)!}{(S-m-j)!(S+m)!}\}^{\frac{1}{2}}|m+j\rangle,$$

$$S_{-}^{j}|m>= \{\frac{(S-m+j)!(S+m)!}{(S-m)!(S+m-j)!}\}^{\frac{1}{2}}|m-j\rangle. \tag{5.21}$$

We assume that the system is evolving under the Hamiltonian (5.20) from the initial state

$$|\psi(0)\rangle = \sum_{m=-S}^{S} \left[\begin{array}{c} 2S \\ m+S \end{array} \right]^{\frac{1}{2}} \frac{\tau^{m+S}}{(1+|\tau|^2)^S}|m\rangle|0\rangle_{ph}, \tag{5.22}$$

where $|0\rangle_{ph}$ is the vacuum state for the field, while $\tau = \tan(\vartheta/2)e^{i\phi}$, as the parameter for the atomic coherent state. For a short time t (i.e., $g_j t \leq 1$), the wave function of the system becomes

$$|\psi(t)\rangle = |\psi(0)\rangle - it\sum_{j=1}^{2r}\frac{g_j}{j!}\{(\hat{a}S_{+})^j + (\hat{a}^{\dagger}S_{-})^j\}|\psi(0)\rangle. \tag{5.23}$$

Using the equations (5.21) and (5.22) we get the following expression

$$|\psi(t)\rangle = |\psi(0)\rangle - it\sum_{j=1}^{2r}\frac{g_j}{\sqrt{j!}}|j\rangle_{ph}\sum_{m=-S}^{S}\frac{\tau^{m+S}}{(1+|\tau|^2)^S}\frac{2S!}{(S-m)!}\times$$
$$[(S+m)!(S+m-2j)!]^{-\frac{1}{2}}|m-j\rangle. \tag{5.24}$$

Suppose at time t the atom is measured to be in its ground state $|-S\rangle$, then the state of the field is given by

$$|\psi_f(t)\rangle \cong \lambda_0|0\rangle - it\sum_{j=1}^{2r}\sqrt{j!}g_j\frac{\tau^j}{(1+|\tau|^2)^S}\left[\begin{array}{c} 2S \\ j \end{array} \right]|j\rangle, \tag{5.25}$$

where $\lambda_0 = (\cos\vartheta/2)^{2S}$. By making the coupling constants $g_j \alpha (2S-j)!\sqrt{j!}Y^{j/2}$ and $2r = M$ (i.e.,$(M+1)$-level atom) the field state is then the $|Y, M>$ of equation (5.1).

To produce the generalized geometric states using the Lie algebra approach, one can see that the coherent phase state bears a close relationship to the $SU(1,1)$ coherent states which are important in the theory of squeezing. The $SU(1,1)$ coherent states$|\xi>$ are defined in terms of the action of the operators $\hat{K}_+, \hat{K}_-$, and $\hat{K}_3$ obeying the commutation relations

(5.26a) $$[\hat{K}_+, \hat{K}_-] = 2\hat{K}_3, \qquad [\hat{K}_3, \hat{K}_\pm] = \pm\hat{K}_\pm$$

The single-mode squeezed states result if we let $\hat{K}_- = \hat{a}^2/2$, with $\hat{K}_+$ being its Hermitian conjugate, by the action of the unitary operator $\exp(\xi\hat{K}_+ - \xi^*\hat{K}_-)$ on the vacuum state $|0\rangle$. An alternative representation of the $SU(1,1)$ algebra for the operators acting on a single field mode is given by

(5.26b) $$\hat{A} = \hat{a}\hat{n}^{\frac{1}{2}}, \qquad \hat{A}^\dagger = \hat{n}^{\frac{1}{2}}\hat{a}^\dagger$$

representating $\hat{K}_-$ and $\hat{K}_+$ respectively with the representation completed by the operator $\hat{n} + \frac{1}{2}$ as $\hat{K}_3$. An $SU(1,1)$ coherent state constructed using these operators has the form

(5.27) $$|\xi\rangle = \exp(\xi\hat{A}^\dagger - \xi^*\hat{A})|0\rangle,$$

which on using the disentangling theorem gives

(5.27a) $$\exp(\xi\hat{A}^\dagger - \xi^*\hat{A}) = \exp\{[e^{i\eta}\tanh r]\hat{A}^\dagger\}\times \exp\{-2[\ln(\cosh r)](\hat{n} + \frac{1}{2})\}\exp\{[e^{-i\eta}\tanh r]\hat{A}\}.$$

With $\xi = r\exp(i\eta)$ this gives

(5.27b) $$|\xi\rangle = (\cosh r)^{-1}\exp\{[e^{i\eta}\tanh r]\hat{A}^\dagger\}|0\rangle = (\cosh r)\sum_{n=0}^{\infty}\frac{[e^{i\eta}\tanh r]^n}{\sqrt{n!}}|n\rangle$$

If we write $Y = exp(i\eta)\tanh r$, then we see that this is simply the coherent phase state or the generalized geometric state $|Y, \infty\rangle$. In principle, this state could be produced by an interaction that involves an intensity-dependent coupling.

5.6. **Even geometric states.** Now we shall consider the properties of a normalized superposition of the two generalized geometric states$|Y, M\rangle$ and $|-Y, M\rangle$ in the form

$$|\psi_e(Y, M)\rangle = \lambda(|Y, M\rangle + |-Y, M\rangle) = \left(\frac{1-|Y|^4}{1-|Y|^{4([M/2]+1)}}\right)^{\frac{1}{2}} \sum_{n=0}^{[M/2]} Y^{2n}|2n\rangle, \tag{5.28}$$

where $[x]$ denotes the largest integer not exceeding x. We refer to these states as the even geometric states by analogy with the terminology applied to superposition of coherent states [42].

With this definition for the even generalized geometric state we can calculate some statistical properties of the field.

$$\begin{aligned} \langle\hat{n}\rangle_e &= \langle\psi_e|\hat{a}^\dagger\hat{a}|\psi_e\rangle \\ &= \left(\frac{1-|Y|^4 \sum_{n=0}^{[M/2]} 2n|Y|^{4n}}{1-|Y|^{4([M/2]+1)}}\right) \\ &= \frac{2|Y|^4\{[M/2]|Y|^{4([M/2]+1)} - ([M/2]+1)|Y|^{4([M/2]+1)}\}}{(1-|Y|^4)(1-|Y|^{4([M/2]+1)})} \end{aligned} \tag{5.29}$$

and

$$\langle\hat{a}^{\dagger 2}\hat{a}^2\rangle_e = \frac{1-|Y|^4}{1-|Y|^{4([M/2]+1)}} \sum_{n=0}^{[M/2]} 2n(2n-1)|Y|^{4n}. \tag{5.30}$$

The expectation value for the field operator $\hat{a}^{2s}$ is given by

$$\langle\psi_e|\hat{a}^{2s}|\psi_e\rangle = \frac{1-|Y|^4}{1-|Y|^{4([M/2]+1)}}(Y^*)^{-2s} \sum_{m=s}^{[M/2]} |Y|^{4m} \left(\frac{2m!}{(2m-2s)!}\right)^{\frac{1}{2}} \tag{5.31}$$

while the expectation values for odd powers vanish.

It is to be expected that this state would give stronger squeezing than the generalized geometric state. For the state (5.28) we calculate the quasiprobability function, and find that

$$(5.32)\quad W^{(e)}(\beta) = \frac{2}{\pi}\exp(-|\beta|^2)\frac{1-|Y|^4}{1-|Y|^{4([M/2]+1)}} \times [\sum_{n=0}^{[M/2]} |Y|^{4n} L_{2n}(4|\beta|^2) + 2\sum_{\substack{m,n\\ m>n}}^{m-n} (4|\beta|^2)(\frac{2n!}{2m!})\frac{1}{2}|Y|^{2(m+n)} \times \cos[2(m-n)(\gamma-\phi)]L_{2n}^{2(m-n)}(4|\beta|^2)]$$

with $Y = |Y|\exp(i\gamma)$,and $\beta = |\beta|\exp(i\phi)$. The investigation of the Wigner function of equation (5.32) for small values of $|Y|(< 0.2)$ does not show any structure to the Gaussian shape apart from a slight anisotropy similar to that of the squeezed states. The effect of the vacuum state is predominant, while the effect of the higher excitations is not pronounced for this case. As $|Y|$ increases, the effects of the higher photon numbers again lead to negative regions. For $|Y| = 2$ we find a complicated structure for the Wigner function, but an enhanced probability for two phases corresponding to opposite directions in the β plane is discernible. We consider the phase properties of this state in a manner analogous to that considered above. Consequently we find that

$$(5.33)\quad P^{(e)}(\theta) = \frac{1}{2\pi}\frac{1-|Y|^4}{1-|Y|^{4([M/2]+1)}} \times \frac{1-|Y|^{4([M/2]+1)}\cos\{2([M/2]+1)\theta\} + |Y|^{4[M/2]+1}}{1-2|Y|^2\cos 2\theta + |Y|^4},$$

while the phase fluctuations in this case are given by

$$(5.34)\quad \langle(\Delta\phi_\theta)^2\rangle = \frac{\pi^2}{3} + \frac{1}{1-|Y|^{4([M/2]+1)}}\sum_{m=1}^{[M/2]}\frac{1}{m^2}\left(|Y|^{2m} - \frac{|Y|^{4([M/2]+1)}}{|Y|^{2m}}\right) \rightarrow \frac{\pi^2}{3} + dilog(1-|Y|^2) \quad as \quad M\rightarrow\infty, \quad |Y|<1.$$

The phase properties for the even geometric state are considered in [37] where the function $P(\theta)$ of equation (5.33) is plotted for different values of M and in the ranges $0 \le |Y| \le 0.99$ and $1.01 < |Y| \le 20$ for $-\pi < \theta < \pi$. The appearance of peaks at $\theta = 0$ and also at $\theta = \pi$ and $\theta = -\pi$. The phase probability distribution is of course 2π interval or window; there are really two peaks. This is reminiscent of the phase distribution found for the squeezed vacuum states [43]. For both the even geometric states and the squeezed vacuum the π periodicity of the phase probability distribution arises from the absence of odd photon number in the expansion of the state. The limits of both large and small $|Y|$ are again number states and this is reflected in the asymptotic form for $P^{(2)}(\theta)$.

6. THE EVEN AND THE ODD NEGATIVE BINOMIAL STATES

Here we shall discuss another type of intermediate states called negative binomial state [5, 9, 10]. This state is always super-Poissonian no matter what parameters one chooses. The interesting property of such state is that under two different limiting conditions it reduces to quasi-thermal or chorennt state. There is also the special case of a negative binomial state with the removal of the $n = 0$ term, known as the logarithmic state, which exhibits some non-classical characteristics. In the following we shall concentrate on the even (odd) negative binomial state, which in fact interpolates between the even (odd) coherent state and the even (odd) quasi-thermal state [9].

6.1. **Definition and properties.** The negative binomial state is defined as [45]

$$|w, q\rangle = \sum_{n=0}^{\infty} \left[\frac{(n+w)!}{n!w!} q^{2n} (1 - |q|^2)^{w+1} \right]^{\frac{1}{2}} |n\rangle \tag{6.1}$$

where $w > 0$ (i.e., w is any real positive number in general), $0 \leq |q|^2 \leq 1$. The mean and the variance of the photon number distribution corresponding to the state (6.1) is given by

$$\langle \hat{n} \rangle = (w+1) \frac{|q|^2}{1 - |q|^2}, \qquad \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = (w+1) \frac{|q|^2}{(1 - |q|^2)^2} \tag{6.2}$$

The state (6.1) reduces to a coherent state in the limiting condition $w \to \infty$, $|q|^2 \to 0$, such that $\langle \hat{n} \rangle \equiv (w+1)|q|^2/(1 - |q|^2)$ is kept constant. However, in another limiting condition, i.e. $w = 0$, the state (6.1) represents a quasi-thermal state which is a pure state having photon number distribution identical to the usual thermal (chaotic) state (mixed state) with the same mean and variance.

We define a general superposition of two negative binomial states as follows

$$|\Psi_g\rangle = N_g[|w,q\rangle + \mu e^{i\psi}|w, qe^{i\phi}\rangle] \tag{6.3}$$

where μ is a real number and N_g is the normalization constant obtained by taking the norm of (6.3), and is given by

$$[N_g^2]^{-1} = 2\,\mathrm{Re}[(1-|q|^2)^{w+1}\{(1-|q|^2)^{-(w+1)}+\mu e^{-i\psi}(1-|q|^2e^{i\phi})^{-(w+1)}\}] \tag{6.3a}$$

For $\mu = 0, |\Psi_g\rangle \rightarrow |w,q\rangle$ and for $\mu = \infty, |\Psi_g\rangle \rightarrow |qe^{i\phi}, w\rangle$. The state (6.3) represents an even negative binomial state $|\Psi_e\rangle$ when $\mu = 1, \psi = 0, \phi = \pi$; an odd negative binomial state $|\Psi_o\rangle$ when $\mu = 1, \psi = \pi, \phi = \pi$; and an oblique negative binomial state $|\Psi_p\rangle$ when $\mu = 1, \psi = 0, \phi = \pi/2$; The normalization constants represented by N_e, N_o, and N_p for the even, odd and oblique negative binomial states respectively are given by the following expressions:

$$\begin{aligned}
[N_e^2]^{-1} &= 2\left[1 + \left(\frac{(1-|q|^2)}{(1+|q|^2)}\right)^{(w+1)}\right] \\
[N_o^2]^{-1} &= 2\left[1 - \left(\frac{(1-|q|^2)}{(1+|q|^2)}\right)^{(w+1)}\right] \\
[N_p^2]^{-1} &= 2\,\mathrm{Re}\left[1 + \left(\frac{(1-|q|^2)}{(1-i|q|^2)}\right)^{(w+1)}\right]
\end{aligned} \tag{6.3b}$$

It is quite straightforward to show that the states $|\Psi_e\rangle$, $|\Psi_o\rangle$, and $|\Psi_p\rangle$ reduce to even, odd, and oblique coherent states respectively in the limiting condition of parameters: $w \rightarrow \infty, |q|^2 \rightarrow 0$, such that $\langle\hat{n}\rangle \equiv (w+1)|q|^2/(1-|q|^2)$ is unchanged. In the other limiting odd and oblique thermal (pure) states. We shall now make use of the states $|\Psi_e\rangle$, $|\Psi_o\rangle$, and $|\Psi_p\rangle$ to calculate the mean photon number of these states. The mean photon number is defined as the expectation value of the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$. It is easy to show that the expression of the mean photon number $\langle\hat{n}\rangle_e, \langle\hat{n}\rangle_o, \langle\hat{n}\rangle_p$ for the even, odd and oblique negative binomial states are

$$
(6.4) \qquad
\begin{aligned}
\langle\hat{n}\rangle_e &= \frac{(w+1)|q|^2[(1-|q|^2)^{-(w+2)} - (1+|q|^2)^{-(w+2)}]}{[(1-|q|^2)^{-(w+1)} + (1+|q|^2)^{-(w+1)}]} \\
\langle\hat{n}\rangle_o &= \frac{(w+1)|q|^2[(1-|q|^2)^{-(w+2)} + (1+|q|^2)^{-(w+2)}]}{[(1-|q|^2)^{-(w+1)} - (1+|q|^2)^{-(w+1)}]} \\
\langle\hat{n}\rangle_p &= \frac{(w+1)|q|^2 \,\mathrm{Re}[(1-|q|^2)^{-(w+2)} + i(1-i|q|^2)^{-(w+2)}]}{\mathrm{Re}[(1-|q|^2)^{-(w+1)} + (1-i|q|^2)^{-(w+1)}]}
\end{aligned}
$$

Again, we obtain $\langle\hat{n}\rangle_e = \langle\hat{n}\rangle \tanh(\langle\hat{n}\rangle), \langle\hat{n}\rangle_o = \langle\hat{n}\rangle \coth(\langle\hat{n}\rangle)$ in the coherent state limit, which as a matter of fact are precisely the expressions of mean photon numbers for the even and odd coherent states respectively [17].

If we examine the second order correlation equation (3.9) aganist the above states, and we find the following expressions

$$
(6.5) \quad
\begin{aligned}
g_e^{(2)}(0) &= \frac{(w+2)}{(w+1)}\left[1 + \frac{(1-|q|^4)^{-(w+1)}\{(1+|q|^2)^{-1} + (1-|q|^2)^{-1}\}^2}{[(1-|q|^2)^{-(w+2)} - (1+|q|^2)^{-(w+2)}]^2}\right] \\
g_o^{(2)}(0) &= \frac{(w+2)}{(w+1)}\left[1 + \frac{(1-|q|^4)^{-(w+1)}\{(1+|q|^2)^{-1} - (1-|q|^2)^{-1}\}^2}{[(1-|q|^2)^{-(w+2)} + (1+|q|^2)^{-(w+2)}]^2}\right] \\
g_p^{(2)}(0) &= \frac{\mathrm{Re}\{(w+1)(w+2)|q|^4[(1-|q|^2)^{-(w+3)} - (1-i|q|^2)^{-(w+3)}]\}}{2(1-|q|^2)^{(w+1)} N_p^2 \langle\hat{n}\rangle_p^2}
\end{aligned}
$$

In the coherent state limit of the parameters w and $|q|^2$, we can recover the expression of the second-order coherence function of even, odd, and oblique coherent states. In fact, in this limit we obtain from (6.5) $g_e^{(2)}(0) = \coth^2(\langle\hat{n}\rangle)$ and $g_o^{(2)}(0) = \tanh^2(\langle\hat{n}\rangle)$, which are exactly (63) and (64) of [17].

Discussion of the second-order coherence function (6.5) is considered in [17] as a function of parameter $q^2 = |q|^2, (0 \le q^2 < 1)$ of even and odd negative binomial states. For the lower value of $g^{(2)}(0)$, the odd state exhibits sub-Poissonian behavior for most of the values of q^2, but the even state always shows super-Poissonian behavior for the complete range of q^2 irrespective of the value of $\langle\hat{n}\rangle$, as could be noted from $g_e^{(2)}(0)$. With increasing $\langle\hat{n}\rangle$ the difference between the values of $g_e^{(2)}(0)$ and $g_o^{(2)}(0)$ reduces considerably, especially at low values of q^2 and their values remain around one. Physically, it means that the non-classical nature of at least one of these states is apparent when the photon occupation number is very small. Nevertheless, the second order coherence function for these states start behaving like ordinary coherent states for the high photon occupation number $\langle \hat{n} \rangle$ (up to a certain range of the parameter q^2). Further it can be also verified that the nature of $g_p^{(2)}(0)$ as a function of q^2 does not show any sub-Poissonian behavior irrespective of any mean photon number $\langle \hat{n} \rangle$ values.

Before proceeding further we would like to give the operator representation of the even and odd negative binomial states in terms of the so-called even and odd thermal states, e.g.

$$\rho_{even}^{nb} = \frac{4N_e^2}{w_e!} \frac{(1-|q|^2)^{(w_e+1)}}{(1-|q|^4)} (|q|^2)^{-w_e} (\hat{a})^{w_e} (\rho_{even}^{th}) (\hat{a}^\dagger)^{w_e} \tag{6.6}$$

when $w \equiv w_e$ is even;

$$\rho_{even}^{nb} = \frac{4N_e^2}{w_o!} \frac{(1-|q|^2)^{(w_o+1)}}{(1-|q|^4)} (|q|^2)^{-(w_o-1)} (\hat{a})^{w_o} (\rho_{odd}^{th}) (\hat{a}^\dagger)^{w_o} \tag{6.7}$$

when $w \equiv w_o$ is odd;

$$\rho_{odd}^{nb} = \frac{4N_o^2}{w_e!} \frac{(1-|q|^2)^{(w_e+1)}}{(1-|q|^4)} (|q|^2)^{-(w_e-1)} (\hat{a})^{w_e} (\rho_{odd}^{th}) (\hat{a}^\dagger)^{w_e} \tag{6.8}$$

when $w \equiv w_e$ is even;

$$\rho_{odd}^{nb} = \frac{4N_o^2}{w_o!} \frac{(1-|q|^2)^{(w_o+1)}}{(1-|q|^4)} (|q|^2)^{-w_o} (a)^{w_o} (\rho_{even}^{th}) (a^\dagger)^{w_o} \tag{6.9}$$

when $w \equiv w_e$ is odd; where $\rho_{even}^{nb}(\rho_{odd}^{nb})$ and $\rho_{even}^{th}(\rho_{odd}^{th})$ are the denisity operators for the even (odd) negative binomial states and even (odd) thermal states respectively. For example, the expression of ρ_{even}^{th} reads as

$$\rho_{even}^{th} = \sum_{n=0}^{\infty} [1-(q^2)^2](q^2)^{2n} |2n\rangle\langle 2n|. \tag{6.10}$$

In equation (6.6-6.9), we have taken only the diagonal terms in the density matrix into account to make their connection with the thermal density of the non-pure state.

6.2. **Squeezing.** To study the squeezing properties of the even and odd negative binomial states, we have to calculate the quadrature variances for the operators of the single mode field which is given by equations (3.11). It is noted that for the present case we always find $\langle X\rangle = 0$, this means that the quadrature variances depends only on $\langle X^2\rangle$. On the contrary we find for the oblique state $\langle X^2\rangle \neq 0$, therefore for both even and odd states we have

$$S_1 = (\Delta X_1)^2 - \frac{1}{4} = \frac{1}{4}(2\hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2} + \hat{a}^2) \tag{6.11}$$

$$S_2 = (\Delta X_2)^2 - \frac{1}{4} = \frac{1}{4}(2\hat{a}^\dagger\hat{a} - \hat{a}^{\dagger 2} - \hat{a}^2) \tag{6.12}$$

In order to calculate normal squeezing, we need to know the expectation values of the even powers of the operators $\hat{a}^\dagger(\hat{a})$ for both even and odd negative binomial states. These expectation values are of the form

$$\langle\Psi_e|\hat{a}^s|\Psi_e\rangle = \frac{(1-|q|^2)^{(w+1)}}{N_e^2}\times \sum_n \frac{[(n+2s+w)!(n+w)!]^{\frac{1}{2}}}{n!w!}|q|^{2n}q^{*2s}[1+(-1)^n] \tag{6.13}$$

for the even state,

$$\langle\Psi_o|\hat{a}^s|\Psi_o\rangle = \frac{(1-|q|^2)^{(w+1)}}{N_o^2}\times \sum_n \frac{[(n+2s+w)!(n+w)!]^{\frac{1}{2}}}{n!w!}|q|^{2n}q^{*2s}[1-(-1)^n] \tag{6.14}$$

for the odd state.

Similarly, the expectation values of $\langle\hat{a}^{2s}\rangle$ can be calculated. We make use of these expressions together with the equations (6.11) and (6.12) to calculate the in-phase squeezing S_1 for both even and odd negative binomial states. For example when we define $q = |q|e^{i\alpha}$ one can obserave no squeezing for $\alpha = 0$.

However if we set $\alpha = \pi/2$ and plot S_1 as a function of the parameter $q^2 = |q|^2$ for $\langle n \rangle = 2$ and 4 respectivily, then one can see from these figures that both even and odd negative binomial states do exhibit squeezing. However, the depth of squeezing and the range of q^2 over which squeezing is observed are very sensitive to the value of $\langle \hat{n} \rangle$, and they decrease with increasing $\langle \hat{n} \rangle$ for the even state, for example. Incidentally, similar behavior has also been observed in the even and odd coherent states. Also, as $\langle \hat{n} \rangle$ increases, the disparity of the squeezing values of even and odd states (as a function of q^2) up to $q^2 = 0.6$ or so diminishes.

6.3. **Wigner function.** Here we study the Wigner function for even, odd and oblique negative binomial states. As well known by studying the Wigner function one can characterize the non-classical nature of the field states.

To find the Wigner function we have to calculate the integral in equation (2.10). Therefore if we insert equation (2.9) into equation (2.10) taking into account the density matrix $\hat{\rho}$ to be $\hat{\rho} = |\Psi_e\rangle\langle\Psi_e|, \hat{\rho} = |\Psi_o\rangle\langle\Psi_o|$ and $\hat{\rho} = |\Psi_p\rangle\langle\Psi_p|$ corresponding to the even, odd, and oblique states respectively, then we obtain

$$W(\beta)|_{even} = \frac{8}{\pi} N_e^2 (1-|q|^2)^{(w+1)} \sum_n \frac{(2n+w)!}{2n!w!} (|q|^2)^{2n} e^{-2|\beta|^2} L_{2n}(4|\beta|^2) \tag{6.15}$$

$$W(\beta)|_{odd} = \frac{8}{\pi} N_o^2 (1-|q|^2)^{(w+1)} \times \sum_n \frac{(2n+w+1)!}{(2n+1)!w!} (-|q|^2)^{2n+1} e^{-2|\beta|^2} L_{2n+1}(4|\beta|^2) \tag{6.16}$$

$$W(\beta)|_{oblique} = \frac{4}{\pi} N_p^2 (1-|q|^2)^{(w+1)} \times \sum_n \frac{(n+w)!}{n!w!} (-|q|^2)^{n} e^{-2|\beta|^2} [1+\cos(n\pi/2)] L_{n}(4|\beta|^2) \tag{6.17}$$

In obtaining the expressions of the Wigner function (as above) we have assumed phase averaging such that only the diagonal terms of the density operator take part in determining these expressions. In other words we are examining a special case (known as mixed states in the literature because of phase averaging) of the pure states defined above. Note that by invoking phase averaging we are able to establish relationships between the density operators of even (odd) negative binomial states and even (odd) thermal states (see (6.6-6.9) above). The equivalence for the Wigner function has precisely motivated us to carry out the study of the phased averaged Wigner function. As we will see such phased averaged Wigner function is a special case of the more generalized Wigner function which interpolates between the Wigner function of the even (odd) thermal state and the phased averaged even (odd) coherent state. We will also see that the phase averaged Wigner function can be put in a closed form.

The Wigner functions for even, odd and oblique negative binomial states are considered for different values of parameters w and $q^2(=|q|^2)$ (keeping $\langle\hat{n}\rangle = (w+1)q^2/(1-q^2)$ unchanged). The Wigner function of the even (odd, oblique) negative binomial state reduces to the Wigner function of an even (odd, oblique) coherent state or an even (odd, oblique) thermal state under the two different limiting conditions of parameters. For example the Wigner function of the even negative binomial state starts with a central peak with a crater-like structure around it near the coherent limit. The depth of the crater reduces considerably as one moves towards the thermal limit. The situation for the odd negative binomial state looks inverted in comparison to the even negative binomial state (note the minus sign appearing outside the formula (6.17)), because the Wigner function of this state starts with an inverted peak surrounded by an inverted crater near its coherent limit and the crater height reduces as one moves towards the thermal limit. The evolution of the Wigner function for the oblique negative binomial state is very intersting. It starts with a doubly folded inverted peak near the coherent state limit; as the parameters are changed towards the thermal limit, one of them starts to unfold towards the positive side and this unfolds.

Since we have used phase averaging of the density operator in deriving the expressions of the Wigner function, they can conveniently be put in a closed form:

$$W(\beta)|_{oblique} = \frac{4}{\pi} N_e^2 (1-|q|^2)^{(w+1)} e^{-2|\beta|^2} [(1+|q|^2)^{-(w+1)} \times {}_1F_1\left(w+1, 1, \frac{4|\beta|^2|q|^2}{1+|q|^2}\right) + \mathrm{Re}(1+i|q|^2)_1^{-(w+1)} \times {}_1F_1\left(w+1, 1, \frac{4i|\beta|^2|q|^2}{1+i|q|^2}\right) \tag{6.18}$$

where ${}_1F_1(a, b; z)$ is the confluent hypergeometric function

$${}_1F_1(a; b; z) = \sum_{r=0} \frac{(a)_r}{(b)_r} \frac{z^r}{r!}. \tag{6.19}$$

When we take the complete density matrix for even, odd, and oblique negative binomial states in calculating the Wigner function, we have to take into account the non-diagonal terms also. In that case the expression of the Wigner function after some simplifications takes the following form

$$W(\beta)|_{oblique} = \frac{1}{\pi}\Big[\sum_n 2(-1)^n e^{-2|\beta|^2} L_n(4|\beta|^2) \rho_q(n,n) + \sum_{m=1} \sum_n 2(-1)^n e^{-2|\beta|^2} \sqrt{\{n!/(m+n)!\}} \times L_n^m(4|\beta|^2)\{[2\beta^*]^m \rho_q(m+n,n) + [2\beta]^m \rho_q(n,n+m)\}\Big] \tag{6.20}$$

where the subscript q stands for even, odd, and oblique negative binomial states respectively. $\rho_q(n,n)$ is the diagonal element of the density operator while $\rho_q(m+n,n)$ or $\rho_q(n,m+n)$ represents the off-diagonal element of the density operator. The definition of the density operator ρ_e, ρ_o, ρ_p for even, odd and oblique negative binomial states respectively has been mentioned earlier. Thus by inserting the appropriate density operator of the state (ρ_e, ρ_o, or ρ_p) in the above equation one can obtain the corresponding Wigner function (W_e, W_o or W_p) of that state.

The Wigner function (as defined by (6.20)) represents the even and the odd coherent state ($\overline{n} = 4, w = 75, q^2 = 0.05$) in phase space. This distribution is a real function which can take negative values in limited regions of the phase space. If we plot the function then one can see two Gaussian peaks are associated with the $|\beta\rangle$ and $|-\beta\rangle(\beta = 2)$ coherent fields and the oscillatory part in between is directly related to the coherence between them. Furthermore we can also observe existence of the oscillations associated with the negative quasi-probabilities values, which is a signature of the non-classical state. Such a state is very different from a statistical mixture involving the same coherent field parts for which the Wigner distribution merely exhibits two separated peaks without superimposed oscillations. In this case of even and odd coherent states the two Gaussian peaks are well separated and compact. Note that the oscillatory part is very much different from each other in two cases. The width of the Gaussian peak in this case is associated with the variance of the coherent field $\sim \sqrt{\overline{n}}$. If we move over to a typical even (odd) negative binomial state with $\overline{n} = 4$ $(w = 5, q^2 = 0.4$), the Wigner function still shows two Gaussian-like peaks and the oscillations in between these peaks. However, the widths of these Gaussian-like peaks are much broader as compared with the coherent state case. This could be because the variance of the negative binomial state ($\sim \overline{n}/(1-q^2)$) is more (super-Poissonian) than the coherent state. The interference between the two negative binomial states is modified considerably, as compared with the coherent state case, because of the change in the statistics or photon number distribution of these states. There is a reduction in the non-classical character of the even state because it is becoming less negative in its quasi-probability distribution. Hence the coherence or quantum interferences are sensitive to the statistics of the constituent states of the "Shrödinger cat". In other words the quasi-probability function for the even (odd) negative binomial state is both qualitatively, as well as quantitatively, different from the even (odd) coherent state.

Finally, we move to the other extreme of the negative binomial state known as the quasi-thermal state (pure state) by setting parameters $w = 0$, $q^2 = 0.8$, $\overline{n} = 4$. In this case it is clear the Gaussian-like peaks have undergone a considerable amount of broadening such that we see only one prominent peak (positive in the even state and negative in the odd state) in the Wigner function of these states along with a few oscillations superimposed on them. Note that the non-classical character of the even quasi-thermal state is reduced considerably in comparison with the odd quasi-thermal state. Thus as we move from the coherent "Shrödinger cat" to a quasi-thermal "Shrödinger cat" we find a reduced non-classical character in the even state and a reduced "coherent" (i.e. less oscillatory behavior) in general for both the even and odd states.

In the coherent limit ($w = 75$, $q^2 = 0.05$, $\overline{n} = 4$) one may observe three Gaussian-like peaks in the quasi-probability distribution. Two of them are small but equal peaks (oblique peaks) symmetrically located opposite to each other. In the meantime the third peak is bigger in height and is located at right angles to the line joining the small peaks. In between the small and big peaks one can observe oscillations which are manifestations of the coherence between the oblique states. For a typically negative binomial state $\left(w = 5,\ q^2 = 0.4,\ \overline{n} = 4\right)$ all the peaks become broader and the oscillations in between them are reduced. Finally in the quasi-thermal limit $\left(w = 0,\ q^2 = 0.8,\ \overline{n} = 4\right)$ we may observe the third peak has overtaken the other two peaks in height as well as in width and the oscillations become less and less prominent. Clearly, here also (as in the even state) the changes in the statistics from Poissonian to super-Poissonian in the component states taking part in the "Schrödinger cat"-like superposition brings a reduction in the non-classical character as well as coherence between them.

6.4. **Possibilities of generating even (odd) negative binomial states.** In the following we shall show the possibility of generating even and odd negative binomial states of the field. For that we first consider the production of an ordinary negative binomial state in the process of parametric amplification by a proper choice of the initial conditions. We consider $SU(1,1)$ coherent states for this purpose. These states are defined by [47] as

$$|\zeta\rangle = (1-|\zeta|^2)^k \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k)}\right]^{\frac{1}{2}} \zeta^n |k, n+k\rangle. \tag{6.21}$$

$|k, n+k\rangle \equiv |k\rangle|n+k\rangle$ represents a two-mode Fock state, i.e. eigenstate of the operator $\hat{a}_1\hat{a}_2$, where $\hat{a}_1$ and $\hat{a}_2$ represent annihilation operators of the two different modes of the electromagnetic field.

The diagonal elements of (6.21) have distributions given by

$$P(n) = (1-|\zeta|^2)^{2k} \frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k)} |\zeta|^{2n} \tag{6.22}$$

which is the same as the negative binomial distribution with $s = 2k - 1$. The $SU(1,1)$ algebra can be realized in terms of two modes $\hat{a}$ and $\hat{b}$ of the field, i.e. $K_+ = \hat{a}^\dagger \hat{b}^\dagger$, $K_- = \hat{a}\hat{b}$, $K_3 = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1)$. So, in terms of the Fock states $|n, m\rangle$of the two-mode radiation field the parameter k equals $(m - n)$ and (6.21) can be written as

$$(6.23) \qquad |\zeta\rangle = (1 - |\zeta|^2)^{(1+s)/2} \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+s+1)}{\Gamma(n+1)\Gamma(s+1)} \right]^{\frac{1}{2}} \zeta^n |n+s, s\rangle$$

or

$$(6.24) \qquad |\zeta\rangle = \exp(\gamma[\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b}]) \frac{\hat{a}^{\dagger s}}{\sqrt{s!}} |0, 0\rangle, \qquad \zeta = \tanh \gamma$$

So the state $|\zeta\rangle$ is essentially the negative binomial state of a two-mode radiation field in which the probability of finding n signal photons obeys the negative binomial distribution. The interaction Hamiltonian of parametric amplification is $i(\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b})$ and it would produce the state (6.24) provided the input to the amplifier is such that the difference between the idler and signal photons is n.

The so-obtained negative binomial state can be used to produce the even and odd negative binomial state by the method of Yurke and Stoler [48] employed for producing superposition of coherent states. For this purpose, we have to allow our ordinary negative binomial state field to propagate through an amplitude dispersive medium. Under suitable conditions we can obtain a quantum superposition of two negative binomial states.

Another method of generating the even (odd) negative binomial state is by allowing the ordinary negative binomial state to interact with a Kerr-like medium in one arm of the Mech-Zehdner interferometer and a resonant two-level atom crossing one of the output ports of the same interferometer. This methodology was reported recently for the possibility of generating even and odd coherent states [49]. Since the negative binomial state represents a coherent state under one of the limiting conditions of its parameters, hence for certain values of parameters one can adopt this method for generating the even (odd) negative binomial states.

7. APPLICATIONS TO DYNAMICAL SYSTEMS

7.1. **The Jaynes-Cumming model.** In the previous sections our concentration was on the statistical properties of even (odd) binomial, as well as negative binomial states, besides the generalized geometric state. However in the present section we shall consider and discuss some dynamical systems, namely the Jaynes-Cumming and resonance fluorescence for a single atom and many cooperative atoms. It would be interesting to point out that the Jaynes-Cummings model of a two-level atom interacting with a single mode of the electromagnetic field which describes the fundamental Bose-Fermi interaction is interesting not only from the theoretical point of view but also the experimental part, where the availability of high-Q super conducting cavities at sub-Kelvin temperature makes it possible to realize it. In this case there is only one Rydberg atom interacting with one single mode of radiation, and the system in this case is known as a micromaser,which in general can be modeled by th Jaynes and Cummings model. The simplest form of interaction between a two-level atom, and a single quantized mode of the electromagnetic field in the case of resonance is described by the Hamiltonian

$$\hat{H} = \omega(\hat{a}^{\dagger}\hat{a} + \frac{\sigma_z}{2}) + \overline{g}(\hat{a}^{\dagger}\sigma_- + \hat{a}\sigma_+). \tag{7.1}$$

where $\overline{g}$ is the coupling between the atom and the field, and $\sigma_z, \sigma_\pm$ are the atomic pseudo-spin operators with the following properties

$$[\sigma_+, \sigma_-] = \sigma_z, \qquad [\sigma_\pm, \sigma_z] = \mp 2\sigma_\pm \tag{7.2}$$

while $\hat{a}$ and $\hat{a}^{\dagger}$ are the usual boson annihilation and creation operators of the field.

i) Even binomial state

In this subsection we are concerned with a discussion of the gradual behavior for the state of the radiation field when it changes from the even(odd) number state to the even (odd)coherent state. Also we shall extend our discussion to include the generalized geometric state [11, 35].

Now suppose we prepare the atom to be initially in the excited state $|e\rangle$ and the field to be initially in the even binomial state; then the initial atom-field state is the product of the atomic superposition states given by

$$|\psi_e(0)\rangle = \gamma \sum_{n=0}^{[\frac{M}{2}]} B_{2n}^M |e, 2n\rangle, \tag{7.3}$$

where B_{2n}^M is the distribution of the photons defined by

$$B_{2n}^M = \left(\begin{array}{c} M \\ 2n \end{array} \right)^{\frac{1}{2}} \eta^{2n} (\sqrt{1 - |\eta|^2})^{(M-2n)} \tag{7.4}$$

For $t > 0$ the state $|\psi_e(t)\rangle$ in the interaction picture may be obtained from the Hamiltonian (7.1) in the form

$$|\psi_e(t)\rangle = \gamma \sum_{n=0}^{[\frac{M}{2}]} B_{2n}^M [\cos \tau\sqrt{2n+1}\ |2n, e\rangle - i \sin \tau\sqrt{2n+1}\ |2n+1, g\rangle \tag{7.5}$$

where $\tau = \overline{g}t$. From equation (7.5) we can easily obtain the expectation value for any atomic or field operator. For example, we can calculate the temporal evolution for $(\hat{a}^\dagger)^{2s}$ to give

$$\begin{aligned} \langle \hat{a}^{\dagger 2s} \rangle = |\gamma|^2 \eta^{*2s} \sum_{n=0}^{[\frac{M}{2}]} \left(\begin{array}{c} M \\ 2n \end{array} \right)^{\frac{1}{2}} \eta^{4n} (1 - |\eta|^2)^{(M-2n)} \left[\frac{(M-2n)!}{(M-2n-2s)!} \right]^{\frac{1}{2}} \\ \times [\cos \tau\sqrt{2n+2s+1}\ \cos \tau\sqrt{2n+1} \\ + (\frac{2n+2s+1}{2n+1})^{\frac{1}{2}}\ \sin \tau\sqrt{2n+2s+1}\ \sin \tau\sqrt{2n+1}] \end{aligned} \tag{7.6}$$

while the calculation of the expectation value of the photon number $\langle \hat{a}^\dagger \hat{a} \rangle$ gives

$$\langle \hat{a}^\dagger \hat{a} \rangle = |\gamma|^2 \sum_{n=0}^{[\frac{M}{2}]} |B_{2n}^M|^2 [2n + \sin^2 \tau\sqrt{2n+1}] \tag{7.7}$$

Finally the expectation value of the second moment of the photon number is given by

$$\langle (\hat{a}^\dagger \hat{a})^2 \rangle = |\gamma|^2 \sum_{n=0}^{[\frac{M}{2}]} |B_{2n}^M|^2 [4n^2 + (4n+1) \sin^2 \tau\sqrt{2n+1}] \tag{7.8}$$

It should be noted that the expectation value of any odd power as well as any power higher than M for the operators $\hat{a}$ and $\hat{a}^\dagger$ vanishes due to the nature of the even binomial state.

1)Time-dependent squeezing

Now let us employ the expectation values of different operators power to discuss the phenomena of squeezing. As usual we shall consider two different kind of squeezing one of them is the amplitude-squared squeezing. To discuss amplitude-squared squeezing we have to calculate $\Delta^2 d_1$ and $\Delta^2 d_2$ which may be written in the form

$$S_1(t) = \frac{1}{4}[2\langle(\hat{a}^\dagger\hat{a})^2\rangle + 2\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}^{\dagger 4} + \hat{a}^4\rangle - \langle(\hat{a}^{\dagger 2} + \hat{a}^2)^2\rangle] < 0 \tag{7.9}$$

and

$$S_2(t) = \frac{1}{4}[2\langle(\hat{a}^\dagger\hat{a})^2\rangle + 2\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}^{\dagger 4} + \hat{a}^4\rangle - \langle(\hat{a}^{\dagger 2} - \hat{a}^2)^2\rangle] < 0 \tag{7.10}$$

where $S_1(t)$ and $S_2(t)$, correspond to $\overline{\Delta d_1}^2$ and $\overline{\Delta d_2}^2$ respectively.

From equations (7.9) and (7.10) we can discuss the amplitude-squared squeezing against the scaled time τ. In our numerical investigation and at time $\tau > 0$ we observe that nonclassical negative values appear in $S_1(t)$ and $S_2(t)$, depend upon the phase of the parameter η, where for a fixed mean photon number $\langle\hat{a}^\dagger\hat{a}\rangle = 5$, it is easy to realize in general that, for short and long periods of time, the amplitude-squared squeezing is pronounced, but this depends on the value of the parameters η and M. This can be observed clearly for the case $\eta = 0.7$ and $M = 10$, however when we increase the value of M and decrease the value of η the system approaches the even-coherent state behavior. This can be noticed for the case $M = 30$ and $\eta = 0.4$. Also it is observed that the phenomenon of normal sqeezing is washed out, and this is due to the highly noisy character of the state produced during the term of interaction.

2)Sub-Poissonian distributions

In order to dicuss the phenomenon of anti-bunching we have to consider the Glauber second order correlation function $g^{(2)}(t)$. From equation (3.9) and for $t > 0$ we may point out that, with the mean photon number $\overline{n} = 5$ and for different values of M and η, we also notice that the function $g^{(2)}(t)$ in general shows oscillatory behavior. Furthermore as we increase the value of M, the correlation functions starts to oscillate between sub-Poissonian and super-Poissonian behavior. This can be seen for $M = 30$. On the other hand the distribution function starts to be super-Poissonian for small value of η and large values of M. This of course is due to the fact that the even binomial state will tend to the even coherent state as $\eta \to 0$ and $M \to \infty$. We can also deduce (for $M = 10$ and $M = 20$) that the state shows antibunching, and as M decreases more antibunching can be observed.

ii)Odd binomial state

Following the same procedure as the previous subsection, we can prepare the field to be initially in the odd binomial state (3.18) and the atom in the atomic superposition coherent state,such that

$$|\theta, \nu\rangle = \cos\theta|e\rangle + e^{-i\nu}\sin\theta|g\rangle, \tag{7.11}$$

where the state $|e\rangle$ and $|g\rangle$ are the atomic excited and ground states respectively, while ν, and θ are two different phases. In this case one can write the time- dependent eigenstate of the system as

$$\begin{aligned} |\psi_{(o)}(t)\rangle = \lambda_2 \sum_{n=0}^{[(M-1)/2]} B_{2n+1}^{M}[h_1(n,t)\,|e, 2n+1\rangle + h_2(n,t)|e, 2n\rangle \\ + h_3(n,t)|g, 2n+1\rangle + h_4(n,t)|g, 2n+2\rangle], \end{aligned} \tag{7.12}$$

where the coefficients $h_i(n, t)$ with $i = 1, 2, 3,$ and 4 are given by

$$\begin{aligned} h_1(n,t) = \cos\theta\cos\tau\sqrt{2n+2}, \quad h_2(n,t) = -ie^{-i\nu}\sin\theta\sin\tau\sqrt{2n+1} \\ h_3(n,t) = e^{-i\nu}\sin\theta\cos\tau\sqrt{2n+1}, \quad h_4(n,t) = -i\cos\theta\sin\tau\sqrt{2n+2} \end{aligned} \tag{7.13}$$

From equation (7.12) we are readily able to calculate the expectation values for different operators. Thus

$$\begin{aligned} \langle \hat{a}^\dagger \hat{a}\rangle = |\eta|^2|\lambda_2|^2(M/2)[1 + (1-2|\eta|^2)^{(M-1)}] \\ + |\lambda_2|^2 \sum_{n=0}^{[(M-1)/2]} |B_{2n+1}^{M}|^2[\cos^2\theta\sin^2\tau\sqrt{2n+2} - \sin^2\theta\sin^2\tau\sqrt{2n+1}] \end{aligned} \tag{7.14}$$

$$(7.15)\quad \langle \hat{a}^{\dagger 2}\hat{a}^{2}\rangle = |\eta|^4|\lambda_2|^2(M/2)(M-1)[1-(1-2|\eta|^2)^{(M-2)}] + 2|\lambda_2|^2\sum_{n=0}^{[(M-1)/2]}|B^M_{2n+1}|^2\times [(2n+2)\cos^2\theta\sin^2\tau\sqrt{2n+2} - 2n\sin^2\theta\sin^2\tau\sqrt{2n+1}]$$

$$(7.16)\quad \langle \hat{a}^{2s}\rangle = \frac{|\lambda_2|^2\eta^{2s}}{(1-|\eta|^2)^s}\sum_{n=0}^{[(M-1)/2]}|B^M_{2n+1}|^2\sqrt{\frac{(M-2n-1)!}{(M-2n-2s-1)!}}\times \{\cos^2\theta[\cos\tau\sqrt{2n+2s+2}\cos\tau\sqrt{2n+2} + \sqrt{\frac{(n+s+1)}{(n+1)}}\sin\tau\sqrt{2n+2s+2}\sin\tau\sqrt{2n+2}] + \sin^2\theta[\cos\tau\sqrt{2n+2s+1}\cos\tau\sqrt{2n+1} + \sqrt{\frac{(2n+1)}{(2n+2s+1)}}\sin\tau\sqrt{2n+2s+1}\sin\tau\sqrt{2n+1}]\}$$

Now we are in position to discuss the time-dependent squeezing as well as normalized second-order correlation function.

1)Time-dependent squeezing

To discuss time-dependent squeezing (normal squeezing) we have to examine the quadrature variances $\overline{\Delta X}^2$ and $\overline{\Delta Y}^2$ against the normalized time τ. In our numerical investigation when we considered the atom to be prepared initially in the ground state $\theta = \frac{\pi}{2}$, taking into account $M = 3$ and $\eta = 0.6$, we find that the squeezing occurs once for a short period of time, although the first quadrature $\overline{\Delta X}^2$ includes a periodic function. In the meantime the squeezing in the second quadrature $\overline{\Delta Y}^2$ appears frequently. On the other hand, when $\theta = 0$ (atom initially in the excited state) we observe no squeezing in both quadratures.

2)Time-dependent correlation function

From the correlation function $g^{(2)}(t)$ equation (3.9), together with equations (7.14) and (7.15), we are able to discuss the antibunching phenomenon. In the numerical study of the function we find for fixed value of the mean photon number $\overline{n} = 5$, and when $\theta = 0$ (atom initially in the excited state) and $\theta = \frac{\pi}{2}$ (atom initially in the ground state), that $g^{(2)}(t)$ is in general oscillatory, and shows sub-Poissonian behavior for both cases when $M = 9$. However, if we increase the value of the parameter M ($M = 19$),we find that the correlation function for the case when the atom is in the excited state still shows sub-Poissonian behavior, while the function for the ground state case starts to oscillate, showing super and sub-Poissonian as well as partial coherence behavior. By increaseing the value of M up to 29, we notice that correlation function in the excited state starts to show super-Poissonian behavior similar to that of the ground state case; however in the ground state case the super-Poissonian behavior is more pronounced than in the excited-state case.

iii)Generalized geometric state

In order to understand the behavior of some quantum systems, where the state of the radiation field changes from the number to the chaotic state, we express the density operator for the generalized geometric state either in the number states $|n\rangle$ or in the coherent state $|\beta\rangle$. Thus, from equation (5.1) and for any operator $\hat{O}$ of the photon number $\widehat{n}$, we find the expectation value in the $|Y, M\rangle$ state is given by

$$\langle \hat{O} \rangle = |\lambda_0|^2 \sum_{n=0}^{M} |Y|^n \langle n|\hat{O}|n\rangle. \tag{7.17}$$

On the other hand one can find the exact solution for the mean atomic inversion for the radiation field initially in the number state $|n\rangle$ and the atom starting in its excited state, which is given by

$$\langle S_z(t)\rangle_n = \frac{1}{2}\cos(2\overline{g}t\sqrt{n+1}) \tag{7.18}$$

Then from equation (7.17) the mean $\langle S_z\rangle_{Y,M}$ in the generalized geometric state is

$$\langle S_z(t)\rangle_{Y,M} = \frac{1}{2}\sum_{n=0}^{M} P_n(|Y|)\cos(2\overline{g}t\sqrt{n+1}), \tag{7.19}$$

where

$$P_n(|Y|) = |Y|^n \left[\frac{1-|Y|}{1-|Y|^{M+1}}\right] \tag{7.20}$$

The expression (7.19) is actually valid for an atom initially in its ground or excited state, $\langle S_z(0)\rangle = \pm\frac{1}{2}$. For other initial conditions the solution (7.20) would have an additional term dependent on $\langle S_z(0)\rangle$ which is not zero for such a case, so terms for $\acute{n} \neq n$ in (7.17) will then contribute. The numerical results for $\langle S_z(t)\rangle_{Y,M}$ against the normalized time $\tau = 2gt$ are presented in [35]. For $|Y| < 1$, namely $|Y| = 0.5$, $\langle S_z(t)\rangle_{Y,M}$ exhibits more irregular behavior as M increases. Note that the case for the chaotic-state field is actually realized for values of $M \geq 10$; so for $|Y| \equiv \overline{n}/(1+\overline{n}) = 0.5, \overline{n} = 1$, and for $|Y| = 0.95$, $\overline{n} = 19$, where the case of relatively weak and strong chaotic fields are discussed, are in good agreement with the results of [50].

For higher $|Y| > 1$, $\langle S_z\rangle_{Y,M}$ tends to show simple (or regular) oscillations, as one would expect analytically, since in this case $|Y, M\rangle \rightarrow$ number state $|M\rangle$ as $|Y| \rightarrow \infty$.

7.2. **Resonance Fluorescence.** The resonance fluorescence phenomenon concerns a radiatively decaying two -level atomic system coupled to an external radiation field in free space. Our concern will be limited to the steady-state regime ($t \rightarrow \infty$) in two different cases: the single atom ($N = 1$), and the thermodynamic limit in a cooperative many atom system, where $N \rightarrow \infty$.

i)Even binomial state

1- Single atom

For an external field in the Fock state $|n\rangle$ one finds in the steady- state case the mean atomic inversion for a single two-level atom interaction with this field takes the form

$$(7.21)\qquad \langle S_z(\infty)\rangle_n = -\frac{1}{2}\sum_{m=0}^{n}(-)^m\frac{n!b^{2m}}{(n-m)!} = -\frac{1}{2}(b^2)^n L_n^{(-n-1)}(-b^{-2})$$

where $b^2 = 2\hbar\hat{g}^2(\Delta^2 + \frac{1}{4}\Gamma^2)^{-1}; \overline{g}$ is the coupling constant and Γ is the Einstein coefficient; Δ is the frequency detuning between the atomic transition frequency and that of the field, and $L_n^{(a)}(x)$ is the generalized Laguerre polynomial.

From the state given by equation (3.7) we find the mean atomic inversion takes the form

$$(7.22)\qquad \langle S_z(\infty)\rangle_{EB} = \sum_{n=0}^{[M/2]} P_{2n}(\eta, M)\langle S_z(\infty)\rangle_{2n}$$

where

$$(7.23)\quad P_{2n}(\eta, M) = |\langle 2n|\psi_e\rangle|^2 = |\lambda_1|^2\frac{M!}{2n!(M-2n)!}\left[\frac{|\eta|^2}{1-|\eta|^2}\right]^{2n}(1-|\eta|^2)^M$$

After some calculations we can write the mean atomic inversion for the even binomial state thus

$$(7.24)\quad \langle S_z(\infty)\rangle_{EB} = -\frac{|\lambda_1|^2}{2}\Big[(b^2|\eta|^2)^M L_M^{(-M-1)}(-b^2|\eta|^2) + (-b^2|\eta|^2)^M L_M^{(-M-1)}(b^2|\eta|^2)\Big].$$

To get the vacuum state we take $\eta = 0$,but to get the even-coherent state we have to take $\eta \to 0$ and $M \to \infty$ such that $M|\eta|^2 \to |\alpha|^2$. In this case we have the following

$$(7.25)\qquad \langle S_z(\infty)\rangle_{EC} = -\frac{1}{2}\sec h|\alpha|^2(1-b^4|\alpha|^2)^{-1}$$

2- Thermodynamic limit

Now let us consider the case of resonance fluorescence in which $\Gamma \to 0$ and $N \to \infty$, such that $(\Gamma N\ )$ is finite. In this case the scaled atomic inversion at exact resonance in the number state field is given by

(7.26)

$$\lim_{N\to\infty}\left[\frac{\langle S_z(\infty)\rangle_n}{N}\right] = -\frac{1}{2}C_n(\frac{1}{2};Z^2) = -\frac{1}{2}\sum_{s=0}^{n}(-)^s\frac{n!}{(n-s)!}\begin{pmatrix}\frac{1}{2}\\ s\end{pmatrix}Z^{-2s}$$

where $z = \Gamma N(2\hbar^{-2}\overline{g}^2)$ and C_n are the Poisson-Charlier polynomials. For the even binomial state we get the expression

$$(7.27)\quad \lim_{N\to\infty}\left[\frac{\langle S_z(\infty)\rangle_{EB}}{N}\right] = -\frac{|\lambda_1|^2}{4}\left[C_M(\frac{1}{2};Z^2|\eta|^2) + C_M(\frac{1}{2};-Z^2|\eta|^2)\right]$$

The behavior of equations (7.24) and (7.27) against $|\eta|$ for different values of M have been considered in [27], where we noted that as we increase the value of M, the curve approaches saturation faster. However, the rate in the thermodynamic limit is slower than for the case of a single atom.

ii)Odd binomial state

1- Single atom

Again we turn our attention to consider the statistical average over the state of the field which is taken to be the odd binomial state. In the steady-state case ($t \to \infty$) we find that the mean atomic inversion takes the form

$$\langle S_z(\infty)\rangle = |\lambda_2|^2\sum_{n=0}^{[\frac{M-1}{2}]}|B_{2n+1}^M|^2\left[\langle S_z(\infty)\rangle_{2n+1}\cos^2\theta - \langle S_z(\infty)\rangle_{2n}\sin^2\theta\right],$$

where $< S_z(\infty) >_{2n}$ and $< S_z(\infty) >_{2n+1}$ are given by equation (7.21) and the atomic is taken in a coherent state $\theta,\phi >$. For the odd coherent state, equation (7.24) reduces to

$$(7.28)\quad \langle S_z(\infty)\rangle_{oc} = -\frac{1}{2}\Big\{\frac{(1-b^2|\alpha|^2\coth|\alpha|^2)}{1-b^4|\alpha|^4}\cos^2\theta - [(1-D'^2)^{-1}(1+b^2D'|\alpha|^2) - b^2\coth|\alpha|^2(1-D'^2)^{-1}\times |\alpha|^2(b^2|\alpha|^2+D')(1-b^4|\alpha|^4)^{-1}]\sin^2\theta\Big\}$$

where D' slands for the differential operator $\frac{d}{d|\alpha|^2}$.

2- Thermodynamic limit

In the case of the odd binomial state, one can see the scaled atomic inversion at exact resonance is given by

$$(7.29)\quad \lim_{N\to\infty}\left[\frac{\langle S_z(\infty)\rangle_{OB}}{N}\right] = \frac{|\lambda_2|^2}{2}\sum_{n=0}^{[\frac{M-1}{2}]}|B^M_{2n+1}|^2\left[C_{2n}(\frac{1}{2};Z^2)\sin^2\theta - C_{2n+1}(\frac{1}{2};Z^2)\cos^2\theta\right]$$

If we plot the figure of the thermodynamic limit equation (7.29) against η, then one can observe for fixed value of $\theta = 0$,and $\frac{\pi}{2}$ with $b = Z^{-1} = 10^{-1}$ that when the atom is in the excited state, the value of $\underset{N\to\infty}{Lim}\left[\frac{<S_z(\infty)>}{N}\right] \simeq 0$ is achieved very slowly and it needs larger values of M and η. This behavior also has been noticed in the case of the generalized geometric state [11, 35], which will be discussed later. For an atom in the ground state, upon increasing the values of M and $\eta, < S_z(\infty) >$ reaches its steady-state value at a slower rate. The same behavior is also observed in [54]. It is clear that the rate in the thermodynamic limit is slower than that of a single atom when the atom is in the ground state, and conversely when the atom is in the excited state. As a special case, the thermodynamic limit for the odd coherent state can be deduced from equation (7.28) to take the form

$$(7.30)\quad \lim_{N\to\infty}\left[\frac{\langle S_z(\infty)\rangle_{oc}}{N}\right] = \frac{1}{2\sinh|\alpha|^2}\times \sum_{n=0}^{\infty}\frac{|\alpha|^{2n+2}}{(2n+1)!}\left[C_{2n}(\frac{1}{2};Z^2)\sin^2\theta - C_{2n+1}(\frac{1}{2};Z^2)\cos^2\theta\right]$$

iii)Geometric state

1- Single atom

In the generalized geometric state the mean atomic inversion for a single two-level atom system is given by

$$(7.31)\qquad \langle S_z(\infty)\rangle_{Y,M} = \sum_{n=0}^{M} P_n(|Y|)\langle S_z(\infty)\rangle_n$$

where $P_n(|Y|)$ and $\langle S_z(\infty)\rangle_n$ are given by equations (7.20) and (7.21), respectively. Here we may mention that $\langle S_z(\infty)\rangle$ reaches its steady value at a slower rate as $|Y|$ increases in the interval $0 \le |Y| < 1$ provided the value of M moves from 2 to 20. In the limit $M \to \infty$ and for $|Y| = \overline{n}/(1+\overline{n}) < 1$ the formula (7.31) gives

$$\begin{aligned}\lim_{M\to\infty}\langle S_z(\infty)\rangle_{Y,M} &= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(\overline{n})^n n!}{(1+\overline{n})^{n+1}}\sum_{m=0}^{n}\frac{(-b^2)^m}{(n-m)!} \\ &= -\left[\frac{1}{1+\overline{n}}\right]\sum_{n=0}^{\infty}\left[\frac{b^2\overline{n}}{1+\overline{n}}\right]^n L_n^{(-n-1)}(-b^{-2}) \\ &\simeq -\frac{1}{1+\overline{n}+\overline{n}b^2}\exp\{b^2[\overline{n}/(1+\overline{n})]\}\end{aligned} \tag{7.32}$$

where the equation (7.32) is only valid for $b^2\overline{n}/(1+\overline{n}) < 1$ (see [51]), which is an alternative form of the derived result using the P representation form such that,

$$\hat{\rho}_{ch} = \int d^2\beta P(\beta)|\beta\rangle\langle\beta|, \tag{7.33}$$

for the single-mode chaotic-state field [52, 53]

$$\langle S_z(\infty)\rangle_{ch} = -\frac{1}{2}(b^2\overline{n})^{-1}[\exp(b^2\overline{n})^{-1}]E_1((b^2\overline{n})^{-1}), \tag{7.34}$$

where $E_1(x) = \int_x^\infty e^{-v}dv/v$ is the exponential integral.

2- Thermodynamic limit ($\mathbf{N} \to \infty$)

For the generalized geometric state case we find the scaled atomic inversion at exact resonance in the number state is given by

$$\underset{N\to\infty}{Lim}\left[\frac{\langle S_z(\infty)\rangle_{Y,M}}{N}\right] = -\frac{1}{2}\sum_{n=0}^{M}P_n(|Y|)C_n(\frac{1}{2};Z^2). \tag{7.35}$$

The case of the chaotic-state field is then obtained by letting

$|Y| = \overline{n}/(1+\overline{n}), M \to \infty :$

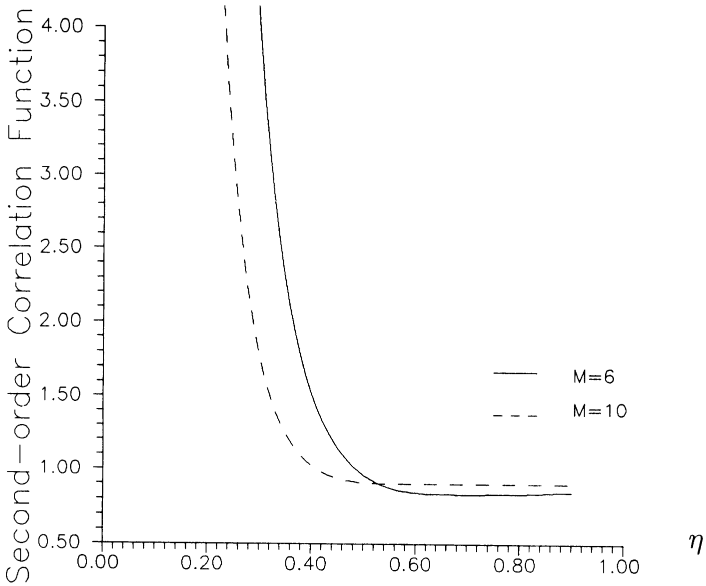
$$\text{(7.36)} \qquad \underset{N\to\infty}{Lim}\left[\frac{\langle S_z(\infty)\rangle_{ch}}{N}\right] = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(\overline{n})^n}{(1+\overline{n})^{n+1}}C_n(\frac{1}{2};Z^2)$$

The scaled atomic inversion in the limit $N \to \infty$, equation (7.35), leads to the same conclusion which applies for the single-atom case, but with lesser saturation value. For $M \to \infty$ (case of chaotic-state field, equation (8.16) for $N \to \infty$) the behavior is in conformity with the results obtained for finite $N \leq 10$ cooperative atoms [52]: the approach to the value of $\underset{N\to\infty}{Lim}\frac{\langle S_z(\infty)\rangle}{N} \simeq 0$ is very slow and requires larger values of $Y = \overline{n}/(\overline{n}+1)$ ($\simeq 0.9$).

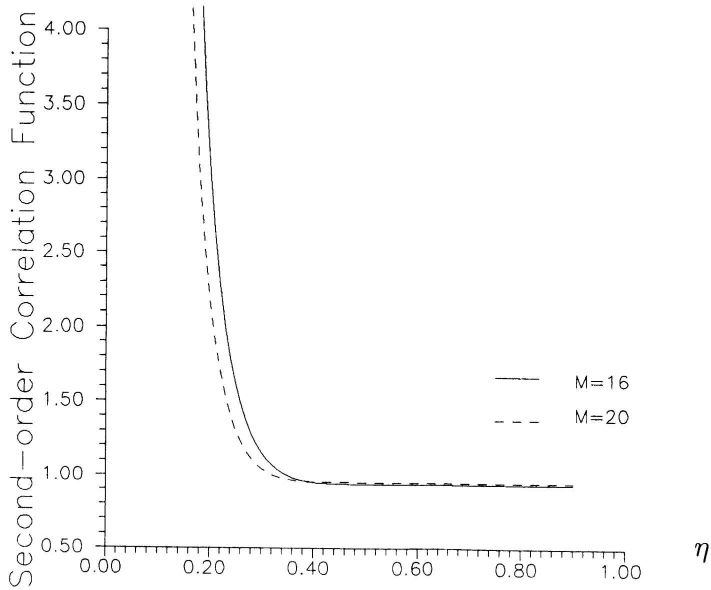
8. CONCLUSION

Superposition of quantum mechanical states of the electromagnetic field have recently received much attention in quantum optics. In the meantime the experimental realizations of nonclassical states of motion of a trapped ion, such as Fock states, coherent states, squeezed states and "Schrödinger cat" states, have been reported [44-47]. In these experiments an ion is laser-cooled in a Paul trap to the ground harmonic state. Then the atomic is put into various quantum states of motion by applications of optical and electric pulses for different durations. Thus the study of non-classical states of light is not a more academic exercise, but it relates to the experimental realm. In this report we have studied some intermediate states, in particular even and odd-coherent states, which interpolate between the even and odd-coherent states and the even and odd Fock states. The even negative binomial states which bridge between the even and odd coherent states and the even and odd pure thermal states have been discussed. The geometric state which is a bridge between the pure thermal state and the Fock state has been introduced and the even-and odd geometric states have been studied.

For these states we have considered the non-classical properties, especially antibunching. Sub-Poissonian states squeezing (normal and lingher) of the field quadratures the quasi-probability distribution functions have been calculated. The non-classical signature shows in attaining negative values for the Wigner-function and oscillations in the photon distribution. Applications of these states to the Jaynes-Cummings model and to the resonances fluorescence have been performed, and the gradual transfer from one limit to the other has been demonstrated.

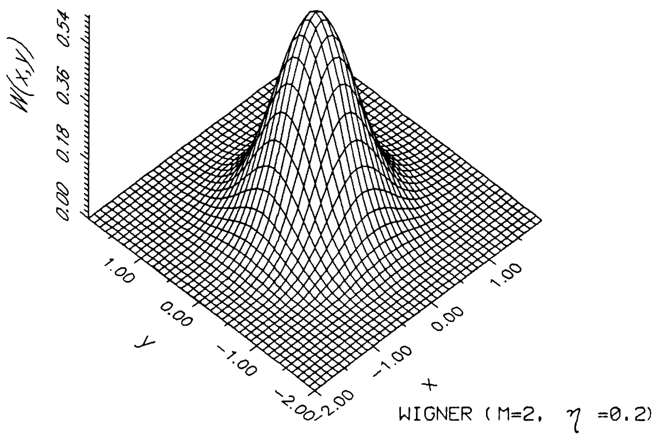


(*a*)

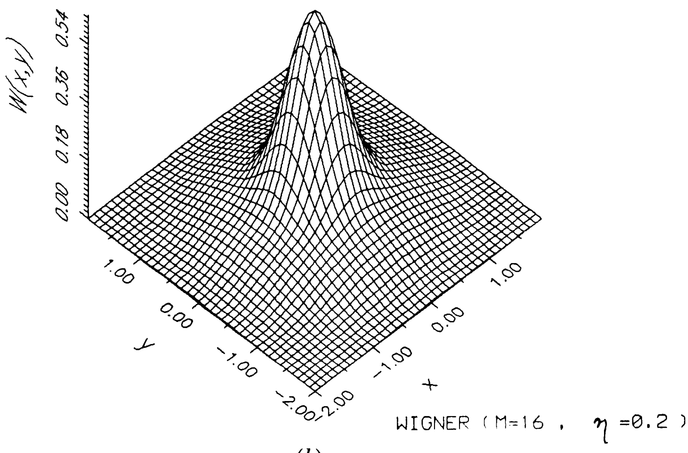


(*b*)

Figure 1. (*a*) $g^2(0)$ plotted against η for $M = 6, 10$. (*b*) As in (*a*) for $M = 16, 20$

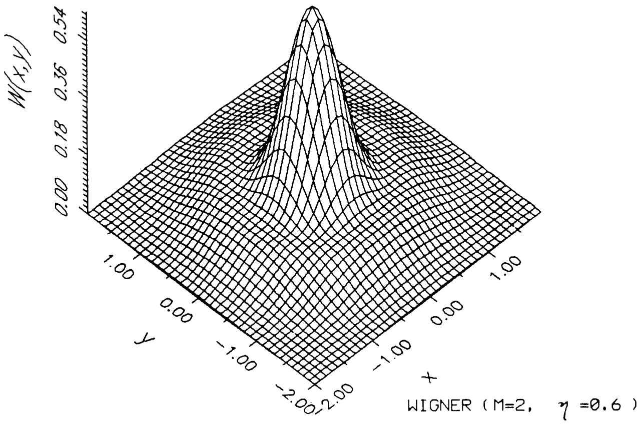


(*a*)

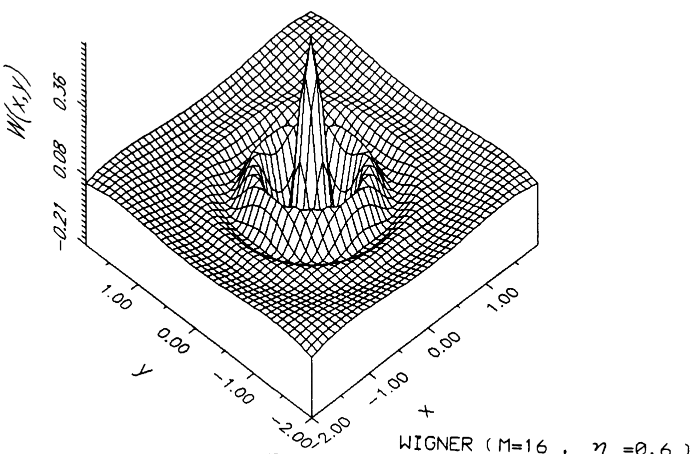


(*b*)

Figure 2. (*a*) Wigner function for the even binomial state with $M = 2$, $\eta = 0.2$
(*b*) Same as (*a*) with $M = 16, \eta = 0.2$

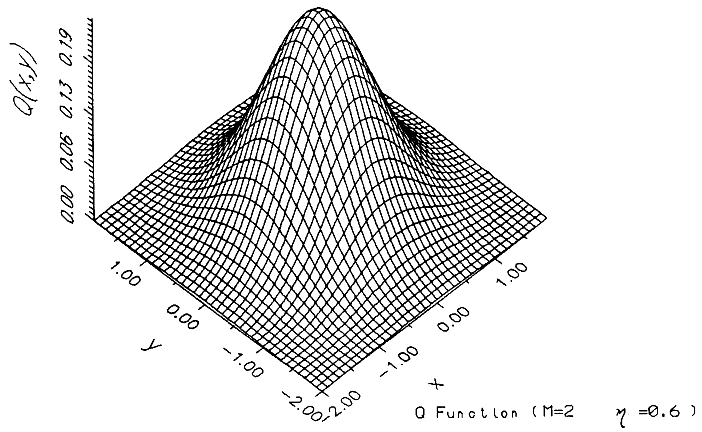


(a)

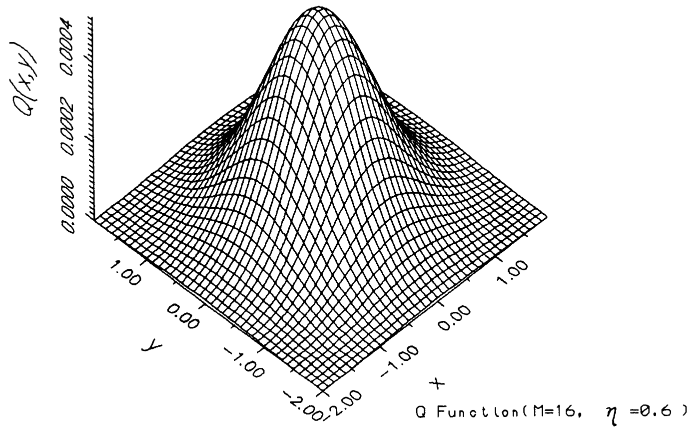


(b)

Figure 3. (a) Same as figure 2 with $M = 2$, $\eta = 0.6$
(b) Same as figure 2 with $M = 16$, $\eta = 0.6$



(a)



(b)

Figure 4. (a) Q function for the even binomial state with $M = 2$, $\eta = 0.6$
(b) As in (a) with $M = 16, \eta = 0.6$

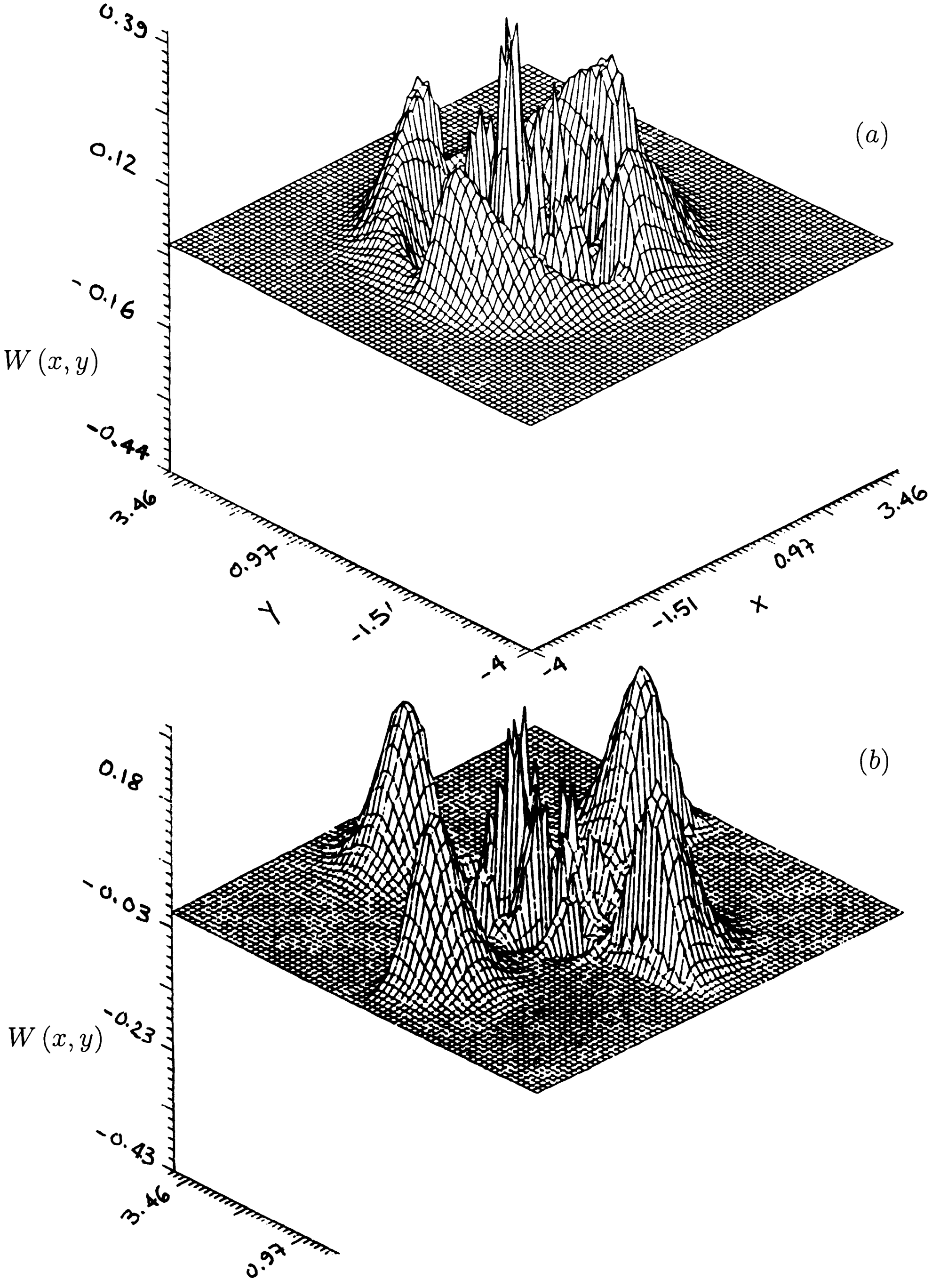


Figure 5. W-function for (a) $M = 5$, $\eta = 0.6$ and (b) $M = 17$, $\eta = 0.6$

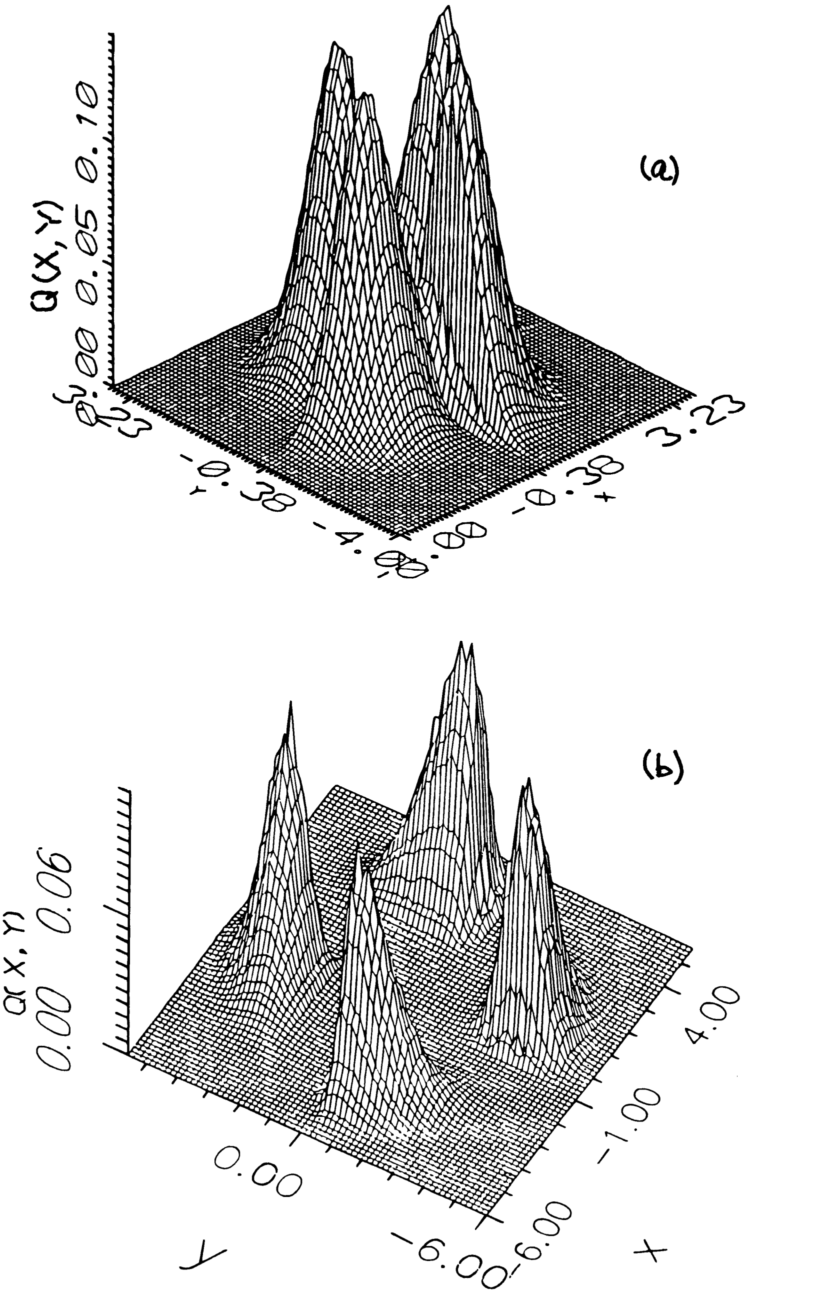


Figure 6. Q-function for (a) $M = 5$, $\eta = 0.6$ (b) $M = 17$, $\eta = 0.9$

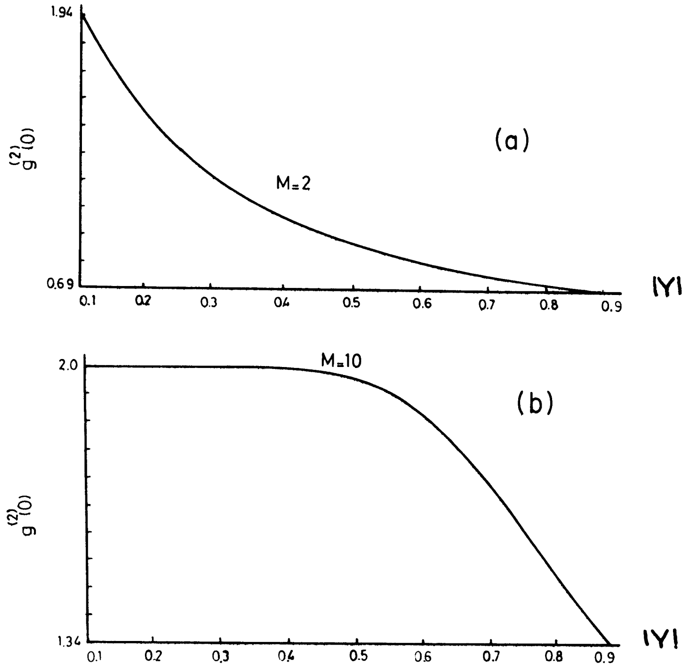


Figure 7. Normalized correlation function $g^{(2)}(0)$ against $|Y| < 1$ (a) for $M = 2$ and (b) for $M = 10$

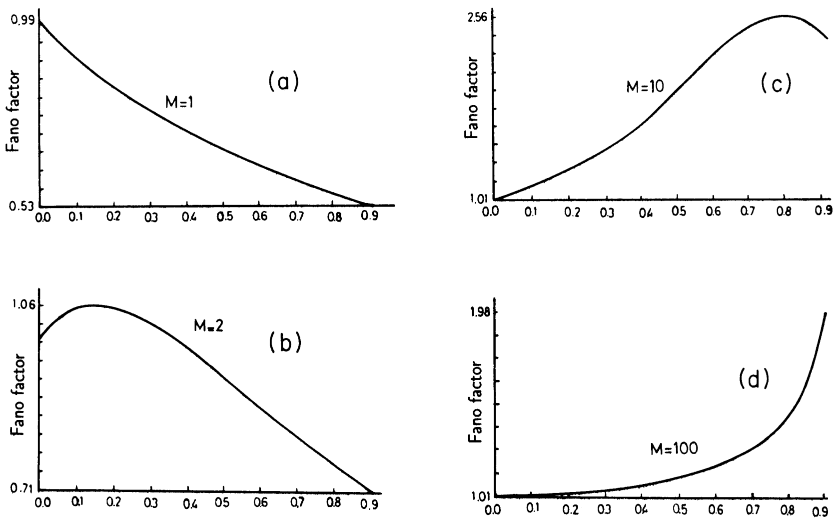


Figure 8. The Fano factor $(\Delta\hat{n})^2 / < \hat{n} >$ against $|Y| < 1$ (a) for $M = 1$, (b) $M = 2, (c)$ $M = 10$, and (d) $M = 100$

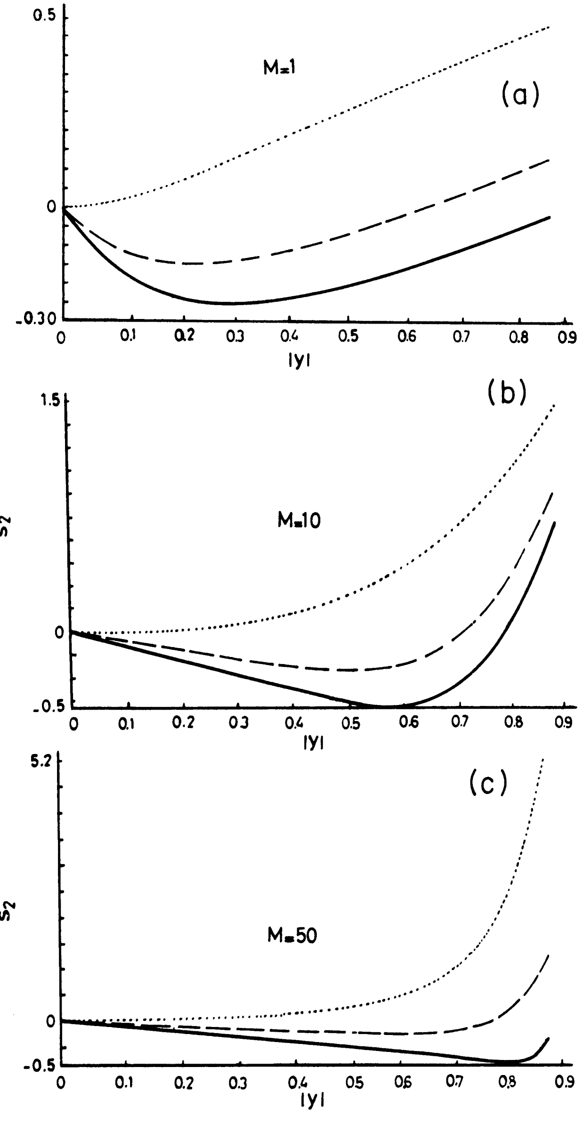


Figure 9. (*a*) The variance $S_1 = 2\,(\Delta X)^2 - 1$ against $|Y| < 1$ for $M = 1$ and different phases $\phi = 0$ (——), $\phi = \pi/4$ ($- - -$), $\phi = \pi/2$ ($\cdots$). (*b*) $S_2 = 2\,(\Delta P)^2 - 1$ for $M = 10$. (*c*) Same as (*b*) for $M = 50$.

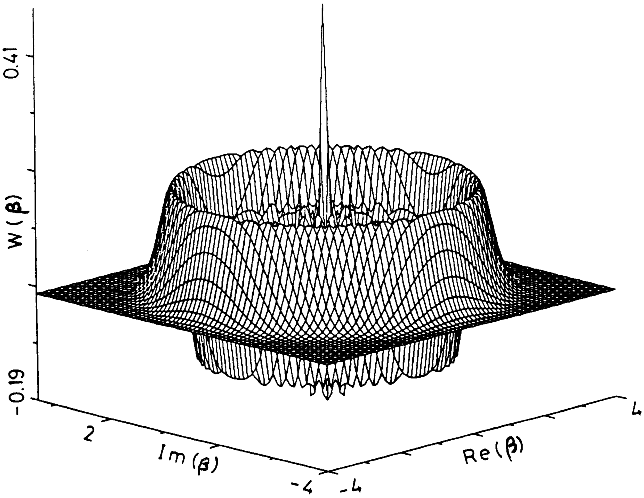


Figure 10. The Wigner function $W(\beta)$ for the case $|Y| = 3$, $M = 10$.

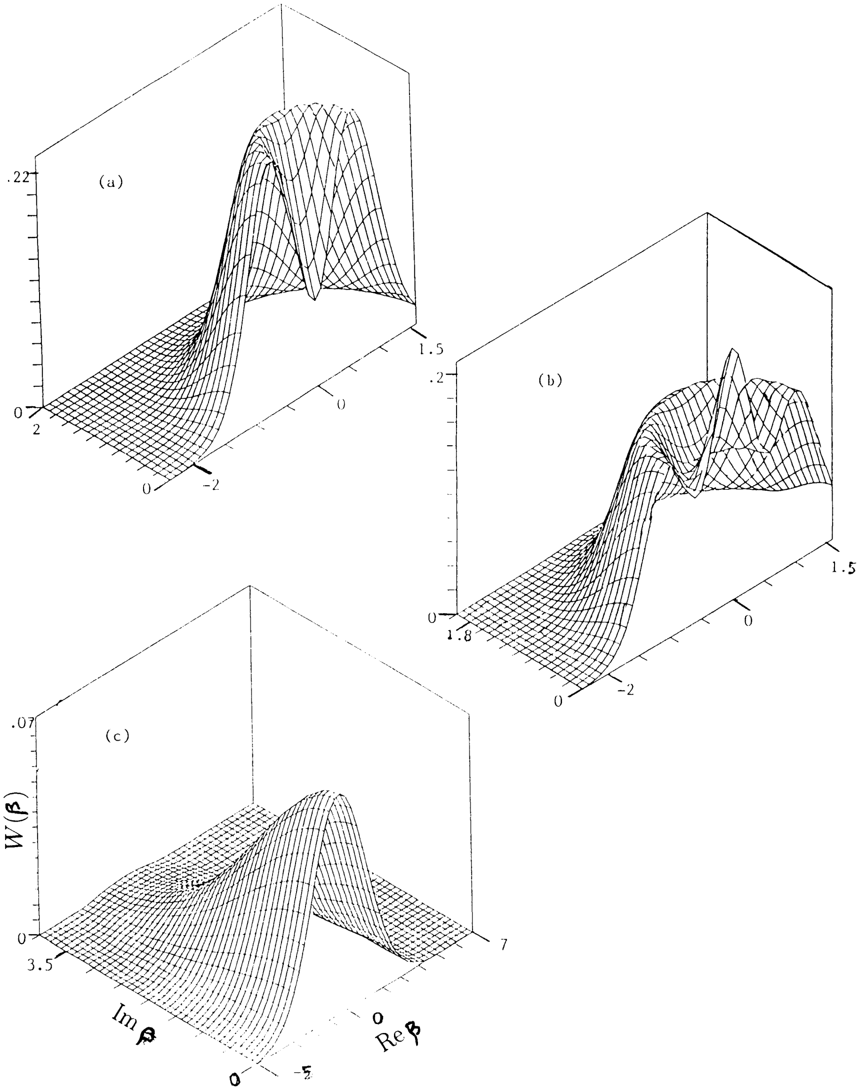


Figure 11. The Wigner function $W(\beta)$ for $|Y| = 0.8$ and (a) $M = 1$, (b) $M = 2$, (c) $M = 20$.

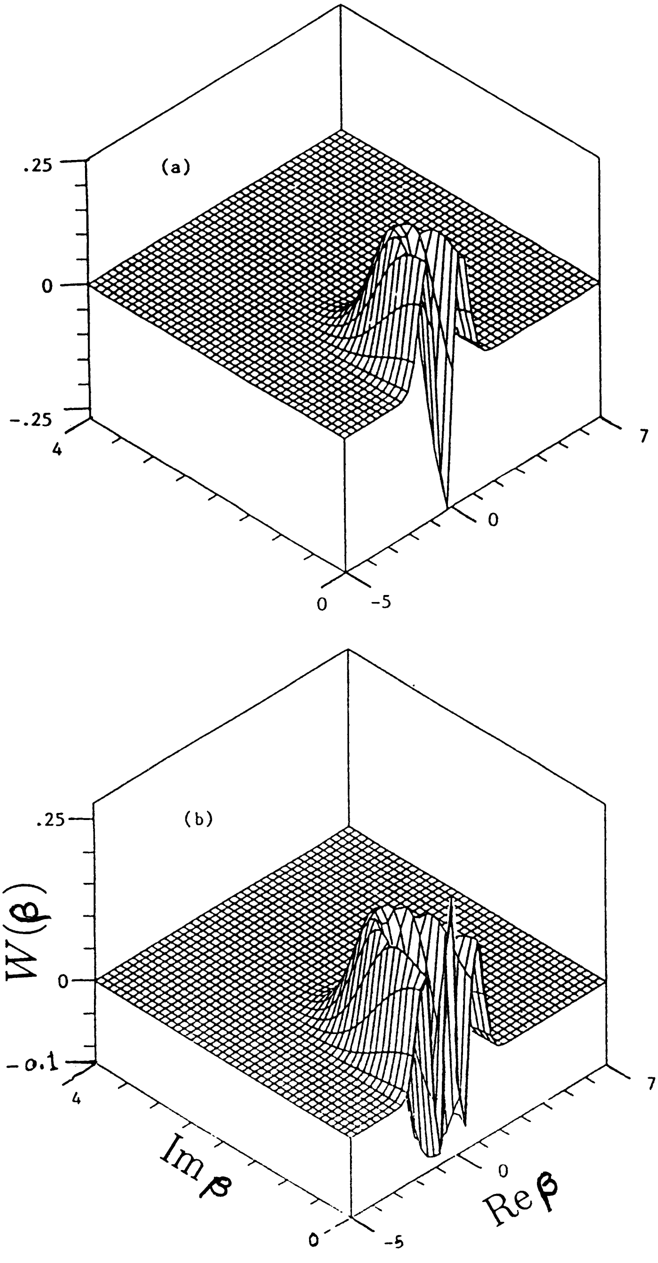


Figure 12. Wigner function $W(\beta)$ for $|Y| = 3$ and (a) $M = 1$, (b) $M = 2$.

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Date received May 25, 2003