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ON A CLASS OF LOCALLY CONFORMAL MANIFOLDS

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ABSTRACT. Geometrical and structural properties are proved for locally conformal almost cosymplectic manifolds.

1. INTRODUCTION

Let M be a $(2m+1)$-dimensional Riemannian manifold. We assume that M is structured to be a locally conformal almost cosymplectic manifold $M(\Omega, \phi, \eta, \xi, f)$, where

(i) Ω is a structure 2-form of rank $2m$,
(ii) ϕ is a tensor field of type (1.1) and of rank $2m$ which satisfies

$$\phi^2 = -\mathrm{Id} + \eta \otimes \xi,$$

(iii) η is the structure 1-form,
(iv) ξ is the structure vector field (Reeb vector field),
(v) f is a nonzero constant.

If $d^\omega = d + e(\omega)$ means the cohomological operator, one has

$$d^{2f\eta}\Omega = 0, \qquad \text{(or equivalently } d\Omega = -2f\eta \wedge \Omega).$$

The vector field $V(V^a)(a, b \in \{1, \cdots 2m\})$ such that

(1) $$dV^a + V^b\theta_b^a = V^a\eta, \quad \eta(V) = C \quad (C = \text{const.}),$$

(where θ_b^a are the local connection forms in the tangent bundle) is a principal vector field on M. Taking the Lie derivative of Ω with respect to V, one derives

$$\mathcal{L}_V\Omega = -2fC\Omega + 2(1+f)\eta \wedge \beta, \tag{2}$$

where

$$\beta = \sum (V^i\omega^{i^*} - V^{i^*}\omega^i), \qquad (i = 1, \cdots m, i^* = i + m), \tag{3}$$

where the ω^a form the associated coframe of the adapted vectorial frame $\mathcal{O} = \text{vect}\{e_a, \xi\}$. By exterior differentiation one gets

$$d(\mathcal{L}_V\Omega) = \eta \wedge \mathcal{L}_V\Omega, \tag{4}$$

which shows that $\mathcal{L}_V\Omega$ is an exterior recurrent form [8] having the structure form η as recurrence form. Next, one has

$$\nabla(\phi V) = \phi V \wedge \xi, \tag{5}$$

(where $\wedge$ stands for the wedge product of vector fields), which proves that ϕV is a Killing vector field having the Reeb vector field ξ as generative [10]. On the other hand, if [,] denotes the Lie bracket, one derives

$$[V, \xi] = 0, \qquad [\phi V, \xi] = 0, \qquad [V, \phi V] = 0, \tag{6}$$

which means that $V, \phi V$, and ξ define a commutative triple [11] [6] [2].
Setting $\|V\|^2 = l$, one derives

$$dl = l\eta, \qquad \text{grad } l = l\xi, \qquad \text{div}(\text{grad } l) = 2ml, \qquad \|\text{grad } l\|^2 = l^2. \tag{7}$$

From (7) it is seen that $\|\text{grad } l\|^2$ and div(grad l) are functions of l; following [12], this proves that l is an isoparametric function.
Some further properties following from (7) are also discussed.

2. PRELIMINARIES

Let (M, g) be a Riemannian C^∞-manifold and let ∇ be the covariant differential operator defined with respect to the metric tensor g. We assume in the sequel that M is oriented and that ∇ is the Levi-Civita connection. Let $\Gamma TM = \Xi(M)$ be the set of sections of the tangent bundle TM, and

$$\flat : TM \xrightarrow{\flat} T^*M \qquad \text{and} \qquad \sharp : TM \xleftarrow{\sharp} T^*M$$

the classical isomorphisms defined by g (i.e. $\flat$ is the index lowering operator, and $\natural$ is the index raising operator).

Following [8], we denote by

$$A^q(M,TM) = \Gamma\mathrm{Hom}(\Lambda^q TM, TM),$$

the set of vector valued q-forms ($q <$ dim M), and we write for the covariant derivative operator with respect to ∇

$$d^\nabla : A^q(M,TM) \to A^{q+1}(M,TM).$$

It should be noticed that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$. We denote by $dp \in A^1(M,TM)$ the canonical vector valued 1-form of M, which is also called the soldering form of M [3]. Since ∇ is symmetric one has that $d^\nabla(dp) = 0$.

The operator

$$d^\omega = d + e(\omega), \tag{8}$$

acting on ΛM is called the cohomology operator [5]. In (8) $e(\omega)$ means the exterior product by the closed 1-form ω, i.e.

$$d^\omega u = du + \omega \wedge u. \tag{9}$$

For any $u \in \Lambda M$ one has

$$d^\omega \circ d^\omega = 0. \tag{10}$$

A form u such that $d^\omega u = 0$ is said to be d^ω-closed.

A function $\epsilon : \mathbb{R}^n \to \mathbb{R}$ is isoparametric [12] if $\|\nabla\epsilon\|^2$ and $\mathrm{div}(\nabla\epsilon)$ are functions of ϵ ($\nabla\epsilon =$gradϵ).

Let $\mathcal{O} = \mathrm{vect}\{e_A | A = 1, \cdots 2m+1\}$ be a local field of adapted vectorial frames over M and let $\mathcal{O}^* = \mathrm{covect}\{\omega^A\}$ be its associated coframe. Then the soldering form dp is expressed by

$$dp = \omega^A \otimes e_A, \tag{11}$$

and E. Cartan's structure equations can be written in indexless manner are

$$\nabla e = \theta \otimes e, \tag{12}$$

$$d\omega = -\theta \wedge \omega, \tag{13}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{14}$$

In the above equations θ (respectively Θ) are the local connection forms in the tangent bundle TM (respectively the curvature 2-forms on M).

3. LOCALLY CONFORMAL MANIFOLDS

Let $M(\Omega, \phi, \eta, \xi, f)$ be a locally conformal almost cosymplectic manifold as defined in Section 1, i.e.

$$d^{2f\eta}\Omega = 0. \tag{15}$$

Consider the vector field

$$V = \sum V^A e_A, \qquad A = 0, 1, \cdots m, \tag{16}$$

and assume that V is a pfaffian transformation, i.e.

$$\mathcal{L}_V \eta = 0 \Longleftrightarrow \eta(V) = C = V^0 = \text{ constant.} \tag{17}$$

Taking the covariant differential of V, and making use of (12) and (17), one finds

$$\nabla V = (dV^a + V^b \theta_b^a) \otimes e_a + V^0 dp - V^b \otimes \xi. \tag{18}$$

We assume in the sequel that the following relation

$$dV^a + V^b \theta_b^a = V^a \eta \tag{19}$$

holds. Under this condition, we agree to call V the principal vector field on M. Under these assumptions, and since in general one has from the structure equations

$$\nabla \xi = dp - \eta \otimes \xi, \tag{20}$$

one derives

$$\nabla V = Cdp + V \wedge \xi - C\eta \otimes \xi, \qquad \text{where} \quad C = V^0, \tag{21}$$

and where $\wedge$ denotes the wedge product of vector fields.

Making use of the operator ϕ (see Section 1), one also derives that

$$\nabla(\phi V) = \phi V \wedge \xi. \tag{22}$$

This proves the fact that ϕV is a Killing vector field having ξ as generative. By (20), (21), and (22), one may write

$$[V,\xi]=0, \qquad [\phi V,\xi]=0, \qquad [V,\phi V]=0, \tag{23}$$

where [,] means the Lie bracket. This proves that $V, \phi V$, and ξ define a commutative triple [11] [6] [2].
On the other hand, since

$$V^\flat=\sum_a V^a\omega^a+V^0\eta, \tag{24}$$

one derives by (19) and (13) that

$$dV^\flat=\eta\wedge V^\flat. \tag{25}$$

This expresses that the dual form $V^\flat$ of the principal vector field V is exterior recurrent [1] with the same recurrent 1-form as $\mathcal{L}_V\Omega$ (see (4). Taking now the Lie derivative of Ω with respect to V, one gets

$$\mathcal{L}_V\Omega=-2fC\Omega+2f\eta\wedge\beta+d\beta, \tag{26}$$

where we have set

$$\beta=\sum(V^i\omega^{i^*}-V^{i^*}\omega^i), \qquad i=1,\cdots m; i^*=i+m. \tag{27}$$

Further, by differentiation of β, one gets by (19) and the structure equations (13) that

$$d\beta=V^i(\omega^a\wedge\theta_a^{i^*}+\eta\wedge\omega^{i^*})-V^{i^*}\left(\omega^a\wedge\theta_a^i+\eta\wedge\omega^i\right). \tag{28}$$

Now, using the Kaehlerian relations

$$\theta_j^i=\theta_{j^*}^{i^*}, \qquad \theta_j^{i^*}=\theta_i^{j^*}, \tag{29}$$

there finally follows that

$$d\beta=2\eta\wedge\beta. \tag{30}$$

Hence, the above equation says that the 1-form β is exterior recurrent with (up to a factor 2) the structure 1-form η as recurrent 1-form.
Under the above conditions, it follows by (30) that (26) that

$$\mathcal{L}_V\Omega=-2fC\Omega+2(f+1)\eta\wedge\beta, \tag{31}$$

and differentiation of (31) gives

$$d(\mathcal{L}_V\Omega) = \eta \wedge \mathcal{L}_V\Omega. \tag{32}$$

Hence, as in the case of (30), one may say that the Lie derivative $\mathcal{L}_V\Omega$ is exterior recurrent with η as recurrence form. It should be noticed that (32) can also be expressed as

$$d^{-\eta}(\mathcal{L}_V\Omega) = 0, \tag{33}$$

and that $\mathcal{L}_V\omega$ is consequently $d^{-\eta}$-closed (8).

Since the q-th covariant differential ∇^q for a vector field Z on M is defined inductively as

$$\nabla^q Z = d^\nabla\left(\nabla^{q-1}\right) Z,$$

one finds in the case under consideration

$$\nabla^2 V = V^\flat \otimes dp + \left(\eta \wedge V^\flat\right) \otimes \xi, \tag{34}$$

and

$$\nabla^3 V = 0. \tag{35}$$

By reference to [7], one may hence say that V is exterior quasi concurrent and a zero element of $A^3(M, TM)$ (see Section 1).

Let $\mathbb{L}$ be the operator of type (1.1) on forms as defined by S. Goldberg [4], i.e.

$$\mathbb{L}u = u \wedge \Omega, \qquad u \in \Lambda^1 M. \tag{36}$$

Setting $u = V^\flat$ one finds by (25)

$$d(\mathbb{L}u) = (1-2f)\eta \wedge \mathbb{L}u, \tag{37}$$

and

$$d(\mathbb{L}^q u) = (1-2f)\eta \wedge \mathbb{L}^q u. \tag{38}$$

From the above formulas it can be noticed that all $(2q+1)$-forms $\mathbb{L}^q u$ are exterior recurrent with the closed form $(1-2f)\eta$ as recurrence form (we recall that $1-2f$ is a constant).

Invoking (19), one now calculates that

$$d\,\|V\|^2 = \|V\|^2\,\eta, \tag{39}$$

and consecutively by (25) one gets

$$\mathcal{L}_V V^\flat = C V^\flat. \tag{40}$$

Following [10] it follows from (40) that V is a selfconformal vector field. By reference to (21) one now remarks that the condition for V to be an affine vector field or a Killing vector field are mutually equivalent (that is $C = 0$), i.e.

$$\mathcal{L}_V V^\flat = 0, \qquad \nabla V = V \wedge \xi.$$

Setting $\|V\|^2 = l$, it follows by (39) that

$$\text{grad } l = l\xi, \tag{41}$$

and so one gets

$$\text{div}(\text{grad } l) = 2ml, \qquad \|\text{grad } l\|^2 = l^2.$$

Hence, since div(grad l) and $\|\text{grad } l\|^2$ are functions of l, then following [12], $l : \mathbb{R} \to \mathbb{R}$ is an isoparametric function.

Continuing along this line of ideas, and operating on (41) with the operator ∇ and taking into account (20), one finds that

$$\nabla \text{ grad } l = l \, dp. \tag{42}$$

Then, if $Z, Z' \in \Xi(M)$ are any vector fields, one gets from (42)

$$\left\langle \nabla_Z \text{ grad } l, Z' \right\rangle = l \left\langle Z, Z' \right\rangle, \tag{43}$$

which shows that grad l defines an infinitesimal concircular transformation [13].

Finally, operating on (42) by d^∇, one gets

$$\nabla^2 \text{ grad } l = l\eta \wedge dp. \tag{44}$$

This proves that grad l is an exterior concurrent vector field [9] [7], and consequently one has

$$\mathcal{R}(\text{grad } l, Z) = 2ml \; g(\text{grad } l, Z), \qquad Z \in \Xi(M), \tag{45}$$

where $\mathcal{R}$ means the Ricci curvature.

Summarizing, we can state the following

Theorem 3.1 *Let $M(\Omega, \phi, \eta, \xi, f)$ be a locally conformal almost cosymplectic manifold and V be the principal vector field on M ($\eta(V) = C = const.$). One has the following properties:*

(i) the Lie derivative of the structure 2-form Ω with respect to V, is exterior recurrent with the structure 1-form η as recurrence form, i.e.

$$d(\mathcal{L}_V \Omega) = \eta \wedge \mathcal{L}_V \Omega;$$

(ii) the vector field ϕV is a Killing vector field with the Reeb vector field $\xi = \eta^\sharp$ as generative;

(iii) $V, \phi V$, and ξ define a commutative triple, i.e.

$$[V, \xi] = 0, \qquad [\phi V, \xi] = 0, \qquad [V, \phi V] = 0,$$

where $[\ ,\ ]$ denotes the Lie bracket;

(iv) Setting $\|V\|^2 = l$, one derives that

$$div(grad\ l) = 2ml, \qquad \|grad\ l\|^2 = l^2,$$

which proves that, according to [12], l is an isoparametric function;

(v) if $\mathbb{L}$ is the operator of S. Goldberg, and setting $u = V^\flat$, then all $(2q+1)$-forms $\mathbb{L}^q u$ are exterior recurrent, having up to a constant η as recurrence form;

(vi) V is a self conformal vector field, i.e.

$$\mathcal{L}_V V^\flat = C V^\flat, \qquad C = const.;$$

(vii) grad l is an exterior concurrent vector field, i.e.

$$\nabla^2 grad\ l = l\eta \wedge dp \Rightarrow \mathcal{R}(grad\ l, Z) = 2ml\ g(grad\ l, Z), \quad Z \in \Xi(M).$$

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