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LUCAS THEOREM FOR GENERALIZED PSEUDO-DERIVATIVES OF ABSTRACT POLYNOMIALS

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ABSTRACT. Let E denote a vector space over an algebracally closed field K of characteristic zero. Our object is to investigate the location of the null sets of generalized pseudo-derivatives (see the definition below) of the product (or quotient) of abstract polynomials. Some special cases of this general problem were studied by Walsh as geometry of polynomials in the complex plane. One of two results deduces a theorem of Zaheer [10, Theorem 3.1] and offers a generalization of Lucas' Theorem (cf. [10, Remark 3.3(1)] and [11, Theorem 4.1].

1. INTRODUCTION

Throughout, unless mentioned otherwise, E denotes a vector space over an algebraically closed field K of characteristic zero. It is known [3] (see also [2, pp. 38-40],[6, pp. 284-255]) that $K = K_0(\mathrm{i})$, where K_0 is a maximal ordered subfield of K and $-i^2$ is the unit element of K. By [10, Remark 1.1], every vector space E can be made into a K-(resp. K_0-) inner product space (written briefly $K-$ (resp. K_0-)i.p.s)

Let $K_\infty \equiv K \cup \{\infty\}$ denote the projective field ([12, p. 352] or [9, p. 116]) obtained by adjoining to K an element ∞ (called scalar infinity). Also, let $E \cup \{\omega\} \equiv E_\omega$ as in [12, p. 372]. For $E = K$, we can use ω and ∞ interchangeably.

The details regarding the rest of the material in this section can be seen in [10, pp. 833-835, 839-843] and [11, pp. 268-272]. It is quite

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interesting to note how certain geometrical configurations in $\mathbb{C}$ have analytical analogues in abstract spaces that are associated with abstract polynomials. The first such instance is in the work of Zervos [12, pp. 352-353] (or [9, p. 116]) in certain fields , as described below.

A subset of A of K_∞ is called *a generalized circular region (g.c.r.)* of K_∞ if either A is one of ϕ, K, K_∞, or if A satisfies the following two properties: (i) $\theta_\zeta(A)$ is K_0-convex for all $\zeta \in K - A$, where $\theta_\zeta(z) = (z - \zeta)^{-1}$ for all $z \in K_\infty$, and (ii) $\infty \in A$ if A is not K_0-convex. The family of all g.c.r.'s will be denoted by $D(K_\infty)$. For $K = \mathbb{C}$, we have the following result [12, p. 352] or [9, p. 116]): The *nontrivial members of* $D(\mathbb{C}_\infty)$ *are the open interior (or exterior) of circles or the open half-planes, adjoined with a connected subset (possibly empty) of their boundary.* The concept of $D(K_\omega)$ was further extended to the family $D(E_\omega)$ of g.c.r.'s of E_ω when E is a K-i.p.s., with an abundance of non-trivial g.c.r.'s [10, Proposition 1.5]. another instance is that of hermitian cones, due to Hormander. Now we describe a still more general concept (due to Zaheer [11]), of the family $D^*(E_\omega)$ in vector spaces, which is even richer than that of $D(E_\omega)$.

Given $S \subseteq E_\omega$, we write

$$G_s(x, y) = \{\rho \in K_\infty : x + \rho y \in S\}, \ \forall x, y \in E. \tag{1.1}$$

We say that S is *a supergeneralized circular region* (briefly, *sg.c.r.*) if $G_s(x, y) \in D(K_\infty)$ for every $x, y \in E$. The family of all sg. c. r.'s will be denoted by $D^*(E_\omega)$. Clearly ϕ, E and E_ω are trivial members of $D^*(E_\omega)$. Properties of ω and ∞ [10, p. 834] imply that (since $G_s(x, 0) = K$ or $G_s(x, 0) = \phi$ depending on whether $x \in S$ or $x \notin S$)

$$\infty \notin G_s(x, 0) \in D(K_\infty), \ \forall x \in S, \tag{1.2}$$

and that

$$\infty \notin G_s(x, y) \ \forall x \in E, y \in E - \{0\} \text{if and only if } \omega \in S. \tag{1.3}$$

Therefore

(1.4) $S \in D^*(E_\omega)$ if and only if $G_s(x,y) \in D(K_\infty)$ $\forall x, y \in E(y \neq 0)$.

The relations between $D(E_\omega)$ and $D^*(E_\omega)$ are given in the following:

Proposition 1.1. [11, pp. 269 -272]. *Let E be a K -i.p.s. Then*

(a) $D(E_\omega) \subsetneqq D^*(E_\omega)$ *if* dim $E \geq 2$

(b) $D(E_\omega) = D^*(E_\omega)$ *if* dim $E = 1$,

(c) $D(K_\omega) = D^*(K_\omega)$, *if $E = K$ is taken as 1-dimensional natural K-i.p.s, where $D(K_\omega) \equiv D(K_\omega)$ as in [11, Remark 3.1]. Hence $D*(\mathbb{C}_\omega)$ coincides with the family $D(\mathbb{C}_\omega)$ of all g.c.r.'s of $\mathbb{C}_\omega$. Note that C is convex if $\infty \notin C \in D(\mathbb{C}_\omega)$.*

Remark 1.2. (i) The above proposition shows that the family $D^*(E_\omega)$ is a natural generalization to vector spaces of the concept of g.c.r.'s [10, Remark 1.6] in the complex plane and that it offers a richer class when E is a K- i.p.s.

(ii) All maximal subspaces and their translations are members of $D^*(E_\omega)$ in a vector space, but are not members of $D(E_\omega)$ when E is K.i.p.s.

(iii) If $S \in D^*(E_\omega)$, then $E_\omega - S$ may not belong to $D^*(E_\omega)$ [11, Remark 3.7 (III)].

(iv) There are sets in $D^*(E_\omega)$ whose complements in E_ω are also in $D^*(E_\omega)$. For example, maximal subspaces and their translations [11, Remark 3.7 (IV)].

A mapping $P : E \to K$ is called *an abstract polynomial* (written briefly *a.p.*) of degree n if for every $x, y \in E$:

$$P(x+\rho y) = \sum_{k=0}^{n} A_k(x,y)\rho^k, \ \forall \rho \in K, \tag{1.5}$$

where the coefficients $A_k(x,y) \in K$ are independent of ρ and $A_n(x,y) \not\equiv 0$. We shall denote by $\mathbb{P}_n(E,K)$ the calss of all a.p.'s of degree n from E to K. In particular, we shall write $\mathbb{P}_n(K) \equiv \mathbb{P}_n(K,K)$ and $\mathbb{P}_n \equiv \mathbb{P}_n(\mathbb{C})$. It is known [5, Theorem 2.2] that $A_0(x,y)$ (resp. $A_n(x,y)$) is independent of y (resp. of x) and $A_k(x,y)$ is an abstract polynomial of degree $n-k$ in x (for each fixed y) and also an abstract homogeneous polynomial of degree k in y (for each fixed x). Consequently, $A_n(x,y) \equiv A_n(0,y) \neq 0$ for some $y \neq 0$ and so the set

$$F(P) = \{h \in E : h \neq 0, A_n(0,h) \neq 0\} \neq \phi.$$

The *null − set* of P is the set $Z(P) = \{x \in E : P(x) = 0\}$.

Note that if E is an n-dimensional vector space over K, and $\{e_1, e_2, ..., e_n\}$ is a basis for E, then each $x \in E$ is written uniquely in the form

$$x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n, \ \ \alpha_1, \alpha_2, ..., \alpha_n \in K.$$

Thus an abstract polynomial P from $E = K^n$ to K can be written as

$$\begin{aligned} P(x) &= P(\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n) \\ &= \sum B_{k_1 k_2 ... k_n} (\alpha_1)^{k_1})(\alpha_2)^{k_2} ... (\alpha_n)^{k_n}, \end{aligned}$$

where the sum is taken for each $k_j = 0, 1, ..., n$, and $k_1 + k_2 + ... + k_n = n$, and where $B_{k_1 k_2 ... k_n}$ are constants with respect to α_j, hence P is the same as a polynomial in n variables over K.

Given $P \in \mathbb{P}_n(E,K)$ (via (1.5)) and $h \in F(P)$, we define for each $k = 1, 2, ..., n$, *the k^{th} pseudo-derivative* $p_h^{(k)}$ of P(relative to h) to be the mapping from E to K given by

$$P_h^{(k)}(x) = k! A_k(x,h), \ \forall x \in E. \tag{1.6}$$

The first few members in (1.6) will be written as P_h', P_h'', etc. It is known that (cf. [10, pp. 841-843]) if $P \in \mathbb{P}_n(E,K)$ and $h \in F(P)$, then $h \in F(P_h^{(k)})$, and

$$P_h^{(k+1)}(x) = (P_h^{(k)})_h'(x), \ \forall x \in E, k = 1, 2, ..., n-1.$$

Given $P \in \mathbb{P}_n(E, K)$ and $h \in F(P)$, since K is algebraically closed, we may write

$$P(x+\rho h) = A_n(x,h)\prod_{j=1}^{n}[\rho - \rho_j(x,h)], \forall \rho \in K, \tag{1.7}$$

where the coefficients $\rho_j(x,h)$ belong to K and are independent of ρ such that $A_n(x,h) \equiv A_n(0,h)$ for all $x \in E$. Let $\Delta(m,n)$ denote the sum of all possible products formed out of the scalars $\rho_j(x,h)$, $j = 1, 2, ...n$, taken m at a time, so that equation (1.7) yields

$$P(x) = A_0(x,h) = (-1)^{n-1}A_n(x,h)\Delta(n,n), \tag{1.8}$$

and

$$P_h'(x) = A_1(x,h) = (-1)^{n-1}A_n(x,h)\Delta(n-1,n). \tag{1.9}$$

Remark 1.5. (I) If $P \in \mathbb{P}_n(E,K)$, $h \in F(P)$, then $P_n^{(k)} \in \mathbb{P}_{n-k}(E,K)$ and the a.p.'s $P_h^{(k)} (k = 1, 2, ..., n)$ deserve to be called *successive pseudo-derivatives* of P.

(II) Let $P_n(K) \equiv P_n(K,K)$ denote the collection of all n^{th} degree (ordinary) polynomials from K to K. If $f^{(k)}$ stands for the k^{th} formal derivative of $f \in P_n(K)$, then

(i) f is necessarily an a.p. of degree n from K to K(i.e. $f \in \mathbb{P}_n(K)$),

(ii) Every nonzero element of K is faithful to f when f is regarded as a member of $\mathbb{P}_n(K)$. That is, $F(f) = K - \{0\}$,

(iii) For each value of $k = 1, 2, ..., n$, the k^{th} pseudo-derivative $f_n^{(k)}$ of f (treated as a member of $\mathbb{P}_n(K)$) and the k^{th} formal derivatives $f^{(k)}$ of f treated as a member of $P_n(K)$) satisfy the relation

$$f_h^{(k)}(x) = h^k f^{(k)}(x), \ \forall x \in E, h \in K - \{0\}.$$

In particular (for $h = 1$), $f_1^{(k)} = f^{(k)}$, and the two notations coincide. The same is true in particular when $K = \mathbb{C}$, but then the formal derivative $f^{(k)}$ becomes the usual k^{th} derivative of f as defined via calculus.

2. GENERALIZED PSEUDO-DERIVATIVES OF ABSTRACT POLYNOMIALS

Before taking up our main result, we define generalized pseudo-derivatives and prove an important property.

Definition 2.1. Given a.p.'s $P_k \in P_{n_k}(E,K)$ and scalars $m_k \in K, k = 1,2,...,q$, set

$$\begin{aligned} Q(x) &= P_1(x).P_2(x)...P_q(x), \\ Q_k(x) &= P_1(x)...P_{k-1}(x).P_{k+1}(x)...P_q(x), \end{aligned}$$

and for each $h \in \bigcap_{k=1}^{q} F(P_k)$, define

$$R_h(x) = \sum_{k=1}^{q} m_k Q_k(x).(P_k)'_h(x), \ \forall x \in E. \tag{2.1}$$

We call $R_h(x)$ *a generalized pseudo-derivative* of the product $Q(x)$.

Note that if $n = n_1 + n_2 + ... + n_q$, then $Q \in \mathbb{P}_n(E,K), Q_k \in \mathbb{P}_{n-n_k}(E,K)$ and $(P_k)'_h$ is an a.p. of degree $n_k - 1$ in $x, 1 \leq k \leq q$. Therefore, $R_h(x)$ is an a.p. of degree at most $n-1$ in x. The following proposition justifies the terminology for $R_h(x)$ to be called *a generalized pseudo-derivative of* $Q(x)$.

Proposition 2.2. *In the notations of Definition 2.1, if* $m_k = 1$ *for* $k = 1,2,...,q$, *then the generalized pseudo-derivative* $R_h(x)$ *of the product* $Q(x)$ *is essentially the first p-derivative* $Q_h(x)$ *of* $Q(x)$. *More precisely,*

$$R_h(x) = Q'_h(x), \ \forall x \in E, \forall h \in \bigcap_{k=1}^{q} F(P_k).$$

Proof. Take an arbitrary but fixed element for each $k (1 \leq k \leq q)$, since $h \in F(P_k)$ and K is algebraically closed, we can write

$$P_k(x+\rho h) = \sum_{m=0}^{n_k} A_{k,m}(x,h)\rho^m, \forall \rho \in K, \tag{2.2}$$

$$= A_{k,n_k}(x,h)\prod_{j=1}^{n_k}(\rho-\rho_{k,j}(x,h)), \forall\rho\in K,$$

where $A_{k,m}(x,h)$ and $\rho_{k,j}(x,h)$ belong to K and are independent of ρ such that $A_{k,n_k}(x,h)\equiv A_{k,n_k}(0,h)\neq 0$ for all $1\le k\le q$. As in (1.8) and (1.9)

$$P_k(x) = A_{k,0}(x,h) = (-1)^{n_k}A_{k,n_k}(x,h)\Delta(n_k,n_k), \tag{2.3}$$

and

$$(P_k)'_h(x) = A_{k,1}(x,h) = (-1)^{n_k-1}A_{k,n_k}(x,h)\Delta(n_k-1,n_k), \tag{2.4}$$

where $\Delta(m,n_k)$ denote the sum of all possible products formed out of the scalars $\rho_{k,j}(x,h), 1\le k\le q, j=1,2,...,n_k$. If we let $r_0=0, r_k=n_1+n_2+...+n_k$ (with $r_q=n$) and define

$$l=\psi(k,j)=r_{k-1}+j,\ \forall j=1,2,...,n_k, 1\le k\le q, \tag{2.5}$$

we may then write

$$\mu_l\equiv\mu_l(x,h)=\rho_{k,j}(x,h), \forall 1\le j\le n_k, 1\le k\le q,$$

so that

$$\begin{aligned} Q_k(x+\rho h) &= \prod_{k=1}^{q}\left(A_{k,n_k}(x,h)\prod_{j=1}^{n_n}\{\rho-\rho_{k,j}(x,h)\}\right)\\ &= \left(\prod_{k=1}^{q}A_{k,n_k}(x,h)\right)\prod_{l=1}^{n}(\rho-\mu_l), \text{ say.}\end{aligned}$$

Next (2.3) and (2.4) give, respectively,

$$\begin{aligned} Q_k(x) &= \prod_{\substack{=1\\ i\neq k}}^{q}\left((-1)^{n_i}A_{i,n_i}(x,h)\prod_{j=1}^{n_i}\rho_{i,,j}\right) \\ &= (-1)^{n-n_k}\mu_1.\mu_2...\mu_{r_{k-1}}.\mu_{r_{k+1}}...\mu_n\left(\prod_{\substack{i=1\\ i\neq k}}^{q}A_{i,n_i}(x,h)\right), \end{aligned} \tag{2.6}$$

and

$$(2.7)\quad (P_k)_h'(x) = (-1)^{n_k-1}A_{k,n_k}(x,h)\left(\sum_{m=1}^{n_k}\prod_{\substack{j=1\\ j\neq m}}^{n_k}\rho_{k,j}(x,h)\right) = (-1)^{n_k-1}A_{k,n_k}(x,h)\left(\sum_{m=1}^{n_k}\left\{\prod_{\substack{j=1\\ j\neq m}}^{n_k}\mu_{r_{k-1}+j}\right\}\right).$$

Consequently, (2.6) and (2.7) imply that

$$\begin{aligned} Q_k(x).(P_k)_h'(x) &= (-1)^{n-1}\left(\prod_{k=1}^{q}A_{k,n_k}(x,h)\right) \times\left(\prod_{m=1}^{n_k}(\mu_1.\mu_2\dots\mu_{r_{k-1}+m-1}).(\mu_{r_{k-1}+m+1}\dots\mu_n)\right)\\ &= (-1)^{n-1}\left(\prod_{k=1}^{q}A_{k,n_k}(x,h)\right) \times\left(\prod_{l=r_{k-1}+1}^{r_k}(\mu_1.\mu_2\dots\mu_{l-1}.\mu_{l+1}\dots\mu_n)\right).\end{aligned}$$

Hence

$$R_h(x) = (-1)^{n-1}\left(\prod_{k=1}^{q}A_{k,n_k}(x,h)\right) \times\left(\sum_{k=1}^{q}m_k\left\{\sum_{l=r_{k-1}+1}^{r_k}\mu_1.\mu_2\dots\mu_{l-1}.\mu_{l+1}\dots\mu_n)\right\}\right).$$

Finally, if $m_k = 1$ for all k, we obtain

$$\begin{aligned}
R_h(x) &= (-1)^{n-1}\left(\prod_{k=1}^{q} A_{k,n_k}(x,h)\right) \\
&\quad \times \left(\sum_{k=1}^{q}\sum_{l=r_{k-1}+1}^{r_k}(\mu_1.\mu_2 \ldots \mu_{l-1}.\mu_{l+1}\ldots\mu_n)\right) \\
&= (-1)^{n-1}\left(\prod_{k=1}^{q} A_{k,n_k}(x,h)\right) \\
&\quad \times \sum_{l=1}^{n}(\mu_1.\mu_2 \ldots \mu_{l-1}.\mu_{l+1}\ldots\mu_n) \\
&= Q_h'(x)
\end{aligned}$$

due to the corresponding formula (2.7) for the polynomial Q.

Remark 2.3. if $q = 1, m_1 = 1$, the above proposition tells us that $R_h(x) = (P_1)_h'(x)$.

3. MAIN RESULTS

After having introduced the basic concepts, we are now ready to prove the main results of this paper. The following theorem generalizes a theorem due to Zaheer (see [10, Theorem 3.1] and [11, Theorem 4.1]) in view of Proposition 1.4.

Theorem 3.1. *Let* $P_k \in \mathbb{P}_{n_k}(E,K)$ $(1 \le k \le q)$ *and* $S \in D^*(E_\omega)$ *with* $\omega \notin S$ *such that* $Z(P_k) \subseteq S$ *for all* k. *If* $R_h(x)$ *is given by* (2.1) *with* $m_k > 0$ *for all* k, *then*

$$Z(R_h) \subseteq S, \ \forall h \in \bigcap_{k=1}^{q} F(P_k)$$

Proof. Arbitrarily choose an element $h \in \bigcap_{k=1}^{q} F(P_k)$. Suppose on the contrary that $R_h(x) = 0$ for some $x \notin S$. Let P be given by (2.2). Since $P_k(x) \neq 0$ for all $k (1 \le k \le q)$, (2.3) gives

$$\rho_{k,j}(x,h) \equiv \rho_{k,j}(\text{say}) \neq 0, \ \forall j = 1,2,\ldots,n_k, 1 \le k \le q.$$

Further, since $P_k(x+\rho_{k,j}h)=0$ for all j and k, clearly then $x+\rho_{k,j}h \in Z(P_k) \subseteq S$ and so $\rho_{k,j} \in G_s(x,h) \equiv G$ (say) for all j and k. Since $S \in D^*(E_\omega)$ and $h \neq 0$, we observe that $G \in D(K_\infty)$ such that $0,\infty \notin G$ (because $x,\omega \notin S$) so that G is K_0-convex and $\rho_{k,j} \neq 0,\infty$ for all k and j. Now consider the mapping $\theta_\zeta(\rho)=1/(\rho-\zeta), \forall \rho \in K_\omega$. For $\zeta=0, \theta_0(\rho)=1/\rho$ and $\theta_0(\rho_{k,j}) \in \theta_0(G)$. But $\theta_0(G)$ is K_0-convex (cf. Definition 2.1) such that $0,\infty \notin \theta_0(G)$ and hence

$$0 \neq \frac{1}{n_k}\sum_{j=1}^{n_k}\frac{1}{\rho_{k,j}} \in \theta_0(G), \quad 1 \leq k \leq q. \tag{3.1}$$

If we let $A=\sum_{k=1}^{q} m_k n_k$, the scalars $\frac{m_k n_k}{A}$, $k=1,2,...,q$, are positive elements of K_0 with sum 1. Now the statement (3.1) and the K_0-convexity of $\theta_0(G)$ imply that $\mu/A \in \theta_0(G)$ where

$$\mu=\sum_{k=1}^{q}\sum_{j=1}^{n_k}\frac{m_k}{\rho_{k,j}} \neq 0. \tag{3.2}$$

By (2.3) and (2.4) we have

$$(P)_h'(x)=\left(\sum_{j=1}^{n_k}\frac{1}{\rho_{k,j}}\right).P_k(x), \ 1 \leq k \leq q. \tag{3.3}$$

Finally, since $Q_k(x).P_k(x)=Q(x)\neq 0$. Definition 2.1 and equations (3.2) and (3.3) imply that

$$R_h(x)=-\left(\sum_{k=1}^{q}\sum_{j=1}^{n_k}\frac{m_k}{\rho_{k,j}}\right).Q(x)\neq 0.$$

This contradicts that $x \in Z(R_h)$, and the proof is complete.

Corollary 3.2. *(Zaheer [11, Theorem 4.1]). Let $P_k \in \mathbb{P}_{n_k}(E,K)$ and $S \in D^*(E_\omega)$, with $\omega \notin S$ such that $Z(P) \subseteq S$, then*

$$Z(P_h') \subseteq S, \quad \forall \quad h \in F(P).$$

Proof. Use Theorem 3.1 and Remark 2.3.

Corollary 3.3. *Let $P_k \in \mathbb{P}_{n_k}(E,K)(1 \leq k \leq q)$ and $S \in D^*(E_\omega)$, with $\omega \notin S$ such that $Z(P_k) \subseteq S$ all k. If $Q(x) = P_1(x).P_2(x)...P_q(x)$ then*

$$Z(Q'_h(x)) \subseteq S, \ \forall \ \ h \in \bigcap_{k=1}^{q} F(P_k).$$

Proof. It is immediate from Proposition 2.2 and Theorem 3.1.

We have studied the generalized p-derivative $R_h(x)$ subject to the condition that all the scalar multipliers m_k are positive. The following theorem primarily deals with a similar study in the case when the m_k's are nonzero elements of K_0 with a vanishing sum. The complex plane version of this theorem leads to certain improvements in Walsh's theorem ([7] or [4, Theorem (20,1)]).

Theorem 3.4. *Let $P_k \in \mathbb{P}_{n_k}(E,K)(1 \leq k \leq q), S_i \in D^*(E_\omega)$ with $\omega \notin S_i(i=1,2)$ and $S_1 \cap S_2 = \phi$, such that*

$$Z(P_k) \subseteq \begin{cases} S_1, 1 \leq k \leq p(< q) \\ S_2, \ p+1 \leq k \leq q \end{cases} \tag{3.4}$$

If $R_h(x)$ is given by (2.1) with

$$m_k \begin{cases} > 0, & 1 \leq k \leq p \\ < 0, & p+1 \leq k \leq q \end{cases} \tag{3.5}$$

Such that

$$\sum_{k=1}^{q} m_k n_k = 0 \tag{3.6}$$

then

$$Z(R_h) \subseteq S_1 \cup S_2, \qquad \forall h \in \bigcap_{k=1}^{q} F(P_k).$$

Proof. Suppose on the contrary that $x \in Z(R_h)$ but $x \notin S_1 \cup S_2$ for some $h \in \bigcap_{k=1}^{q} F(Pk)$. Let P_k be given by (2.2), wherein we agree to write $\rho_{k,j} \equiv \rho_{k,j}(x,h)$, $A_{k,m} \equiv A_{k,m}(x,h)$. We know (cf. (3.5) and (2.3)) that

$$P_k(x) = (-1)^{n_k} A_{k,n_k}.\rho_{k,1}.\rho_{k,2}...\rho_{k,n} \neq 0$$

for all k. Therefore, $\rho_{k,j} \neq 0$ for all $1 \leq j \leq n_k$ and $1 \leq k \leq q$ $1 \leq k \leq q$. Now, applying same arguments as in the proof of Theorem 3.1 to each of the statements in (3.5) for S_1 and S_2 (replacing S in turn), we easily conclude (cf. (3.1)) that

$$0 \neq \frac{1}{n_k} \sum_{j=1}^{n_k} \frac{1}{\rho_{k,j}} \in \begin{cases} \theta_0(G_1) & \text{for } 1 \leq k \leq p \\ \theta_0(G_2) & \text{for } p+1 \leq k \leq q \end{cases}$$

where $\theta_0(\rho) = 1/\rho$ for all $\rho \in K_\infty$ and $G_i \equiv G_{S_i}(x,h) \in D(K_\infty)$ such that $\theta_0(G_i)$ is K_0-convex and $0, \infty \notin \theta_0(G_i)$ for $i = 1, 2$. Let

$$A_1 = \sum_{k=1}^{p} m_k n_k, \quad A_2 = \sum_{k=p+1}^{q} m_k n_k.$$

In view of (3.6) the scalars $m_k n_k / A_1$(resp. $m_k n_k / A_2$) are positive elements of K_0 for $k = 1, 2, ..., p$ (resp. $k = p+1, ..., q$) with sum 1. Now (3.8) and K_0-convexity of $\theta_0(G_i)$ imply that $\mu_i / A_i \in \theta_0(G_i)$ for $i = 1, 2$, where

$$\mu_1 = \sum_{k=1}^{p} \sum_{j=1}^{n_k} \frac{m_k}{\rho_{k,j}}, \quad \mu_2 = \sum_{k=p+1}^{p} \sum_{j=1}^{n_k} \frac{m_k}{\rho_{k,j}}. \tag{3.7}$$

Therefore,

$$0, \infty \neq \mu_i / A_i = \theta_0(\rho_i) = \frac{1}{\rho_i} \text{ for some } \rho_i \in G_i (i = 1, 2). \tag{3.8}$$

That is, $\rho_i = A_i / \mu_i \in G_i$ and so $x + (A_i/\mu_i)h \in S_i$ for $i = 1, 2$. We claim that $\mu_1 + \mu_2 \neq 0$, for otherwise (cf. (3.7)), $A_1 + A_2 = 0$ and $A_1/\mu_1 = A_2/\mu_2 \in G_1 \cap G_2$. Consequently $X + (A_1/\mu_1)h \in S_1 \cap S_2$, contradicting that S_1 and S_2 are disjoint. Hence

$$\mu_1 + \mu_2 = \sum_{k=1}^{p} \sum_{j=1}^{n_k} \frac{m_k}{\rho_{k,j}} \neq 0.$$

Finally, since $P_k(x) \neq 0$ for all k, we obtain (cf. (3.4))

$$R_h(x) = -\left(\sum_{k=1}^{p} \sum_{j=1}^{n_k} \frac{m_k}{\rho_{k,j}} \right) \left(\prod_{k=1}^{q} P_k(x) \right) \neq 0. \tag{3.9}$$

This contradicts that $x \in Z(R_h)$, and the proof is complete,

The above theorem deduces as corollary the following result which is an improved version of the second part of two-circle theorem due to Walsh [4, Theorem (20,1)].

Corollary 3.5. Let $f_i \in P_n(K)$ be n^{th} degree ordinary polynomials from K to K and $C_i \in D(K_\omega)$ with $C_1 \cap C_2 = \phi$ and $\omega \notin C_i (i = 1,2)$. If $Z(f_i) \subset C_i$ for $i = 1, 2$, then all the finite zeros of the formal derivative of the quotient $f = f_1/f_2$ lie in $C_1 \cup C_2$.

Proof. By Remark (1.5) (II) $f_i \in P_n(K), F(f_i) = K - \{0\}$ and the $R_h(x)$ of Theorem 3.4 (with $p = 1, q = 2$ and $m_1 = -m_2 = 1$) is given by

$$\begin{aligned} R_h(x) &= (f_1)'_h(x)f_2(x) - (f_2)'_h(x)f_1(x) \\ &= h[f'_1(x)f_2(x) - f'_2(x)f_1(x)] \\ &= hf'(x)[f_2(x)]^2, \end{aligned} \tag{3.10}$$

where $f'(x)$ denotes the formal derivative of the quotient f_1/f_2. Since $R_h(x)$, the a.p.'s f_i and the $C_i \in D^*(K_\omega) \equiv D(K_\omega)$ satisfy the hypotheses of Theorem 3.4 for $E = K$, we conclude that $Z(R_h) \subseteq C_1 \cup C_2$ for all $h \in \bigcap_{i=1}^2 F(f_i)$. The equality (3.12) then say that the finite zeros of f' lie in $C_1 \cup C_2$, as was to be proved.

For $K = \mathbb{C}$ and the C_i taken as closed interior of circles (a special subclass of $D(\mathbb{C}_\omega) \equiv D^*(\mathbb{C}_\omega)$, the above corollary is a result due to Walsh (cf. [4, the second part of Theorem (20,1)]) on the finite zeros of the derivative of the quotient of two polynomials of the same degree.

4. SOME GENERAL EXAMPLES

In this section, we discuss some general examples to support the validity of hypotheses and the degree of generality of our main theorems.

Example 4.1. Let E be a vector space over K of arbitrary dimension. Consider the hyperplane $S = a + E_0$ where E_0 is a maximal subspace of E and $a \in E$. Then $S \in D^*(E_\omega)$ by Remark 1.5 (II) and $\omega \notin S$. Given an element $z \notin E_0$ (it is possible to choose one), every element $x \in E$ has the unique representation [1, p. 80] $x = y + tz$ for some $y \in E_0$ and $t \in K$. With this representation, let

$$a = y_0 + t_0 z, \text{ where } y_0 \in E_0 \text{ and } t_0 \in K,$$

so that

$$x \in E_0 \Leftrightarrow t = 0 \text{ and } x \in S \Leftrightarrow t = t_0. \tag{4.1}$$

For each $k = 1, 2, ..., q$ define

$$P_k(x) = (t - t_0)^{n_k}, \forall x = y + tz \in E$$

where $y \in E_0$ and $t \in K$. Then(cf. (4.1))

$$P_k(x) = 0 \Leftrightarrow t = t_0 \Leftrightarrow x \in S.$$

Hence

$$Z(P_k) \subseteq S, \quad \forall 1 \leq k \leq q.$$

For elements $h = y' + t'z \in E(y' \in E_0, t' \in K)$, and for $1 \leq k \leq q$, we have

$$\begin{aligned} P_k(x + \rho h) &= (t + \rho t' - t_0) \\ &= \sum_{j=0}^{n_k} A_{k,j}(x, h)\rho^j, \end{aligned}$$

where the coefficients $A_{k,0}(x, h) = (t - t_0)^{n_k}$ and $A_{k,j}(x, h) = C(n_k, j)(t')^j(t - t_0)^{n_k - j}, 1 \leq j \leq n_k$, belong to K and are independent of ρ such that

$$A_{k,n_k}(x, h) \equiv A_{k,n_k}(0, h) = (t')^{n_k} \not\equiv 0$$

Thus, for $1 \leq k \leq q$, $P_k \in \mathbb{P}_{n_k}(E, K)$ satisfies (4.2) and

$$\begin{aligned} F(P_k) &= \{h \in E : A_{k,n_k}(0, h)(t')^{n_k} \in K - \{0\}\} \\ &= \{h \in E : h \notin E_0\} \\ &= E - E_0 \neq \phi. \end{aligned}$$

So that

$$\bigcap_{k=1}^{q} F(P_k) = E - E_0 \neq 0. \tag{4.2}$$

We have therefore shown that *for every hyperplane* S (*a member of* $D^*(E_\omega)$ *there exist a.p.'s* $P_k \in \mathbb{P}_{n_k}(E, K)$ *satisfying all the hypotheses of Theorem 3.1*. Since $S \subseteq E$, the statement of these theorems are neither vacuous nor trivial.

Example 4.2. Let E be a vector space over K of arbitrary dimension. Consider the sets $S_i = a_i + E_0$ where E_0 is a maximal subspace of E and $a_i \in E (i = 1, 2)$ with $a_1 - a_2 \notin E_0$. Then $S_i \in D^*(E_\omega)$ by Remark 1.5 such that $\omega \notin S_i$ and $S_1 \cap S_2 = \phi$. Given $z \notin E_0$ (it is possible to choose one), every element $x \in E$ has the unique representation $x = y + tz$ for some $y \in E_0$ and $t \in K$. With this representation let

$$a_i = y_i + t_i z, \text{ where } y_i \in E_0, \text{ and } t_i \in K(i = 1, 2),$$

so that

$$x \in E_0 \Leftrightarrow t = 0 \text{ and } x \in S_i \Leftrightarrow \ t = t_i \text{ for } i = 1, 2. \tag{4.3}$$

For each $k = 1, 2, ..., q$, we define

$$P_k(x) = \begin{cases} (t - t_1)^{n_k} & \text{for } 1 \le k \le p(< q), \\ (t - t_2)^{n_k} & \text{for } p + 1 \le k \le q, \end{cases}$$

for all $x = y + tz$, where $y \in E_0$ and $t \in K$. Then (cf. (4.4))

$$\begin{aligned} P_k(x) &= 0 \Leftrightarrow t = t_1 \Leftrightarrow x \in S_1 \text{ for } 1 \le k \le p, \\ P_k(x) &= 0 \Leftrightarrow t = t_2 \Leftrightarrow x \in S_2 \text{ for } p + 1 \le k \le p, \end{aligned}$$

and so

$$Z(P_k) \subseteq \begin{cases} S_1 \text{ for } 1 \le k \le p(< q), \\ S_2 \text{ for } p + 1 \le k \le q, \end{cases} \tag{4.4}$$

For elements $h = y' + t'z \in E \quad (y' \in E_0, \quad t' \in K)$, we have

$$P_k(x + \rho h) = \begin{cases} (t + \rho t' - t_1)^{n_k} & \text{for } 1 \le k \le p(< q), \\ (t + \rho t' - t_2)^{n_k} & \text{for } 1 + p \le k \le q, \end{cases}$$

If we continue, as in the above example, with t_0 replaced in turn by t_1 and t_2 in the respective ranges when $1 \le k \le p$ and $p + 1 \le k \le q$, we would then arrive at the conclusion that $P_k \in \mathbb{P}_{n_k}(E, K)$ and satisfy

(4.3) and (4.5) for the full range $1 \leq k \leq q$, with $S_1 \cap S_2 = \phi$. That is, for the hyperplanes S_1 and S_2 (members of $D * (E_\omega)$) there exist a.p.'s $P_k \in \mathbb{P}_{n_k}(E, K)$ satisfying all the hypotheses of Theorem 3.4. Since S_i and $S_1 \cup S_2$ are proper subsets of E, the statement of Theorem 3.4 is neither vacuous nor trivial.

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