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A FINITE DIFFERENCE METHOD FOR APPROXIMATING THE SOLUTION OF A CERTAIN CLASS OF SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. The objective of this paper is to provide a finite difference approximation for the solution of the second order boundary-value problem

$$\frac{1}{p(x)}(p(x)y')' - q(x)y = f(x) \quad x \in (0,1), \quad \lim_{x\to 0+} p(x)y'(x) = 0, \quad y(1) = 0$$

with appropriate conditions on the real valued functions $p(x), q(x)$ and $f(x)$. It is shown that an $O(h^2)$ approximation can be obtained using the finite differences scheme presented in this paper.

1. INTRODUCTION

In this paper we consider the boundary value problem

$$\frac{1}{p(x)}(p(x)y')' - q(x)y = f(x) \quad x \in (0,1), \tag{1.1}$$

$$\lim_{x\to 0+} p(x)y' = 0, \tag{1.2}$$

$$y(1) = 0, \tag{1.3}$$

$$p(x) \geq 0, p(0) = 0 \quad \text{and} \quad p(x) \text{ is increasing in a neighborhood } [0,\delta] \text{ of } 0, \tag{1.4}$$

$$p^{-1}(x) \in L^1_{loc}(0,1], \tag{1.5}$$

$$q(x), f(x) \in C[0,1] \quad \text{and} \quad |p'(x)| \leq M_1 < \infty, \tag{1.6}$$

$$q(x) \geq 0, \tag{1.7}$$

$$p'(x)\int_x^1 \frac{1}{p(\tau)}d\tau < M_2 \quad \text{for all } x \in (0,1). \tag{1.8}$$

We will discuss a finite difference approximation to the solution of the above problem. Problems of the form (1.1) are encountered in the study of radially or axially symmetric problems in which case the problem can be reduced to a one- dimensional problem with $p(x) = x^{\alpha}$. More general functions $p(x)$ are encountered in connection with string equations in dynamics and statistics [5]. Such problems are also encountered in stochastic control problems when studying the steady state properties of systems driven by noise which is proportional to the state or which are nonlinear functions of the state [3]. For example, the duffing equation driven by Brownian noise

$$dx = x^3 dt + x d\beta_t$$

where $d\beta_t$ is a Brownian motion, has a steady state Fokker-Planck equation of the form

$$\frac{e^{-3x}}{x^2}(x^2 e^{3x} p')' + 3p = 0.$$

Moreover, the Hermite and Laguarre operators provide other examples of singular two point boundary value problems [8]. There is a growing literature on numerical treatment of singular two-point boundary value problems ([1],[3],[6],[10],[14] and references therein). In particular, finite difference methods were discussed in the papers by Chawla et al [2], Doedel et al [4], Jamet [9], Nassif [11] and others. In [2], Chawla considers the special case with $p(x) = x^{\alpha}$, $\alpha \geq 1$, however, $f(x)$ in the right hand side of (1.1) is non-linear. In this paper, we extend the work in [2] by applying Chawla's finite difference scheme to the boundary value problem (1.1)-(1.3) with more general functions $p(x)$. We prove $O(h^2)$ convergence under less differentiability conditions on $q(x)$ and $f(x)$.

This paper is organized as follows. In Section 2 we develop the finite difference scheme that is used to approximate the solution to the problem. Convergence of the finite difference scheme and order of convergence is treated in Section 3. In Section 4 we prove the existence and uniqueness of the solution to problem (1.1)-(1.3) and that this solution is continuous on [0,1]. Finally, Section 5 provides numerical examples that demonstrate $O(h^2)$ convergence.

2. THE FINITE DIFFERENCE SCHEME

In this section we develop the finite difference scheme that we will employ to approximate the solution to the problem (1.1)-(1.3). We formally use Taylor series expansions, change of order of integration and integration by parts. Justification for using these techniques will be given in Section 4. Rewriting (1.1) in the form

$$(py')' = p(qy + f) \tag{2.1}$$

and integrating and applying the boundary conditions (1.2) and (1.3) we get

$$y(x) = -\int_x^1 \frac{1}{p(\tau)} \int_0^\tau p(t)q(t)y(t) + f(t))dtd\tau. \tag{2.2}$$

Interchanging the order of integration in (2.2) we get

$$\begin{aligned} y(x) &= -\int_0^x \left(\int_x^1 \frac{1}{p(\tau)}d\tau\right) p(t)(q(t)y(t) + f(t))dt \\ &\quad -\int_x^1 \left(\int_t^1 \frac{1}{p(\tau)}d\tau\right) p(t)(q(t)y(t) + f(t))dt. \end{aligned} \tag{2.3}$$

Written differently, (2.3) becomes

$$y(x) = -\int_0^1 K(x,t)p(t)(q(t)y(t) + f(t))dt \tag{2.4}$$

where $K(x,t)$ is the kernel and is given by

$$K(x,t) = \begin{cases} \int_x^1 \frac{1}{p(\tau)}d\tau & t \le x \\ \int_t^1 \frac{1}{p(\tau)}d\tau & t > x. \end{cases} \tag{2.5}$$

Now for $N \ge 2$ we consider a uniform mesh over the interval [0,1]. Let $x_k = kh$, $k = 0, \ldots, N$, $h = 1/N$. Let $y_k = y(x_k)$. For ease of notation, let $g(t) = q(t)y(t) + f(t)$. Using (2.4) and (2.5) we obtain for $x = x_k$ and $x = x_{k+1}$, $k \ge 1$

$$\begin{aligned} y_{k+1} - y_k &= \int_{x_k}^{x_{k+1}} \frac{1}{p(\tau)}d\tau \int_0^{x_k} p(t)g(t)dt \\ &\quad + \int_{x_k}^{x_{k+1}} \left(\int_t^{x_{k+1}} \frac{1}{p\tau}d\tau\right) p(t)g(t)dt. \end{aligned} \tag{2.6}$$

Similarly, for $k \geq 2$ we have

$$
\begin{aligned}
y_k - y_{k-1} &= \int_{x_{k-1}}^{x_k} \frac{1}{p(\tau)} d\tau \int_0^{x_k} p(t)g(t)dt \\
&\quad - \int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} \frac{1}{p\tau} d\tau \right) p(t)g(t)dt. \qquad (2.7)
\end{aligned}
$$

In order to simplify the notation, we let

$$\psi_k(t) = \int_t^{x_{k+1}} \frac{1}{p(\tau)} d\tau, \quad k \geq 1. \qquad (2.8)$$

Eliminating $\int_0^{x_k} p(t)g(t)dt$ from (2.6) and (2.7) we obtain for $k \geq 2$

$$
\begin{aligned}
\Delta y_k &\equiv -\frac{y_{k+1}}{\psi_k(x_k)} + \left(\frac{1}{\psi_k(x_k)} + \frac{1}{\psi_{k-1}(x_{k-1})} \right) y_k - \frac{y_{k-1}}{\psi_{k-1}(x_{k-1})} \\
&= -\int_{x_{k-1}}^{x_{k+1}} p(t)U_k(t)g(t)dt \qquad (2.9)
\end{aligned}
$$

where

$$U_k(t) = \begin{cases} 1 - \psi_{k-1}(t)/\psi_{k-1}(x_{k-1}) & \text{if } x_{k-1} \leq t \leq x_k \\ \psi_k(t)/\psi_k(x_k) & \text{if } x_k \leq t \leq x_{k+1}. \end{cases} \qquad (2.10)$$

Now writing the Taylor polynomial for $y(t)$ in Equation (2.9) in the interval $[x_{k-1}, x_{k+1}]$ around x_k we obtain

$$\Delta y_k = -A_k y_k - B_k - C_k \quad \text{for } k \geq 2 \qquad (2.11)$$

where

$$
\begin{aligned}
A_k &= \int_{x_{k-1}}^{x_{k+1}} p(t)U_k(t)q(t)dt, \\
B_k &= \int_{x_{k-1}}^{x_{k+1}} p(t)U_k(t)q(t)(t - x_k)y'(\xi_k)dt, \ x_{k-1} \leq \xi_k \leq x_{k+1}, \\
C_k &= \int_{x_{k-1}}^{x_{k+1}} p(t)U_k(t)f(t)dt,
\end{aligned}
$$

and $U_k(t)$ is given by (2.10).

The discretisation at $k = 1$ may be obtained from (2.6) as follows:

$$y_2 - y_1 = \psi_1(x_1)\int_0^{x_1} p(t)g(t)dt + \int_{x_1}^{x_2} \psi_1(t)p(t)g(t)dt, \tag{2.12}$$

or

$$\frac{y_2}{\psi_1(x_1)} - \frac{y_1}{\psi_1(x_1)} = \int_0^{x_2} p(t)U_1(t)g(t)dt \tag{2.13}$$

where

$$U_1(t) = \begin{cases} 1 & \text{if } 0 \le t \le x_1 \\ \psi_1(t)/\psi_1(x_1) & \text{if} x_1 \le t \le x_2. \end{cases} \tag{2.14}$$

By expanding $y(t)$ in the interval $[0, x_2]$ about $t = x_1$, we may rewrite (2.13) as

$$\frac{y_1}{\psi_1(x_1)} - \frac{y_2}{\psi_1(x_1)} = -A_1 y_1 - B_1 - C_1 \tag{2.15}$$

where

$$\begin{aligned} A_1 &= \int_0^{x^2} p(t)U_1(t)q(t)dt, \\ B_1 &= \int_0^{x_2} p(t)U_1(t)q(t)(t - x_1)y'(\xi_1)dt,\ 0 \le \xi_1 \le x_2, \\ C_1 &= \int_0^{x_2} p(t)U_1(t)f(t)dt. \end{aligned}$$

We note that $A_1 \ge 0$.

In matrix notation, (2.11) and (2.15) may be expressed as

$$HY + AY = B(h) + C \tag{2.16}$$

where $H = (h_{ij})$ is a tridiagonal matrix whose elements are

$$\begin{aligned} h_{11} &= \frac{1}{\psi_1(x_1)},\ h_{ii} = \frac{1}{\psi_{i-1}(x_{i-1})} + \frac{1}{\psi_i(x_i)},\quad i = 2(1)N-1 \\ h_{i,i-1} &= -\frac{1}{\psi_{i-1}(x_{i-1})},\quad i = 2(1)N-1 \\ h_{i,i+1} &= -\frac{1}{\psi_i(x_i)},\quad i = 1(1)N-2. \end{aligned}$$

And
$$\begin{aligned} Y &= (y_1, \ldots, y_{N-1})^T \\ A &= \text{diag}\,(A_i), \quad i = 1(1)N-1 \\ B(h) &= (-B_1, \ldots, -B_{N-1})^T \\ C &= (-C_1, \ldots, -C_{N-1})^T. \end{aligned}$$

The method we consider determines an approximation $\tilde{Y}$ for Y by solving the $(N-1)$ by $(N-1)$ linear system

$$HY\tilde{Y} + A\tilde{Y} = C. \tag{2.17}$$

To approximate y_0 we proceed as follows

$$y'(x) = \frac{1}{p(x)} \int_0^x p(t)(q(t)y + f(t))dt.$$

Integrating both sides from $x = 0$ to $x = x_1$ we get

$$y_1 - y_0 = \int_0^{x_1} \int_0^x \frac{p(t)}{p(x)} g(t)dtdx = \int_0^{x_1} \int_t^{x_1} \frac{1}{p(x)} dxp(t)g(t)dt.$$

Expanding $y(t)$ in the interval $[0, x_1]$ about $t = x_1$, we get

$$\begin{aligned} y_1 - y_0 \quad = \quad & \int_0^{x_1} p(t)q(t) \int_t^{x_1} \frac{1}{p(x)} dxdty_1 \\ & + \int_0^{x_1} p(t)q(t) \int_t^{x_1} \frac{1}{p(x)} dx(t - x_1)y'(\xi_0)dt \\ & + \int_0^{x_1} p(t)f(t) \int_t^{x_1} \frac{1}{p(x)} dxdt \quad \text{where } 0 \leq \xi_0 \leq x_1. \end{aligned}$$

If we take $x_1 < \delta$ then $\int_t^{x_1} \frac{1}{p(x)} dx < \frac{\delta}{p(t)}$. We will show in Section 4 (Corollary 4.1) that $y \in C^1[0, 1]$. Therefore, the second and third terms in the above expression are of $O(h^3)$ and $O(h^2)$ respectively. Hence, the expression

$$\hat{y}_0 = \hat{y}_1 - \left\{ \int_0^{x_1} p(t)q(t) \int_t^{x_1} \frac{1}{p(x)} dxdt \right\} \tilde{y}_1 \tag{2.18}$$

may be used to compute an $O(h^2)$ approximation $\tilde{y}_0$ for y_0.

To summarize, the finite difference scheme is given by (2.17) for $k \geq 1$ and by (2.22) for $k = 0$.

3. CONVERGENCE OF THE FINITE DIFFERENCE SCHEME AND RATE OF CONVERGENCE

Theorem 3.1. *The finite difference scheme represented by (2.17) and (2.22) is convergent of* $O(h^2)$.

Proof. Let $E = (e_1, \ldots, e_{N-1})^t = Y - \tilde{Y}$. From (2.16) and (2.17) we get the error equation

$$(H + A)E = B(h). \tag{3.1}$$

It can be easily checked that H is irreducible and irreducibly diagonally dominant (see [12], pp. 47-55). Moreover, since the diagonal elements of H are positive and the off diagonal elements are negative, H is an M-matrix i.e., H is invertible and $H^{-1} \geq 0$. Now, since A is a nonnegative diagonal matrix and H is an M-matrix, then $H + A$ is also an M-matrix and $(H + A)^{-1} \leq H^{-1}$. Now from (3.1) we have

$$\|E\|_\infty \leq \|H^{-1}|B(h)|\|_\infty \tag{3.2}$$

in the uniform norm, where

$$|B(h)| = (| - B_1|, \ldots | - B_{N-1}|)^T.$$

Since H is symmetric and tridiagonal, it can be checked that $H^{-1} = (h_{ij}^{-1})$ is given by

$$h_{ij}^{-1} = \begin{cases} \int_{x_j}^1 \frac{1}{p(\tau)} d\tau, & i \leq j \\ \int_{x_i}^1 \frac{1}{p(\tau)} d\tau, & i \geq j. \end{cases} \tag{3.3}$$

The proof will be complete if we show that $\|E\|_\infty$ is of $O(h^2)$. Since $y \in C^1[0,1]$ by corollary 4.1 and since $q \in C[0,1]$, we may conclude form (3.2) that

$$e_k \leq c \sum_{1}^{N-1} S_k$$

where c is a constant and

$$S_k = \int_{x_{k-1}}^{x_k} p(t) U_k(t)(t - x_k) dt \int_{x_k}^1 \frac{1}{p(t)} dt. \tag{3.4}$$

In what follows we show that S_k is of $O(h^3)$ for $k = 1(1)N-1$ and that E is of $O(h^2)$. Now,

$$\begin{aligned} S_k &\leq \int_{x_{k-1}}^{x_k} p(t)U_k(t)(t-x_k)\int_t^1 \frac{1}{p(\tau)}d\tau dt + \int_{x_k}^{x_{k+1}} p(t)U_k(t)(t-x_k)dt \\ (3.5) \quad & \left\{\int_{x_k}^{x_{k+1}} \frac{1}{p(t)}dt + \int_{x_{k+1}}^1 \frac{1}{p(t)}dt\right\} \\ &= S_k^1 + S_k^2 + S_k^3 \end{aligned}$$

where

$$\begin{aligned} S_k^1 &= \int_{x_{k-1}}^{x_k} p(t)U_k(t)(t-x_k)\int_t^1 \frac{1}{p(\tau)}d\tau dt, \\ (3.6) \qquad S_k^2 &= \int_{x_k}^{x_{k+1}} p(t)U_k(t)(t-x_k)dt \int_{x_k}^{x_{k+1}} \frac{1}{p(t)}dt, \\ S_k^3 &= \int_{x_k}^{x_{k+1}} p(t)U_k(t)(t-x_k)dt \int_{x_{k+1}}^1 \frac{1}{p(t)}dt. \end{aligned}$$

If $x_k \geq \delta$, then using Lemma (4.2) we can show that $S_k^2 = O(h^3)$. If $x_k < \delta$, then without loss of generality we may assume that $x_{k+1} \leq \delta$ (for otherwise we split the integral at $x = \delta$). Using integration by parts, the definition of U_k and assumption (1.4) we have

$$\begin{aligned} S_k^2 &\leq h\left\{\left\{[\int_{x_k}^t p(\tau)d\tau \int_t^{x_{k+1}} \frac{1}{p(\tau)}d\tau]_{x_k}^{x_{k+1}} + \int_{x_k}^{x_{k+1}}\int_{x_k}^t p(\tau)d\tau \frac{1}{p(t)}dt\right\}\right. \\ (3.7) \leq & \ h\left\{0 + \int_{x_k}^{x_{k+1}} p(t)\frac{1}{p(t)}\int_{x_k}^t d\tau dt\right\} \\ &= O(h^3). \end{aligned}$$

We conclude this proof by showing that $S_k^1 + S_k^3$ is of $O(h^3)$. Using integration by parts and observing that $U_k \leq 1$ we have

$$\begin{aligned} S_k^1 + S_k^3 &\leq \frac{1}{2}(t-x_k)^2 p(t)\int_t^1 \frac{1}{p(\tau)}d\tau\ ]_{x_{k-1}}^{x_k} + \int_{x_{k-1}}^{x_k} \frac{1}{2}(t-x_k)^2 dt \\ &\quad - \int_{x_{k-1}}^{x_k} \frac{1}{2}(t-x_k)^2 p'(t)\int_t^1 \frac{1}{p(\tau)}d\tau dt \\ &\quad + \frac{1}{2}(t-x_k)^2 p(t)\int_t^1 \frac{1}{p(\tau)}d\tau\ ]_{x_k}^{x_{k+1}} \end{aligned}$$

$$
\begin{aligned}
&+\int_{x_k}^{x_{k+1}} \frac{1}{2}(t-x_k)^2 dt - \int_{x_k}^{x_{k+1}} \frac{1}{2}(t-x_k)^2 p'(t)\int_t^1 \frac{1}{p(\tau)}d\tau dt \\
= \; & \frac{1}{2}h^2\left\{p(x_{k+1})\int_{x_{k+1}}^1 \frac{1}{p(\tau)}d\tau - p(x_{k-1})\int_{x_{k-1}}^1 \frac{1}{p(\tau)}d\tau\right\} + O(h^3) \\
= \; & \frac{1}{2}h^2\left\{2h\left(p(t)\int_t^1 \frac{1}{p(\tau)}d\tau\right)'\right\} + O(h^3) \\
& \qquad \text{for some } t \in (x_{k-1}, x_{k+1}) \\
= \; & h^3\left\{-1 + p'(t)\int_t^1 \frac{1}{p(\tau)}d\tau\right\} + O(h^3) \\
= \; & O(h^3) \text{ because } p'(t)\int_t^1 \frac{1}{p(\tau)}d\tau \text{ is bounded by} \\
& \qquad \text{assumption (1.8).}
\end{aligned}
$$

4. EXISTENCE AND UNIQUENESS OF THE SOLUTION

In this section we establish existence and uniqueness of the solution to our problem (1.1)- (1.3) with the assumptions (1.4)-(1.8) and that this solution is continuously differentiable on [0,1]. To this end we define the Banach space

$$\dot{C}[0,1] = \{y \in C[0,1] : y(1) = 0\}.$$

Now we defined the operator $L : \dot{C}[0,1] \to C[0,1]$ by

$$
(4.1) \quad \begin{aligned} & D(L) = \{y \in \dot{C}[0,1] : Ly \in C[0,1], \lim_{x\to 0+} p(x)y' = 0\} \\ & Ly = (py')'. \end{aligned}
$$

We assume that L is densely defined. Now we define the operator $K : \dot{C}[0,1] \to C[0,1]$ by

$$
(4.2) \quad \begin{aligned} D(K) &= D(L) \\ Ky &= (py')' - pqy. \end{aligned}
$$

Then (1.1) may be written as

$$
(4.3) \qquad \frac{1}{p}Ky = f.
$$

To prove existence and uniqueness of a solution for (1.1)-(1.3), it is sufficient to show that $K^{-1}p$ exists as an operator from $C[0,1]$ to $\dot{C}[0,1]$. Before we show this we prove the following three lemmas.

Lemma 4.1. *If* $|p'(x)| \leq M_1$, $M_1 > 0$ *and* $p(x) \geq 0 \ \ \forall x \in [0,1]$, *then*

(1) $p(x) > 0$ *for* $x \in (0,1]$.

(2) $\int_0^1 \frac{1}{p(x)} dx = \infty$.

Proof. (1) Assume $p(x_0) = 0$ for some $x_0 \in (0,1)$, then

$$|p(x)| = |x - x_0||p'(\xi)| \leq x - x_0|M_1 \quad (\text{where } x_0 \leq \xi \leq 1), \tag{3.9}$$

therefore

$$\int_{x_0}^1 \frac{1}{|p(x)|} dx \geq \frac{1}{M_1} \int_{x_0}^1 \frac{1}{|x - x_0|} dx = \infty. \tag{3.10}$$

This contradicts condition (1.5).

The case $x_0 = 1$ can be treated similarly.

(2) This is easily seen in view of (1.4) and (1) above.

Lemma 4.2. $p(x)$ *is bounded away from zero on the subinterval* $[\delta, 1]$ *of (0,1], i.e.* $\exists\, c > 0$ *such that* $p(x) \geq c \ \ \forall x \in [\delta, 1]$.

Proof. By Lemma 4.1, we have $p(x) > 0$ for $x \in (0,1]$. Assume that $p(x)$ is not bounded from below on $[\delta, 1]$. Then $\exists$ a sequence $\{ x_i\}_N \subset [\delta, 1]$ such that $p(x_i) \to 0$ as $i \to \infty$. But then there exists a subsequence $\{x_i\}_{N_1 \subset N}$ such that $x_i \to x_0 \in [\delta, 1]$. Thus $p(x_0) = 0$ which is not possible by Lemma 4.1.

Lemma 4.3. $L^{-1}p$ *is a compact operator from* $C[0,1]$ *to* $\dot{C}[0,1]$.

Proof. Let $f \in C[0,1]$ and y be a solution for

$$\frac{1}{p(x)} (py')' = f(x), \tag{4.4}$$

we show first that $y \in \dot{C}[0,1]$. From (4.4) we have

$$y' = \frac{1}{p(x)} \int_0^x pf dt \text{ and } y = \int_x^1 \frac{1}{p(t)} \int_0^t p(\tau) f(\tau) d\tau dt, \quad \text{thus } y(1) = 0. \tag{4.5}$$

Therefore y' is bounded on $[\delta, 1]$. Since $p(x)$ is increasing in the interval $[0, \delta]$ we have

$$\begin{aligned} \lim_{x \to 0+} y' &= \lim_{x \to 0+} \frac{1}{p(x)} \int_0^x p(t) f(t) dt \\ &\leq \lim_{x \to 0+} \frac{1}{p(x)} p(x) \int_0^x dt \|f\|_\infty \\ &= 0. \end{aligned} \tag{4.6}$$

Following the argument used in [3], we concluded that y' is bounded on (0,1] and hence y is uniformly bounded on (0,1]. Therefore $y(0^+)$. Therefore $y(0^+)$ exists and is unique. We define $y(0) = y(0^+)$ and hence conclude that $y \in \dot{C}[0,1]$. To show that $L^{-1}p$ is compact we note that $L^{-1}p$ is an integral operator with kernel

$$M(x,t) = \begin{cases} p(t) \int_x^1 \int_x^1 \frac{1}{p(\tau)} d\tau & t \leq x \\ p(t) \int_t^1 \frac{1}{p(\tau)} d\tau & t > x. \end{cases}$$

We will show that $M(x,t)$ is uniformly bounded and that $\int_0^1 |M_x(x,t)| dt$ is uniformly bounded in x. It then follows from Arzela-Ascoli's theorem that $L^{-1}p$ is compact.

Now $M(x,t) \leq p(t) \int_t^1 \frac{1}{p(\tau)} d\tau = p(t) \left\{ \int_1^\delta \frac{1}{p(\tau)} d\tau + \int_\delta^1 \frac{1}{p(\tau)} d\tau \right\}$, with the second part of the last parenthesis bounded by Lemma 4.2. As for the first part we have

$$p(t) \int_t^\delta \frac{1}{p(\tau)} d\tau \leq p(t) \frac{1}{p(t)} \int_t^\delta d\tau = (\delta - t).$$

Therefore, the right hand side of the above inequality is bounded on $[0, \delta]$. Thus $M(x,t)$ is uniformly bounded on [0,1]. Also since

$$\int_0^1 |M_x(x,t)| dt = \int_0^x \frac{p(t)}{p(x)} dt = \frac{\int_0^\delta p(t) dt + \int_\delta^x p(t) dt}{p(x)},$$

a similar argument shows that $\int_0^1 |M_x(x,t)|dt$ is uniformly bounded in x. Thus $L^{-1}p$ is compact.

Theorem 4.1. *$K^{-1}p$ exists as an operator from $C[0,1]$ to $\dot{C}[0,1]$.*

Proof. Since $K^{-1}p = (1 - L^{-1}pq)^{-1}L^{-1}p$ and $L^{-1}p$ is compact we need only to show that $(1 - L^{-1}pq)^{-1}$ is bounded. Since multiplication by the continuous function q is a continuous operator, the operator $L^{-1}pq$ is compact. Hence it suffices to show that 1 is not an eigenvalue for $L^{-1}pq$. So let $L^{-1}pqy = y$, then $(L - pq)y = 0$. Therefore,

$$0 = ((L - pq)y, y) = -\int_0^1 py'^2 dx - \int_0^1 pqy^2 dx$$

where $(f,g) = \int_0^1 f(x)g(x)dx$. We conclude that $py'^2 = 0$ and hence $y' \equiv 0$ (by Lemma 4.2). Since $y \in D(L)$ we must also have $y(1) = 0$. Therefore $y \equiv 0$. This completes the proof of the theorem.

Corollary 4.1. *If y is a solution of (1.1)-(1.3) then $y \in C^1[0,1]$.*

Proof. The proof is a simple adaptation of the arguments used to derive (4.5) and (4.6) above.

5. NUMERICAL RESULTS

In this section we provide numerical examples which verify $O(h^2)$ convergence of the finite difference scheme employed in Section 2. The computer application program "Mathematica" was used to execute the algorithms that were used with the numerical examples.

Three examples were used with three different functions $p(x)$. Uniform mesh was used with $N = 16, 32$ and 64. Following are the examples and the numerical results obtained.

Example 1

$$\begin{aligned} \left(\left(\sin\frac{\pi}{2}x\right)y'\right)' - \frac{\pi^2}{2}\left(\sin\frac{\pi}{2}x\right)y &= -\frac{\pi^2}{2}\sin\pi x, \quad 0 \le x \le 1 \\ \lim_{x\to 0+}\left(\sin\frac{\pi}{2}x\right)y' &= 0, \end{aligned} \tag{5.1}$$

$$y(1) = 0.$$

It is easily checked that $y(x) = \cos\frac{\pi}{2}x$ is an exact solution for the above problem. Table I below shows $O(h^2)$ convergence of the finite difference scheme used. In this table, the vectors Y and $\tilde{Y}$ are the exact and approximate solutions as defined by (2.20) and (2.21) respectively.

TABLE I: Numerical Results for Example 1

N	$\|Y - \tilde{Y}\|_\infty$	h^2
16	7.8 (-4)*	3.9 (-3)
32	2.0 (-4)	9.8 (-4)
64	5.0 (-5)	2.4 (-4)

*The numbers in parenthesis are powers of 10, e.g. 7.8 (-4) = 7.8×10^{-4}.

Example 2

$$\left((2x - x^2)^{3/2}y'\right)' - (2x - x^2)^{3/2}y = (2 - 6x - x^2 + x^3)(2x - x^2)^{3/2}$$

$$(5.2) \qquad +(6 - 15x + 9x^2)x^{3/2}(2 - x)^{1/2}, \quad 0 \le x \le 1$$

$$\lim_{x \to 0+}(2x - x^2)^{3/2}y' = 0,$$

$$y(1) = 0.$$

The exact solution for (5.2) is $y(x) = x^2(1 - x)$. Numerical results showing $O(h^2)$ convergence are given in Table II.

TABLE II: Numerical Results for Example 2

N	$\|Y - \tilde{Y}\|_\infty$	h^2
16	2.2 (-3)	3.9 (-3)
32	6.1 (-4)	9.8 (-4)
64	1.6 (-4)	2.4 (-4)

Example 3

$$(5.3) \qquad (p(x)y')' - q(x)p(x)y = p(x)f(x), \quad 0 \le x \le 1$$

$$\begin{aligned} \lim_{x \to 0+} p(x)y' &= 0, \\ y(1) &= 0, \end{aligned}$$

where

$$\begin{aligned} p(x) &= \sqrt{x}(\sqrt{x}+1) \text{ In } (\sqrt{x}+1) \\ q(x) &= 1 \\ f(x) &= 3 - \frac{15}{2}x - x^2 + x^3 + \frac{\sqrt{x}(2-3x)}{2(\sqrt{x}+1)} \left(1 + \frac{1}{\text{In } (\sqrt{x}+1)}\right). \end{aligned}$$

The exact solution for (5.3) is $y(x) = x^2(1-x)$.

Numerical results for Example 3 are given in Table III

Table III: Numerical Results for Example 3

N	$\|\|Y - \tilde{Y}\|\|_\infty$	h^2
16	2.1 (-3)	3.9 (-3)
32	5.8 (-4)	9.8 (-4)
64	1.5 (-4)	2.4 (-4)

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