Arab J. Math. Sc.
Volume 15, Number 1, June 2009
pp 53-64

POTENTIAL THEORY ON FINITE GRAPHS

K. ABODAYEH AND V. ANANDAM

ABSTRACT. If Δ is the Laplacian matrix defined on a finite graph, it is proved here that proper symmetric submatrices of Δ are invertible. Based on this lemma, some of the significant results in the theory of electrical networks and of random walks (like Dirichlet problem, balayage, equilibrium principle etc. which are associated with potential theory on finite graphs) can be proved immediately.

1. INTRODUCTION

Some of the important results and concepts in the theory of electrical networks (and of random walks) can be related to the potential theoretic study of functions on finite graphs. In this category appear the Dirichlet problem and mixed boundary-value problems, the equilibrium principle, the condenser principle etc. (For these topics, see Brelot [5] in the context of classical potential theory and [2] in the context of infinite trees). They are all in some way connected to the Laplace operator which can be conveniently represented by a square matrix of order n in a finite graph with n vertices. The Laplacian matrix itself is singular. However, we prove that any proper symmetric submatrix of the Laplacian is non-singular. As consequences we are able to immediately

2000 Mathematics Subject Classification. 31C20, 31D05, 15A09.
Key words and phrases. Laplacian matrix on a finite graph, Boundary-value problems, Green function, Balayage.
The first author was supported by Prince Sultan University, Prince Salaman Research and Translation Center, under Contract IBRP-MATH-2008-5-3

derive the following: the existence of solutions to the Poisson-Dirichlet problem, the Neumann problem and other mixed boundary-value problems; the equilibrium principle and the condenser principle, the existence of the Green function and the notion of balayage. These boundary value problems have already been solved in Bendito et al. [3] by constructing appropriate Green functions for each problem; this involves the use of equilibrium measures obtained as the solution of some linear programming problems. In this note, however, these and other related problems are solved only by using the fact that symmetric proper submatrices of the Laplacian matrix are invertible.

2. Preliminaries

Let X be a connected directed graph without loops and an arbitrary orientation, having n vertices $x_1, \dots, x_n$ and m edges $e_1, \dots, e_m$. Let $x \sim y$ denote that there is an edge $e = [x, y]$ connecting x and y; if e is directed from x to y, let us write $h(e) = y$ and $t(e) = x$. The incidence matrix of the graph is $D = (d_{ij})$ with n rows and m columns, where $d_{ij} = 1$ if $h(e_j) = x_i$, $d_{ij} = -1$ if $t(e_j) = x_i$, and $d_{ij} = 0$ if x_i is not an end vertex of e_j.

Let C_0 (respectively, C_1) denote the real valued functions defined on the set of vertices (respectively, edges). Considering C_0 and C_1 as column vectors, we get the mappings $D : C_1 \to C_0$ and $D^t : C_0 \to C_1$ which is the transpose of D, defined as follows: If $\varphi \in C_1$, then $D\varphi(x) = \sum_{x=h(e)} \varphi(e) - \sum_{x=t(e)} \varphi(e)$ and if $u \in C_0$, then $D^t u(e) = u(h(e)) - u(t(e))$.

Hence, if $u \in \ker D^t$, then $u(x) = u(y)$ if $x \sim y$. Since x is connected, this implies that if $u \in \ker D^t$, then u is a constant, so that $\ker D^t$ has dimension 1. Hence $\dim(\operatorname{Im} D^t) = n - 1$. Let now $\mathbf{Z} = \ker D$ and by considering the inner product defined on C_1, let us write $C_1 = Z \oplus B$. Now $\dim B = \dim(\operatorname{Im} D) =$ rank of $D =$ rank of $D^t = \dim(\operatorname{Im} D^t) = n - 1$. Moreover, if $f \in \operatorname{Im} D^t$, that is $f = D^t\varphi$, and if g is arbitrary in $\mathbf{Z}$, then $(g, f) = (g, D^t\varphi) = (Dg, \varphi) = (0, \varphi) = 0$, so that $\operatorname{Im} D^t \subset B$. Since $\dim(\operatorname{Im} D^t) = n - 1 = \dim B$ also, we conclude that $B = \operatorname{Im} D^t$.

Let us consider now a finite electrical network, with $t(e)$ representing the conductance on the edge e. Then the basic problem in the electrical network is the Kirchhoff's problem which can be stated as follows, see Biggs [4, p.659]: Given the external current $c \in C_1$, find $\varphi \in C_0$ such that $DTD^t\varphi = Dc$, where T is the diagonal matrix of order m whose diagonal entries are $t(e)$.

Note $DTD^t\varphi(x) = 0$ expands as $\sum_{y\sim x} t(x,y)[\varphi(x) - \varphi(y)] = 0$ where $t(x,y) =$ the conductance $t(e)$ on the edge e joining x and y. If we write $t(x) = \sum_{x\sim y} t(x,y)$, then $t(x) > 0$ for every vertex x and $\varphi(x) = \sum_{y\sim x} \frac{t(x,y)}{t(x)}\varphi(y)$ expresses the mean-value property of φ. In analogy with the notation used in the study of the real part of an analytic function, where the mean-value property plays an important role, let us write $-\Delta = DTD^t$.

Thus, if $\varphi \in C_0$, then $\Delta\varphi \in C_0$ where $\Delta\varphi(x) = \sum t(x,y)[\varphi(y) - \varphi(x)]$ for each vertex x. (Note that the sign $x \sim y$ under the summation sign $\sum$ is not needed, since $t(x,y) > 0$ if and only if $x \sim y$.) Moreover, in its matrix form $\Delta = (C_{ij})$ is a symmetric $n \times n$ matrix where $C_{ii} = -\sum_{j\neq i} t(x_i,x_j) < 0$ for every i, $C_{ij} = t(x_i,x_j) \geq 0$ if $i \neq j$, $\sum_j C_{ij} = \sum_i C_{ji} = 0$; further since rank of $D =$ rank of $D^t = n-1$ and since T is a non-singular matrix, the rank of Δ is $n-1$. Since $t(x,y) = t(y,x)$, we also find that $\sum_x \Delta\varphi(x) = 0$ for every $\varphi \in C_0$; for $\sum_x \Delta\varphi(x) = (\Delta\varphi, 1) = (\varphi, \Delta 1) = (\varphi, 0) = 0$.

Thus, for a given $f \in C_0$, if we can find $\varphi \in C_0$ such that $\Delta\varphi = f$, then necessarily $\sum_x f(x) = 0$. We shall prove that this condition on f is sufficient to find a solution φ of the equation $\Delta\varphi = f$. Of course, φ will be determined only up to an additive constant since $\ker\Delta$ is the set of constant functions in C_0. In particular, the Kirchhoff's problem $(-\Delta)\varphi = DTD^t\varphi = Dc$ has a solution for a given $c \in C_1$ since $\sum_x Dc(x) = \sum_x[\sum_{h(e)=x} c(e) - \sum_{t(e)=x} c(e)] = 0$ (proved in Biggs [4] where we have a matrix solution and a tree solution to this problem).

3. A basic lemma

Let X be a finite, connected graph without loops and with an arbitrary orientation. For a subset $E \subset X$, a vertex x is said to be an interior vertex of E if and only if x and all its neighbours in X are in E, written as $x \in \overset{o}{E}$. $\partial E = E \backslash \overset{o}{E}$ is referred to as the boundary of E. The definition of the boundary of F given in Chung et al. [6] and Bendito et al. [3] is different from the one given here. In the earlier definition, the boundary of F lies outside F. But in many potential theoretic problems like balayage, Dirichlet and Neumann problems, we deal with functions defined only on F, their values outside F being irrelevant. Hence, the boundary given here is differently defined. For a given real valued function u on X, we write $\Delta u(x) = \sum_{y\in X} t(x,y)[u(y)-u(x)]$, for any x in X; if E is a proper subset of X and if $s \in \partial E$, then we write $\frac{\partial u}{\partial n^-}(s) = \sum_{y\in E} t(s,y)[u(y)-u(s)]$.

A proper subset F of X itself can be considered as a restricted (may be not connected) graph of X; that is, e is an edge in the graph F if and only if both the ends of e are vertices in F. Let us denote by Δ^*_F the Laplacian restricted to F; that is, for any real valued function u on F, if $x \in \overset{o}{E}$, then $\Delta^*_F u(x) = \Delta u(x)$ and if $s \in \partial F$, then $\Delta^*_F u(s) = \frac{\partial u}{\partial n^-}(s)$.

Lemma 3.1. *Let u be a real valued function on X. Let $A = \{y : \Delta u(y) < 0\}$. Then $u(x) \leq \max_{y\in A} u(y)$ for all $x \in X$.*

Proof. If $A = \emptyset$, then $\Delta u \geq 0$ on X. This implies, since X is connected, that u is constant and the lemma is trivial. Same conclusion if $A = X$. Assume therefore that A is a non-empty proper subset of X.

Let $\beta = \max_{y\in A} u(y)$ and $\alpha = \max_{x\in X} u(x)$. Since $\beta \leq \alpha$, to prove the lemma we have to show that the assumption $\beta < \alpha$ leads to a contradiction. Assume therefore that $\beta < \alpha$. Suppose $u(x_0) = \alpha$ for some $x_0 \in X \setminus A$. Then $0 \leq \Delta u(x_0) = \sum t(x_0,y)[u(y) - u(x_0)]$,which implies (since $u(x_0)$ is the maximum value) that $u(y) = u(x_0) = \alpha$ for all $y \sim x_0$.

Now, if $x_0 \in \partial(X \setminus A)$, then there exits some $y \in A$ such that $y \sim x_0$ and hence $u(y) = \alpha$. This shows that $\alpha \leq \beta$ contradicting the assumption that $\beta < \alpha$. Consequently, all the neighbours of x_0 are in $X \setminus A$ itself and $u(z) = \alpha$ if $z \sim x_0$. Repeat the argument starting with $u(z) = \alpha$. Then, not to contradict the assumption, we should have all the neighbours of z in $X \setminus A$ itself.

Choose some $y \in A$. Since X is connected, there is a path $\{x_0, x_1, \ldots, x_n = y\}$ connecting x_0 and y. Let i be the largest index such that for all $j < i$, x_j is in the interior of $X \setminus A$. Note $1 \leq i \leq n-1$. Since x_{i-1} is in the interior of

$X \setminus A$ and $x_i \sim x_{i-1}$, we should have $x_i \in X \setminus A$. But x_i is not in the interior of $X \setminus A$ so that $x_i \in \partial(X \setminus A)$. Hence there is some $y_1 \in A$ such that $x_i \sim y_1$. Now from what we have proved earlier $u(x_i) = \alpha$ so that $u(y_1) = \alpha$ also. This means $\beta \geq \alpha$ since $y \in A$, a contradiction.

Thus, the assumption $\alpha > \beta$ is not tenable, so that $\alpha = \beta$. That is, $\max_{x \in X} u(x) = \max_{y \in A} u(y)$. □

We single out now two remarks coming out of the above lemma.

Remark 1 *Let u be a real valued function on X such that $\Delta u = 0$ on a subset $F \neq X$. Then, for any $x \in X$,*

$$\inf_{y \in X \setminus F} u(y) \leq u(x) \leq \sup_{y \in X \setminus F} u(y).$$

Remark 2 *(Uniqueness) Let u be a real valued function on X such that $\Delta u = 0$ on a proper subset F of X and $u = 0$ on $X \setminus F$. Then $u \equiv 0$.*

For $1 \leq k \leq n-1$, let Δ_k denote a $k \times k$ symmetric submatrix of Δ, obtained by deleting $n-k$ rows and the corresponding $n-k$ columns from Δ. Then, Remark 2 has an equivalent formulation.

Theorem 3.1. *For any k, $1 \leq k \leq n-1$, Δ_k is a non-singular matrix.*

Proof. Let the vertices corresponding to the rows in Δ_k form the proper subset F of X. Let $u = (u_1, \ldots, u_k)^t$ be a column vector such that $\Delta_k u = 0$.

Let $v = (v_1, \ldots, v_n)^t$ be a column vector of order n obtained from u by introducing 0 for the components missing in u. Now consider the column vector Δv. It can be calculated that $\Delta v(x) = \Delta_k u(x) = 0$ if $x \in F$. Thus, v is a function defined on X such that $\Delta v = 0$ on F and $v = 0$ on$X \setminus F$. Then, by Remark 2, $v \equiv 0$. Consequently $u \equiv 0$.

Thus, we have proved that if u is a solution for the homogeneous system $\Delta_k u = 0$, then $u \equiv 0$. This implies that Δ_k is a non-singular matrix. □

4. Some consequences of the lemma

In this section, we derive some potential -theoretic consequences of Lemma 3.1 and Theorem 3.1. Suppose F is a subset of X. Let us write Δ_F to denote the submatrix of Δ obtained by deleting the rows and the columns corresponding to the vertices not found in F.

The equilibrium principle, given in Bendito et al. [3], states that given a proper subset $F \subset X$, there exists a unique function $\gamma^F \geq 0$ on X such that $\Delta\gamma^F = -1$ and $F = \{x : \gamma^F(x) \neq 0\}$.

Theorem 4.1. *(Generalized Equilibrium Principle) Let F be a proper subset of X. Then, given $f \geq 0$ on F, there exists a unique u on X such that $u \geq 0$ on X, $\Delta u = -f$ on F and $u = 0$ on $X \setminus F$. Moreover, if $f > 0$ on F, then $u > 0$ on F.*

Proof. Since Δ_F is non-singular, there exists a unique function v on F such that $\Delta_F v = -f$ on F. Define u on X such that $u = v$ on F and $u = 0$ on $X \setminus F$. Then $\Delta u = -f$ on F. Consequently, if $A = \{y : \Delta u(y) > 0\}$, then $A \subset X \setminus F$ and for any $x \in X$,

$$\begin{aligned} u(x) &\geq \min_{y\in A} u(y) && \text{(Lemma 3.1)} \\ &= 0 && \text{(Since } u = 0 \text{ on } X \setminus F \supset A). \end{aligned}$$

Suppose now $f > 0$ on F. Let u attain its minimum value on F at a vertex x_0. Since $u \geq 0$ on F, if $u(x_0) > 0$, then $u > 0$ on F. On the contrary, if $u(x_0) = 0$ then $\Delta u(x_0) = -f(x_0) < 0$ reads us $\sum t(x_0, y)[u(y) - u(x_0)] = \Delta u(x_0) = -f(x_0) < 0$, a contradiction since the left side equals $\sum t(x_0, y)u(y) \geq 0$.

The uniqueness of the solution u follows from Remark 2. □

We immediately have a solution to the Kirchhoff's problem as given in Biggs [4].

Theorem 4.2. *(Kirchhoff's Problem) Let $f(x)$ be a real-valued function on X. Let z be a fixed vertex in X. Then there exists a unique function u on X such that $\Delta u(x) = f(x)$ if $x \neq z$ and $\Delta u(z) = -\sum_{x\neq z} f(x)$.*

Proof. Take $F = X \setminus \{z\}$. Then by Theorem 4.1, there exists a unique function u on X such that $\Delta u(x) = f(x)$ if $x \in F$ and $u = 0$ on $X \setminus F$. That is, $\Delta u(x) = f(x)$ if $x \neq z$ and $u(z) = 0$. Now for any real- valued function v on X, $\sum_{x\in X} \Delta v(x) = 0$. Hence $\Delta u(z) = -\sum_{x\neq z} \Delta u(x) = -\sum_{x\neq z} f(x)$. □

Let F be a proper subset of X. Let f be a real-valued function on ∂F. Then the Dirichlet problem searches for a function u on F such that $u = f$ on ∂F and $\Delta u = 0$ on $\overset{o}{F}$. On the other hand the Poisson problem searches for a solution of v on X such that $\Delta v = g$ on X, where g is defined on X; we know a solution to the Poisson problem from 4.2 above, provided $\sum_{x\in X} g(x) = 0$. A combination of these two problems, Poisson-Dirichlet problem, is solved in Bendito et al. [3] by using appropriate Green functions. Here we get a solution as a consequence of Theorem 4.1.

Theorem 4.3. *(Poisson-Dirichlet Problem) Let F be a proper subset of X. Let f be a real-valued function on F and g be a real-valued function on $X \setminus F$. Then there exists a unique function u on X such that $\Delta u = f$ on F and $u = g$ on $X \setminus F$.*

Proof. Let us assume, without loss of generality, that g is defined on X. Let $h = \Delta g$ on X. Then by Theorem 4.1, there exists v on X such that $\Delta v = f - h$ on F and $v = 0$ on $X \setminus F$. Define $u = v + g$ on X. Then, on F, $\Delta u = \Delta v + \Delta g = (f - h) + h = f$; and on $X \setminus F$, $u = 0 + g$. Thus, u on X satisfies the conditions $\Delta u = f$ on F and $u = g$ on $X \setminus F$. The uniqueness follows from Remark 2. □

Note 1 *For the Dirichlet problem, g is given on the boundary ∂F of a proper subset F of X. However, by giving arbitrary values, we can assume g is defined on $X \setminus \overset{o}{F}$. Then there exists a unique solution u such that $\Delta u = 0$ on $\overset{o}{F}$ and $u = g$ on ∂F.*

Note 2 *To determine the unique solution u in Theorem 4.3, we proceed as follows: Let the number of vertices in F be k. Since $u = g$ on $X \setminus F$, we have only to find the k values for u on F. Denoting them by $(u_1, u_2, \ldots, u_k)$, if we* *take out the k-linear equations in $\Delta u = f$ on F, then we have a linear system given by*

$$\Delta_F(u_1, u_2, \ldots, u_k)^t = (g_1, g_2, \ldots, g_k)^t,$$

where the right side is known. We also know that Δ_F is invertible. Hence $u_1, u_2, \ldots, u_k$ can be calculated.

We shall present the solution to a mixed boundary-value problem on a proper subset $F \subset X$, comprising Poisson, Dirichlet and Neumann problems (see Bendito et al.[3]).

Theorem 4.4. *(Mixed boundary-value problem) Let F be a proper subset of X. Let A and B be non-empty disjoint subsets such that $A \cup B = \partial F$. Let f be a real-valued function F. Then there exists a unique function u on F such that $\Delta u(x) = f(x)$ if $x \in \overset{o}{F}$, $u(s) = f(s)$ if $s \in A$ and $\frac{\partial u}{\partial n^-}(s) = f(s)$ if $s \in B$.*

Proof. Let Δ_F^* be the Laplacian restricted to F. Assume that f is arbitrarily extended outside F to cover $X \backslash F$. Then, by 4.1, there exists a unique function u on F such that $u = f$ on A and $\Delta_F^* = f$ on $F \setminus A$. Since $\Delta u(x) = \Delta_F^* u(x)$ if $x \in \overset{o}{F}$ and $\Delta_F^* u(s) = \frac{\partial u}{\partial n^-}(s)$ if $s \in \partial F$, we have the following properties for u, namely $\Delta u(x) = f(x)$ if $x \in \overset{o}{F}$, $u(s) = f(s)$ if $s \in A$ and $\frac{\partial u}{\partial n^-}(s) = f(s)$ if $s \in B$. □

Note 3 *If $A = \emptyset$ and $f = 0$ on $\overset{o}{F}$, the above mixed boundary value problem turns into the Neumann problem which has a solution if and only if $\sum_{s \in \partial F} f(s) = 0$, as remarked in the paragraph after Theorem 4.2; when a solution exists, it is unique up to an additive constant.*

Theorem 4.5. *(Condenser Principle) Let A and B be two non-empty disjoint subsets of X. Let $F = X \setminus (A \cup B) \neq \emptyset$. Let a and b be two real numbers, $a < b$. Then there exists a unique φ on X such that $a \leq \varphi(x) \leq b$ if $x \in X$,*

$$\begin{aligned} \varphi(x) = a \quad & \text{and } \Delta\varphi(x) \geq 0 \quad \text{if } x \in A, \\ \varphi(x) = b \quad & \text{and } \Delta\varphi(x) \leq 0 \quad \text{if } x \in B, \\ & \text{and } \Delta\varphi(x) = 0 \quad \text{if } x \in F. \end{aligned}$$

Proof. In Note 3, take $f = 0$, $g(x) = a$ if $x \in A$ and $g(x) = b$ if $x \in B$. Then there exists a unique function φ on X such that $\varphi(x) = a$ if $x \in A$, $\varphi(x) = b$ if $x \in B$, and $\Delta\varphi(x) = 0$ if $x \in F$. By Remark 1, $a \leq \varphi(x) \leq b$ if $x \in X$. Consequently, the inequalities $\Delta\varphi \geq 0$ on A and $\Delta\varphi \leq 0$ on B follow from the definition of Δ. □

Theorem 4.6. *(Balayage) Let F be a proper subset of X and let u be a real-valued function on X such that $\Delta u \leq 0$ on $X \setminus F$. Then there exists a unique function v on X such that $v \leq u$ on X, $v = u$ on F, $\Delta v = 0$ on $X \setminus F$ so that $\sum_{y \in X \setminus \overset{o}{F}} \Delta u(y) = \sum_{s \in \partial F} \Delta v(s)$.*

Proof. By Theorem 4.1, there exists φ on X such that $\Delta\varphi = \Delta u$ on $X \setminus F$ and $\varphi = 0$ on F. Let $v = u - \varphi$ on X. Then, $v = u$ on F and $\Delta v = 0$ on $X \setminus F$. Let $w = u - v$ on X. Then $\Delta w = \Delta u - \Delta v \leq 0$ on $X \setminus F$ by hypothesis, and $w = 0$ on F. Hence by Lemma 3.1, $w \geq 0$ on X, that is $u \geq v$ on X. Further, since $\sum_{x \in X} \Delta u(x) = \sum_{x \in X} \Delta v(x) = 0$ and since $\sum_{x \in \overset{o}{F}} \Delta u(x) = \sum_{x \in \overset{o}{F}} \Delta v(x)$, we have

$$\begin{aligned} \sum_{x \in X \setminus \overset{o}{F}} \Delta u(x) &= \sum_{x \in X \setminus \overset{o}{F}} \Delta v(x) \\ &= \sum_{x \in \partial F} \Delta v(x) + \sum_{x \in X \setminus F} \Delta v(x) \\ &= \sum_{x \in \partial F} \Delta v(x) + 0 \qquad \square \end{aligned}$$

Note 4 If we interpret $-\Delta u(x)$ as the charge associated with u at the vertex x, then the last assertion indicates that the total charge of u outside F has been swept onto ∂F to obtain v.

Theorem 4.7. *(Domination Principle) Let u and v be two real-valued functions on X. Let F be a non-empty subset of X. suppose $u \geq v$ on F and $\Delta u \leq \Delta v$ on $X \setminus F$. Then $u \geq v$ on X.*

Proof. Let $w = u - v$ on X. Then $w \geq 0$ on F and $\Delta w \leq 0$ on $X \setminus F$. Hence, by Lemma 3.1, $w \geq 0$ on X. □

Theorem 4.8. *(Green Function) Let F be a proper subset of X. Then, if $y \in \overset{o}{F}$, there exists a unique function $G_y^F(x) \geq 0$ on X such that $\Delta G_y^F(x) = -\delta_y(x)$ if $x \in \overset{o}{F}$, $G_y^F(s) = 0$ if $s \in X \setminus \overset{o}{F}$, and $G_y^F(x) \leq G_y^F(y)$ for any $x \in X$.*

Proof. Let $g(x) = -\delta_y(x)$ if $x \in \overset{o}{F}$. Then by Theorem 4.1, there exists a unique function on X, noted $G_y^F(x)$, such that $\Delta G_y^F(x) = g(x)$ on $\overset{o}{F}$ and $G_y^F(s) = 0$ if $s \in X \setminus \overset{o}{F}$. Moreover, since $\Delta G_y^F \geq 0$ on $\overset{o}{F}$ and $G_y^F = 0$ on $X \setminus \overset{o}{F}$, we conclude (by Lemma 3.1) that $G_y^F \geq 0$ on X. (It can be seen that $G_y^F > 0$ on $\overset{o}{F}$ if $\overset{o}{F}$ is connected.)

To prove the last assertion, note that $G_y^F(x) = G_y^F(y)$ if $x = y$; and if $x \neq y$, then $\Delta G_y^F(x) \geq 0$. Consequently, note that $\Delta G_y^F(y) < 0$. Hence, by Lemma 3.1, $G_y^F(x) \leq G_y^F(y)$ for all x in X. □

Theorem 4.9. *(Poisson Kernel) Let F be a proper subset of X. Let $G_x^F(y)$ be the Green function on F for $x \in \overset{o}{F}$, $y \in F$. Let $f(s)$ be defined on ∂F. Then the Dirichlet solution in F with boundary value $f(s)$ is $\sum_s f(s) \frac{\partial G_x^F}{\partial n^-}(s)$.*

Proof. Let Δ_F^* be the Laplacian restricted to Δ. Since Δ_F^* is symmetric, for any two real-valued functions u and v on F, we have $(u, \Delta_F^* v) = (\Delta_F^* u, v)$, which can be written as

$$\sum_{y \in F} [u(y)\Delta_F^* v(y) - v(y)\Delta_F^* u(y)] = 0,$$

or, if we express Δ_F^* in terms of Δ and the inner normal derivatives at the boundary vertices of F (see Abodayeh and Anandam [1, Theorem 4, p.419]), can be written as

$$\sum_{y \in \overset{o}{F}} [u(y)\Delta v(y) - v(y)\overset{o}{F} u(y)] = -\sum_{s \in \partial F} [u(s)\frac{\partial v}{\partial n^-}(s) - v(s)\frac{\partial u}{\partial n^-}(s)].$$

Let now $h(x)$ be the Dirichlet solution in F with boundary value $f(S)$ (see Note 1). Take $v = h$ and $u = G_x^F$ in the above equation. Since $\Delta h(y) = 0$ if $y \in \overset{o}{F}$, $h(s) = f(s)$ if $s \in \partial F$, $G_x^F(s) = 0$ if $x \in \overset{o}{F}$ and $s \in \partial F$, and

$\Delta G_x^F = -\delta_x$ if $x \in \overset{o}{F}$, we obtain

$$h(s) = \sum_s f(s) \frac{\partial G_x^F}{\partial n^-}(s). \qquad \square$$

Note 5 *For $x \in \overset{o}{F}$ and $s \in \partial F$, $p(x,s) = \frac{\partial G_x^F}{\partial n^-}(s)$ is the Poisson kernel for F.*

Note 6 *Alternatively, for $x \in \overset{o}{F}$ and $s \in \partial F$, the Poisson kernel $p(x,s)$ for fixed $s \in \partial F$, can be defined as the Dirichlet solution on F with boundary value $\delta_s(z)$ for $z \in \partial F$. This way, we can avoid introducing the Green identity to define the Poisson kernel.*

Note 7 *If $a, b \in \overset{o}{F}$, then $G_a^F(b) = G_b^F(a)$. This follows immediately from the above Green identity involving u and v, if we set $u(x) = G_a^F(x)$ and $v(x) = G_b^F(x)$.*

Acknowledgement. The first author is very grateful to Prince Sultan university for the enormous help that they made by supporting this project financially.

References

[1] K. Abodayeh and V. Anandam, Dirichlet problem and Green's formulas on trees, *Hiroshima Math.J.,* **35** (2005), 413-424.

[2] V. Anandam, and I. Bajunaid, Some aspects of the classical potential theory, *Hiroshima Math.J.,* **37** (2007), 277 - 314.

[3] E. Bendito, A. Carmona and A.M. Encinas, Solving boundary value problems on networks using equilibrium measures, *Journal of Functional Analysis,* **171** (2000), 155-176.

[4] N. Biggs, Algebraic potential theory on graphs, *Bull.London Math.Soc.,* **29** (1997), 641 - 682.

[5] M. Brelot, Éléments de la théorie classique du potentiel, 3^e *édition, CDU, Paris,* 1965.

[6] F. Chung and S.T. Yau, Discrete Green's functions, *J.Comb. Theory,* **91** (2000), 191 - 214.

K. ABODAYEH

Department of Mathematical Sciences,
Prince Sultan University,
P.O. Box 66833, Riyadh 11586,
Saudi Arabia.
E-mail: kamal@oyp.psu.edu.sa

V. ANANDAM

Ramanujan Institute for Advanced Study in Mathematics,
University of Madras,
Chennai- 600005, India.
E-mail: bhickooa@yahoo.co.uk

Date received : December 24, 2008