Arab J. Math. Sc.
Volume 13, December 2007
pp.27-38

ON LIMITING CASE OF THE SOBOLEV THEOREM FOR B-RIESZ POTENTIAL IN B-MORREY SPACES

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ABSTRACT.We consider the generalized shift operator, associated with the Laplace-Bessel differential operator $\Delta_B = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}$, $\gamma > 0$. We study the fractional maximal operator M_γ^α (fractional B-maximal operator) and Riesz potential I_γ^α (B-Riesz potential), associated with the generalized shift operator and its modified version $\widetilde{I}_\gamma^\alpha$ (modified B-Riesz potential) in the B-Morrey space.

1. INTRODUCTION

The maximal operator, Riesz potential and related topics associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}, \quad \gamma > 0$$

have been investigated by many researchers, see B. Muckenhoupt and E.Stein [10], I. Kipriyanov [9], K. Trimeche [16], L. Lyakhov [8], K. Stempak [15], A.D. Gadjiev and I.A. Aliev [2], V.S. Guliyev [3, 4, 5], V.S. Guliyev and J.J. Hasanov [6], A. Serbetci, I. Ekincioglu [13] and others.

In this paper we consider the generalized shift operator, generated by the Laplace-Bessel differential operator Δ_B in terms of which we study boundedness of the modified B-Riesz potential in the B-Morrey space to B-BMO space.

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[1]*partially supported by the INTAS grant Ref. Nr 06-1000015-5635.*

Mathematics Subject Classification: Primary 42B20, 42B25, 42B35

Key Words: B-maximal operator, fractional B-maximal operator, B-Riesz potential, B-Morrey space, B-BMO space.

The structure of the paper is as follows. In Section 1 we present some definitions and auxiliary results. In Section 2 we study some embeddings into the B-Morrey spaces. The statement of main results of the paper is the inequality of Sobolev-Morrey type in limiting case for the B-Riesz potential, established in Section 3.

2. PRELIMINARIES

Suppose that $\mathbb{R}^n$ is n-dimensional Euclidean space, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x|^2 = \sum_{i=1}^n x_i^2$, $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x = (x', x_n) \in \mathbb{R}^n$, $\mathbb{R}_+^n = \{x = (x', x_n) \in \mathbb{R}^n; x_n > 0\}$, $E(x,r) = \{y \in \mathbb{R}_+^n \; ; \; |x - y| < r\}$, $E(0,r) = E_r$, $A^* = \mathbb{R}_+^n \backslash A$, $\gamma > 0$.

For measurable $E \subset \mathbb{R}_+^n$ suppose $|E|_\gamma = \int_E x_n^\gamma dx$, then $|E_r|_\gamma = \omega(n,\gamma) r^{n+\gamma}$, where

$$\omega(n,\gamma) = \int_{E_1} x_n^\gamma dx = \frac{\pi^{\frac{n-1}{2}}}{2} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)}.$$

Let's denote by T^y the generalized shift operator (B–shift operator) acting according to the law

$$T^y f(x) = C_\gamma \int_0^\pi f\left(x' - y', (x_n, y_n)_\beta\right) \, d\nu(\beta),$$

where $(x_n, y_n)_\beta = (x_n^2 - 2x_n y_n \cos\beta_n + y_n^2)^{\frac{1}{2}}$, $d\nu(\beta) = \sin^{\gamma-1}\beta \, d\beta$ and

$$C_\gamma = \pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)} = \frac{2}{\pi}\omega(2,\gamma).$$

We remark that the generalized shift operator T^y is closely connected with the Bessel differential operator B (see [7]).

The translation operator T^y generates the corresponding B-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}_+^n} f(y) [T^y g(x)] y_n^\gamma dy.$$

Lemma 2.1. *Let* $0 < \alpha < n+\gamma$. *Then for* $2|x| \le |y|$ *the following inequality is valid*

$$\left|T^y|x|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma}\right| \le 2^{n+\gamma-\alpha+1}|y|^{\alpha-n-\gamma-1}|x|. \tag{2.1}$$

Proof. Let's show that

$$\left|T^y|x|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma}\right|$$
$$\le C_\gamma \int_0^\pi \left|\left|(x'-y',(x_n,y_n)_\beta)\right|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma}\right| d\nu(\beta).$$

First we estimate

$$\left|\left|(x'-y',(x_n,y_n)_\beta)\right|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma}\right|.$$

By the theorem about mean value we get

$$\left|\left|(x'-y',(x_n,y_n)_\beta)\right|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma}\right|$$
$$\le \left|\left|(x'-y',(x_n,y_n)_\beta)\right| - |y|\right| \cdot \xi^{\alpha-n-\gamma-1},$$

where $\min\{|(x'-y',(x_n,y_n)_\beta)|, |y|\} \le \xi \le \max\{|(x'-y',(x_n,y_n)_\beta)|, |y|\}$.

We note that

$$\left|(x'-y',(x_n,y_n)_\beta)\right| \le |x|+|y| \le \frac{3}{2}|y|,$$
$$\left|(x'-y',(x_n,y_n)_\beta)\right| \ge |x-y| \ge |y|-|x| \ge \frac{1}{2}|y|$$

and

$$\left|(x'-y',(x_n,y_n)_\beta)\right| - |y| \le |x|+|y|-|y| \le |x|$$
$$|y| - \left|(x'-y',(x_n,y_n)_\beta)\right| \le |y|-|x-y| \le |x|.$$

Hence

$$\frac{1}{2}|y| \le \left|(x'-y',(x_n,y_n)_\beta)\right| \le \frac{3}{2}|y|,$$
$$\left|\left|(x'-y',(x_n,y_n)_\beta)\right| - |y|\right| \le |x|.$$

Thus we obtain (2.1). □

3. SOME DEFINITIONS AND AUXILIARY RESULTS

Let $L_{p,\gamma}(\mathbb{R}^n_+)$ be the space of measurable functions on $\mathbb{R}^n_+$ with finite norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\mathbb{R}^n_+)} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p x_n^\gamma dx\right)^{1/p}, \quad 1 \le p < \infty.$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_+)$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_\infty} = \underset{x\in\mathbb{R}^n_+}{esssup}\,|f(x)|.$$

Definition 3.1. Let $1 \le p < \infty$. We denote by $WL_{p,\gamma}(\mathbb{R}^n_+)$ the weak $L_{p,\gamma}$ space defined as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_+$ with the finite norms

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r \left|\{x \in \mathbb{R}^n_+ \ : \ |f(x)| > r\}\right|_\gamma^{1/p}.$$

Definition 3.2. [3] Let $1 \le p < \infty$, $0 \le \lambda \le n+\gamma$. We denote by $L_{p,\lambda,\gamma}(\mathbb{R}^n_+)$ Morrey space ($\equiv B$-Morrey space), associated with the Laplace-Bessel differential operator as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_+$, with the finite norm

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{t>0,\, x\in\mathbb{R}^n_+} \left(t^{-\lambda}\int_{E_t} (T^y\,|f(x)|)^p\, y_n^\gamma dy\right)^{1/p}.$$

Note that

$$L_{p,0,\gamma}(\mathbb{R}^n_+) = L_{p,\gamma}(\mathbb{R}^n_+),$$

and if $\lambda < 0$ or $\lambda > n+\gamma$, then $L_{p,\lambda,\gamma}(\mathbb{R}^n_+) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n_+$.

[4] Let $1 \le p < \infty$. Then

$$L_{p,n+\gamma,\gamma}(\mathbb{R}^n_+) = L_\infty(\mathbb{R}^n_+) \text{ and } \|f\|_{L_{p,n+\gamma,\gamma}} = \omega(n,\gamma)^{1/p}\,\|f\|_{L_\infty}.$$

Definition 3.3. [6] Let $1 \le p < \infty$,$0 \le \lambda \le n+\gamma$. We denote by $WL_{p,\lambda,\gamma}(\mathbb{R}^n_+)$ the weak B-Morrey space as the set of locally integrable functions $f(x)$,$x \in \mathbb{R}^n_+$ with finite norm

$$\|f\|_{WL_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}^n_+} \left(t^{-\lambda}\int_{\{y\in E_t:\, T^y|f(x)|>r\}} y_n^\gamma dy\right)^{1/p}.$$

We note that

$$L_{p,\lambda,\gamma}(\mathbb{R}^n_+) \subset WL_{p,\lambda,\gamma}(\mathbb{R}^n_+) \text{ and } \|f\|_{WL_{p,\lambda,\gamma}} \le \|f\|_{L_{p,\lambda,\gamma}}.$$

Definition 3.4. [3] We denote by $BMO_\gamma(\mathbb{R}^n_+)$ B–BMO space the set of locally integrable functions $f(x), x \in \mathbb{R}^n_+$, with finite norms

$$\|f\|_{*,\gamma} = \sup_{r>0,\ x\in\mathbb{R}^n_+} |E_r|_\gamma^{-1} \int_{E_r} |T^y f(x) - f_{E_r}(x)| y_n^\gamma dy < \infty,$$

where $f_{E_r}(x) = |E_r|_\gamma^{-1} \int_{E_r} T^y f(x) y_n^\gamma dy$.

We define the B-maximal operator M_γ as

$$M_\gamma f(x) = \sup_{r>0} |E_r|_\gamma^{-1} \int_{E_r} T^y |f(x)| y_n^\gamma dy$$

and fractional B-maximal operator M_γ^α as

$$M_\gamma^\alpha f(x) = \sup_{r>0} |E_r|_\gamma^{\frac{\alpha}{n+\gamma}-1} \int_{E_r} T^y |f(x)| y_n^\gamma dy.$$

In [6] the following theorem was proved.

Theorem 3.1. *1. Let $f \in L_{1,\lambda,\gamma}(\mathbb{R}^n_+)$, $0 \le \lambda < n+\gamma$, then $M_\gamma f \in WL_{1,\lambda,\gamma}(\mathbb{R}^n_+)$ and*

$$\|M_\gamma f\|_{WL_{1,\lambda,\gamma}} \le C_{1,\lambda,\gamma} \|f\|_{L_{1,\lambda,\gamma}},$$

where $C_{1,\lambda,\gamma}$ depends only on λ,γ and n.

2. Let $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_+)$, $1 < p < \infty$,$0 \le \lambda < n+\gamma$, then $M_\gamma f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_+)$ and

$$\|M_\gamma f\|_{L_{p,\lambda,\gamma}} \le C_{p,\lambda,\gamma} \|f\|_{L_{p,\lambda,\gamma}},$$

where $C_{p,\lambda,\gamma}$ depends only on p,λ,γ and n.

Corollary 3.1. [3, 5]

1. Let $f \in L_{1,\gamma}(\mathbb{R}^n_+)$, then $M_\gamma f \in WL_{1,\gamma}(\mathbb{R}^n_+)$ and

$$\|M_\gamma f\|_{WL_{1,\gamma}} \le C_{1,\gamma} \|f\|_{L_{1,\gamma}},$$

where $C_{1,\gamma}$ depends only on γ and n.

2. Let $f \in L_{p,\gamma}(\mathbb{R}^n_+)$, $1 < p < \infty$, then $M_\gamma f \in L_{p,\gamma}(\mathbb{R}^n_+)$ and

$$\|M_\gamma f\|_{L_{p,\gamma}} \leq C_{p,\gamma} \|f\|_{L_{p,\gamma}},$$

where $C_{p,\gamma}$ depends only on p, γ and n.

We consider the B-Riesz potential

$$I^\alpha_\gamma f(x) = \int_{\mathbb{R}^n_+} T^y |x|^{\alpha-n-\gamma} f(y) y_n^\gamma dy, \quad 0 < \alpha < n+\gamma,$$

and the modified B-Riesz potential

$$\widetilde{I}^\alpha_\gamma f(x) = \int_{\mathbb{R}^n_+} \left(T^y |x|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma} \chi_{E_1^*}(y) \right) f(y) y_n^\gamma dy,$$

where $E_1^* = \mathbb{R}^n_+ \backslash E_1$.

In [6] the following theorem was proved.

Theorem 3.2. *Let $0 < \alpha < n+\gamma$, $0 \leq \lambda < n+\gamma-\alpha$ and $1 \leq p < \frac{n+\gamma-\lambda}{\alpha}$.*

1) If $p = 1$, $1 - \frac{1}{q} = \frac{\alpha}{n+\gamma-\lambda}$, then the operator I^α_γ is bounded from $L_{1,\lambda,\gamma}(\mathbb{R}^n_+)$ to $WL_{q,\lambda,\gamma}(\mathbb{R}^n_+)$.

2) If $1 < p < \frac{n+\gamma-\lambda}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+\gamma-\lambda}$, then the operator I^α_γ is bounded from $L_{p,\lambda,\gamma}(\mathbb{R}^n_+)$ to $L_{q,\lambda,\gamma}(\mathbb{R}^n_+)$.

Corollary 3.2. *Let $0 < \alpha < n+\gamma$, $0 \leq \lambda < n+\gamma-\alpha$ and $1 \leq p < \frac{n+\gamma-\lambda}{\alpha}$.*

1) If $p = 1$, $1 - \frac{1}{q} = \frac{\alpha}{n+\gamma-\lambda}$, then the operator M^α_γ is bounded from $L_{1,\lambda,\gamma}(\mathbb{R}^n_+)$ to $WL_{q,\lambda,\gamma}(\mathbb{R}^n_+)$.

2) If $1 < p < \frac{n+\gamma-\lambda}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+\gamma-\lambda}$, then the operator M^α_γ is bounded from $L_{p,\lambda,\gamma}(\mathbb{R}^n_+)$ to $L_{q,\lambda,\gamma}(\mathbb{R}^n_+)$.

Corollary 3.3. [2] *Let $0 < \alpha < n+\gamma$ and $1 \leq p < \frac{n+\gamma}{\alpha}$.*

1) If $p = 1$, $1 - \frac{1}{q} = \frac{\alpha}{n+\gamma}$, then the operator I^α_γ is bounded from $L_{1,\gamma}(\mathbb{R}^n_+)$ to $WL_{q,\gamma}(\mathbb{R}^n_+)$.

2) If $1 < p < \frac{n+\gamma}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+\gamma}$, then the operator I^α_γ is bounded from $L_{p,\gamma}(\mathbb{R}^n_+)$ to $L_{q,\gamma}(\mathbb{R}^n_+)$.

4. STATEMENT OF MAIN RESULTS

The following theorems from our main result in which we obtain the fractional B-maximal operator M_γ^α to be bounded from the Morrey spaces $L_{p,\lambda,\gamma}(\mathbb{R}_+^n)$ to $L_\infty(\mathbb{R}_+^n)$ and the modified fractional integral operator $\widetilde{I}_\gamma^\alpha$ to be bounded from the spaces $L_{p,\lambda,\gamma}(\mathbb{R}_+^n)$ to $BMO_\gamma(\mathbb{R}_+^n)$ under the limiting case $p = \frac{n+\gamma-\lambda}{\alpha}$.

Theorem 4.1. *Let $0 < \alpha < n+\gamma$, $0 \le \lambda < n+\gamma-\alpha$ and $1 < p = \frac{n+\gamma-\lambda}{\alpha}$, then the operator M_γ^α is bounded from $L_{p,\lambda,\gamma}(\mathbb{R}_+^n)$ to $L_\infty(\mathbb{R}_+^n)$.*

Theorem 4.2. *Let $0 < \alpha < n+\gamma$, $0 \le \lambda < n+\gamma-\alpha$ and $1 < p = \frac{n+\gamma-\lambda}{\alpha}$, then the operator $\widetilde{I}_\gamma^\alpha$ is bounded from $L_{p,\lambda,\gamma}(\mathbb{R}_+^n)$ to $BMO_\gamma(\mathbb{R}_+^n)$.*

Moreover, if for $f \in L_{p,\lambda,\gamma}(\mathbb{R}_+^n)$ the integral $I_\gamma^\alpha f$ exists almost everywhere, then $I_\gamma^\alpha \in BMO_\gamma(\mathbb{R}_+^n)$ and the following inequality is valid

$$\|I_\gamma^\alpha f\|_{BMO_\gamma} \le C\|f\|_{L_{p,\lambda,\gamma}},$$

where $C > 0$ is independent of f.

Corollary 4.1. [3, 5] *Let $0 < \alpha < n+\gamma$ and $1 < p = \frac{n+\gamma}{\alpha}$, then the operator $\widetilde{I}_\gamma^\alpha$ is bounded from $L_{p,\gamma}(\mathbb{R}_+^n)$ to $BMO_\gamma(\mathbb{R}_+^n)$.*

Moreover, if for $f \in L_{p,\gamma}(\mathbb{R}_+^n)$ the integral $I_\gamma^\alpha f$ exists almost everywhere, then $I_\gamma^\alpha \in BMO_\gamma(\mathbb{R}_+^n)$ and the following inequality is valid

$$\|I_\gamma^\alpha f\|_{BMO_\gamma} \le C\|f\|_{L_{p,\gamma}},$$

where $C > 0$ is independent of f.

5. PROOF OF THEOREMS

Proof of Theorem 4.1. Let $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_+)$, $1 < p = \frac{n+\gamma-\lambda}{\alpha}$. Then applying Hölder's inequality we have

$$t^{\alpha-n-\gamma} \int\limits_{E_t} T^y |f(x)| y_n^\gamma dy$$

$$\leq t^{\alpha-n-\gamma} \left(\int\limits_{E_t} (T^y |f(x)|)^p \, y_n^\gamma dy \right)^{1/p} |E_t|_\gamma^{1/p'}$$

$$\leq \omega(n,\gamma)^{1/p'} \left(t^{-\lambda} \int\limits_{E_t} (T^y \, |f(x)|)^p \, y_n^\gamma dy \right)^{1/p}$$

$$\leq \omega(n,\gamma)^{1/p'} \, \|f\|_{L_{p,\lambda,\gamma}}.$$

Theorem 4.1 is proved. □

Proof of Theorem 4.2. Let $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_+)$, $1 < p = \frac{n+\gamma-\lambda}{\alpha}$. For given $t > 0$ we denote

$$f_1(z) = f(z)\chi_{E_{2t}}(z), \quad f_2(z) = f(z) - f_1(z), \tag{5.1}$$

where $\chi_{E_{2t}}$ is the characteristic function of the set E_{2t}. Then

$$\widetilde{I}^\alpha_\gamma f(z) = \widetilde{I}^\alpha_\gamma f_1(z) + \widetilde{I}^\alpha_\gamma f_2(z) = F_1(z) + F_2(z),$$

where

$$F_1(z) = \int\limits_{E_{2t}} \left(T^y |z|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma} \chi_{E_1^*}(y) \right) f(y) y_n^\gamma dy,$$

$$F_2(z) = \int\limits_{E_{2t}^*} \left(T^y |z|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma} \chi_{E_1^*}(y) \right) f(y) y_n^\gamma dy.$$

We note that the function f_1 has compact (bounded) support and thus

$$a_1 = - \int\limits_{E_{2t} \setminus E_{\min\{1,2t\}}} |y|^{\alpha-n-\gamma} f(y) y_n^\gamma dy$$

is finite.

We note also that

$$F_1(z) - a_1 = \int\limits_{E_{2t}} T^y |z|^{\alpha-n-\gamma} f(y) y_n^\gamma dy$$

$$- \int\limits_{E_{2t} \backslash E_{\min\{1,2t\}}} |y|^{\alpha-n-\gamma} f(y) y_n^\gamma dy$$

$$+ \int\limits_{E_{2t} \backslash E_{\min\{1,2t\}}} |y|^{\alpha-n-\gamma} f(y) y_n^\gamma dy$$

$$= \int\limits_{\mathbb{R}_+^n} T^y |z|^{\alpha-n-\gamma} f_1(y) y_n^\gamma dy.$$

Therefore

$$|F_1(z) - a_1| \leq \int\limits_{\mathbb{R}_+^n} |y|^{\alpha-n-\gamma} T^y |f_1(z)| y_n^\gamma dy$$

$$= \int\limits_{\{y \in \mathbb{R}_+^n : T^y|z| < 2t\}} |y|^{\alpha-n-\gamma} T^y |f(z)| y_n^\gamma dy.$$

Further, for $z \in E_t$, $T^y|z| < 2t$ we have

$$|y| \leq |z| + |z-y| \leq |z| + T^y|z| < 3t.$$

Consequently

$$|F_1(z) - a_1| \leq \int_{E_{3t}} |y|^{\alpha-n-\gamma} T^y \, |f(z)| \, y_n^\gamma dy, \tag{5.2}$$

if $z \in E_t$.

By Theorem 3.1 and inequality (5.2) for $\alpha p = n + \gamma - \lambda$ we get

$$|E_t|_\gamma^{-1} \int_{E_t} |T^x F_1(z) - a_1| \, z_n^\gamma dz$$

$$\leq |E_t|_\gamma^{-1} \int_{E_t} T^x \left(\int_{E_{3t}} |y|^{\alpha-n-\gamma} \, T^y |f(z)| y_n^\gamma dy \right) z_n^\gamma dz$$

$$\leq \frac{2^{n+\gamma-\alpha} 3^\alpha}{2^\alpha - 1} t^{\alpha-n-\gamma} \cdot t^{(n+\gamma)/p'} \left(\int_{E_t} T^x \left(M_\gamma(f(z)) \right)^p z_n^\gamma dz \right)^{1/p}$$

$$\leq C_1 \|f\|_{L_{p,\lambda,\gamma}}, \tag{5.3}$$

where $C_1 = \frac{2^{n+\gamma-\alpha}3^{\alpha}}{2^{\alpha}-1}\omega(n,\gamma)^{1/p'} \cdot C_{p,\lambda,\gamma}$.

We denote by

$$a_2 = \int\limits_{E_{\max\{1,2t\}}\backslash E_{2t}} |y|^{\alpha-n-\gamma} f(y) y_n^{\gamma} dy.$$

Let's estimate $|F_2(z) - a_2|$ for $z \in E_t$:

$$|F_2(z) - a_2| \le \int\limits_{E_{2t}^*} |f(y)| \left| T^y |z|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma} \right| y_n^{\gamma} dy.$$

Applying Lemma 2.1 and Hölder's inequality we have

$$|F_2(z) - a_2| \le 2^{n+\gamma-\alpha+1} |z| \int\limits_{E_{2t}^*} |f(y)| |y|^{\alpha-n-\gamma-1} y_n^{\gamma} dy$$

$$\le 2^{n+\gamma-\alpha+1} |z| \left(\int_{E_t^*} |y|^{-\beta} |f(y)|^p y_n^{\gamma} dy \right)^{1/p}$$

$$\times \left(\int_{E_t^*} |y|^{\left(\frac{\beta}{p}+\alpha-n-\gamma-1\right)p'} y_n^{\gamma} dy \right)^{1/p'} = 2^{n+\gamma-\alpha+1} |z| I_1 \cdot I_2,$$

where $\lambda < \beta < \lambda + p$.

For I_1 we get

$$I_1 = \Big(\sum_{j=0}^{\infty} \int_{E_{2^{j+1}t} \backslash E_{2^j t}} \left(T^y \left| f(x) \right| \right)^p |y|^{-\beta} y_n^{\gamma} dy \Big)^{1/p}$$

$$\le 2^{\frac{\lambda}{p}} t^{\frac{\lambda-\beta}{p}} \|f\|_{L_{p,\lambda,\gamma}} \Big(\sum_{j=0}^{\infty} 2^{(\lambda-\beta)j} \Big)^{1/p} = C_2 t^{\frac{\lambda-\beta}{p}} \|f\|_{L_{p,\lambda,\gamma}}, \tag{5.4}$$

where $C_2 = \left(\frac{2^{\beta}}{2^{\beta-\lambda}-1} \right)^{1/p}$.

For I_2 we obtain

$$I_2 = \left(\int_{\mathbb{S}_+^{n-1}} \xi_n^{\gamma} d\xi \int_t^{\infty} r^{n+\gamma-1+\left(\frac{\beta}{p}+\alpha-n-\gamma-1\right)p'} dr \right)^{\frac{1}{p'}} = C_3 t^{\frac{\beta-\lambda}{p}-1},$$

where $C_3 = (\omega(n,\gamma))^{1/p'} (p'^{1/p'} \left(1 - \frac{\beta-\lambda}{p} \right)^{-1/p'}$.

Then for $z \in E_t$

$$|F_2(z) - a_2| \leq C_4 |z| t^{-1} \|f\|_{L_{p,\lambda,\gamma}} \leq C_4 \|f\|_{L_{p,\lambda,\gamma}},$$

where $C_4 = C_2 \cdot C_3 \cdot 2^{n+\gamma-\alpha+1}$.

Thus for $\alpha p = n + \gamma - \lambda$ and for all $x \in \mathbb{R}^n_+$, $z \in E_t$ we obtain

$$|T^x F_2(z) - a_2| \leq T^x \, |F_2(z) - a_2| \leq C_4 \|f\|_{L_{p,\lambda,\gamma}}. \tag{5.5}$$

We denote by

$$a_f = a_1 + a_2 = \int\limits_{E_{\max\{1,2t\}}} |y|^{\alpha-n-\gamma} f(y) y_n^\gamma dy.$$

Finally, from (5.3) and (5.5) we have

$$\sup_{x,t} |E_t|_\gamma^{-1} \int\limits_{E_t} \left| T^x \widetilde{I}_\gamma^\alpha f(z) - a_f \right| z_n^\gamma dz \leq (C_1 + C_4) \|f\|_{L_{p,\lambda,\gamma}}.$$

Thus

$$\left\| \widetilde{I}_\gamma^\alpha f \right\|_{BMO_\gamma} \leq 2 \sup_{x,t} |E_t|_\gamma^{-1} \int\limits_{E_t} \left| T^z \widetilde{I}_\gamma^\alpha f(x) - a_f \right| z_n^\gamma dz \leq C_5 \|f\|_{L_{p,\lambda,\gamma}}.$$

Theorem 4.2 is proved. □

Acknowledgements. The authors express their thanks to Prof. V.S. Guliyev for helpful comments on the manuscript of this paper. Furthermore, the authors are thankful to referee for his valuable comments and suggestions.

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Date received October 17, 2007