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APPLICATIONS OF BRIOT - BOUQUET DIFFERENTIAL SUBORDINATION TO SOME CLASSES OF MEROMORPHIC FUNCTIONS

F. M. AL-OBOUDI AND H. A. AL-ZKERI

ABSTRACT. The object of this paper is to define and study some classes of meromorphic functions using a new differential operator. Inclusion relations, integral operators, and other results are found by the application of Briot - Boquet differential subordination

1. INTRODUCTION

Let A be the set of all functions analytic in the unit disc $E = \{z : |z| < 1\}$. Let $g, G \in A$. We say that g is subordinate to G written $g \prec G$, if G is univalent, $g(0) = G(0)$ and $g(E) \subset G(E)$.

Let Σ be the set of all meromorphic functions f in E and having in $D = E|\{0\}$, the the Laurent expansion $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$. The convolution or Hadamard Product $f * g$ of two functions f and g in Σ is defined as follows

If $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k$, then

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k.$$

If follows from the definition of the Hadamard Product that

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$$z(f*g)'(z) = (zf'*g)(z) = (f*zg')(z).$$

In [1], the first author defined the operator $D_\lambda^n f$, which generalizes the well-known Salagean operator [7]. In this paper we modify the operator $D_\lambda^n f$ for meromorphic functions as follows

$$\begin{aligned} D_\lambda^0 f(z) &= f(z), \\ D_\lambda^1 f(z) &= (1-\lambda)f(z) + \lambda\frac{(z^2 f(z))'}{z}, \ \lambda \geq 0, \\ &= D_\lambda f(z), \\ D_\lambda^2 f(z) &= D_\lambda(D_\lambda^1 f(z)), \ldots \\ D_\lambda^n f(z) &= D_\lambda(D_\lambda^{n-1} f(z)) = (1-\lambda)D_\lambda^{n-1} f(z) \\ &\qquad + \lambda\frac{(z^2 D_\lambda^{n-1} f(z))'}{z}, \quad n \in \mathbb{N} \end{aligned} \tag{1.1}$$

Using the operator $D_\lambda^n f$, we introduce three subclasses $\Sigma\,(m;n;\lambda;h)$, $Q\,(m;n;\lambda;h)$ and $T\,(m;n;\lambda;h)$ of meromorphic functions and investigate certain properties of functions belonging to these classes. We require the following lemmas to prove the results of this paper.

Lemma 1.1 ([3]). *Let $\beta, \sigma \in C$. Let $h \in A$ be convex univalent in E with* $\mathrm{Re}\,[\beta h\,(z) + \sigma] > 0$, *$z \in E$, $h\,(0) = 1$ and $P \in A$ with $P(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in E, then*

$$P(z) + \frac{zP'(z)}{\beta\, P(z) + \sigma} \prec h(z) \to P(z) \prec h(z).$$

Lemma 1.2 ([5]). *Let $\beta, \sigma \in C$. Let $h \in A$ be convex univalent in E with $h\,(0) = 1$ and* $\mathrm{Re}\,[\beta h\,(z) + \sigma] > 0$, *$z \in E$, and $q \in A$ with $q\,(0) = 1$ and $q(z) \prec h(z)$, $z \in E$. If $P(z) = 1 + p_1 z + +p_2 z^2 + \cdots$ is analytic in E, then*

$$P(z) + \frac{zP'(z)}{\beta\, q(z) + \sigma} \prec h(z) \to P(z) \prec h(z)$$

2. THE CLASS $\Sigma(m;n;\lambda;h)$

Definition 2.1. Let $f=\{f_1,f_2,...,f_m\}$, $f_i\in\Sigma$, $1\le i\le m$ be such that

$$\frac{-z[D_\lambda^n f_i(z)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n f_j(z)} \prec h(z),\ z\in E,\ i=\overline{1,m},$$

where $z\sum\limits_{j=1}^{m} D_\lambda^n f_j(z)\neq 0$ in E, h is convex univalent in E with $h(0)=1$.

We say that $f=\{f_1,f_2,...,f_m\}$ belongs to the class $\Sigma(m;n;\lambda;h)$.

Remark 2.1. If $m=1$ and $\lambda=0$ or $n=0$ and $h(z)=\frac{1-z}{1+z}$, then $D_\lambda^n f(z)=f(z)$ and $\Sigma(1;n;0;h)=\Sigma^*$, the class of meromorphic starlike functions which has been studied by Clunie [2], Pommerenke [6] and others.

Theorem 2.1. *Let* $f=\{f_1,f_2,...,f_m\}\in\sum(m;n;\lambda;h)$ *and* $F(z)=\frac{1}{m}\sum\limits_{i=1}^{m} f_i(z)$. *Then* F *satisfies the condition*

$$\frac{-z[D_\lambda^n F(z)]'}{D_\lambda^n F(z)}\prec h(z),\qquad z\in E \tag{2.1}$$

Proof. Let $f=\{f_1,f_2,...,f_m\}\in\sum(m;n;\lambda;h)$. Then for any $z_0\in E$,

$$\frac{-z_0[D_\lambda^n f_i(z_0)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n f_j(z_0)}\in h(E)$$

and hence equal $h(w_i)$ (say) for some $w_i\in E$, $i=\overline{1,m}$. Hence

$$-\frac{\sum\limits_{i=1}^{m} z_0[D_\lambda^n f_i(z_0)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n f_j(z_0)}=\sum_{i=1}^{m} h(w_i).$$

Let $f(z)=\frac{1}{z}+\sum\limits_{k=1}^{\infty} a_k z^k$. Then, from (1,1) we see that

$$\begin{aligned} D_\lambda^n f &= f(z)*\left\{\frac{1}{z}+\sum_{k=1}^{\infty}[1+\lambda(k+1)]^n z^k\right\} \\ &= (f*k_n)(z), \end{aligned}$$

where

$$k_n(z) = \frac{1}{z} + \sum_{k=1}^{\infty} [1+\lambda(k+1)]^n z^k, \qquad n \in \mathbb{N}_0. \tag{2.2}$$

Hence

$$\frac{-z_0[D_\lambda^n F(z_o)]'}{D_\lambda^n F(z_0)} = \frac{-z_0[k_n * \sum_{i=1}^{m} f_i(z_0)]'}{k_n * \sum_{j=1}^{m} f_j(z_0)}.$$

Since

$$D_\lambda^n \sum_{i=1}^{m} f_i(z) = \sum_{i=1}^{m} D_\lambda^n f_i(z),$$

we have

$$\begin{aligned}\frac{-z_0[D_\lambda^n F(z_0)]'}{D_\lambda^n F(z_0)} &= \frac{1}{m}\left[\frac{-z_0 \sum_{i=1}^{m} [D_\lambda^n f_i(z_0)]'}{\frac{1}{m}\sum_{i=1}^{m} D_\lambda^n f_i(z_0)}\right] \\ &= \frac{1}{m}\sum_{i=1}^{m} h(w_i) = h(w_0).\end{aligned}$$

for some $w_0 \in E$, since h is convex in E. This completes the proof of the theorem. □

Remark 2.2. If $f = \{f_1, f_2, ..., f_m\} \in \sum(m; n; \lambda; h)$ and $h(z) = \frac{1-z}{1+z}$, then Theorem 2.1 shows that $D_\lambda^n F \in \Sigma^*$, where $F(z) = \frac{1}{m}\sum_{i=1}^{m} f_i(z)$, and hence $D_\lambda^n f_i$ are close-to-convex meromorphic functions [4].

Theorem 2.2. *Let* $f = \{f_1, f_2, ..., f_m\} \in \sum(m; n; \lambda; h)$. *Define*

$$F_i(z) = \frac{\gamma+1}{z^{\gamma+2}} \int_0^z t^{\gamma+1} f_i(t)dt, \qquad (\gamma \in C, \operatorname{Re}\gamma > 0), \qquad i = \overline{1,m}.$$

If $\operatorname{Re} h$ *is bounded in* E *and* $\operatorname{Re}(\gamma+2) > \max_{z\in E} \operatorname{Re} h(z)$, *then* $F = \{F_1, F_2, \ldots, F_m\} \in \sum(m; n; \lambda; h)$.

Proof. From the definition of $F_i(z)$ it follows that

$$zF_i'(z) + (\gamma+2)F_i(z) = (\gamma+1)f_i(z)$$

and on taking convolution with k_n, given by (2.2), we obtain

$$z[D_\lambda^n F_i(z)]' + (\gamma+2)D_\lambda^n F_i(z) = (\gamma+1)D_\lambda^n f_i(z), \qquad i=\overline{1,m}. \tag{2.3}$$

Let

$$P_i(z) = \frac{mz[D_\lambda^n F_i(z)]'}{\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)}. \tag{2.4}$$

From (2.3) we get

$$\frac{P_i(z)}{m}\sum_{j=1}^{m} D_\lambda^n F_j(z) + (\gamma+2)D_\lambda^n F_i(z) = (\gamma+1)D_\lambda^n f_i(z) \tag{2.5}$$

Differentiating the equality (2.5) with respect to z, we obtain

$$\frac{P_i'(z)}{m}\sum_{j=1}^{m} D_\lambda^n F_j(z) + \frac{P_i(z)}{m}\sum_{j=1}^{m} [D_\lambda^n F_j(z)]' + (\gamma+2)[D_\lambda^n F_i(z)]' = (\gamma+1)[D_\lambda^n f_i(z)]'.$$

From (2.4) we have

$$P_i'(z)\frac{\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)}{m} + \frac{P_i(z)}{m}\frac{\sum\limits_{i=1}^{m} P_i(z)\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)}{mz} + (\gamma+2)\frac{P_i(z)\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)}{mz} = (\gamma+1)[D_\lambda^n f_i(z)]'.$$

Hence

$$P_i'(z) + \frac{P_i(z)}{mz}\sum_{i=1}^{m} P_i(z) + (\gamma+2)\frac{P_i(z)}{z} = \frac{(\gamma+1)[D_\lambda^n f_i(z)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)}.$$

Then

$$\frac{-zP_i'(z)}{\frac{1}{m}\sum\limits_{i=1}^{m} P_i(z) + \gamma + 2} - P_i(z) = \frac{-(\gamma+1)z[D_\lambda^n f_i(z)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)} \cdot \frac{1}{\frac{1}{m}\sum\limits_{i=1}^{m} P_i(z) + \gamma + 2}$$
$$= \frac{-(\gamma+1)z[D_\lambda^n f_i(z)]'}{\frac{1}{m}\left[\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)\cdot\sum\limits_{i=1}^{m} P_i(z) + (\gamma+2)\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)\right]}.$$

From (2.5) we have

$$\frac{-zP_i'(z)}{\frac{1}{m}\sum\limits_{i=1}^{m} P_i(z)+\gamma+2} - P_i(z) = \frac{-(\gamma+1)z[D_\lambda^n f_i(z)]'}{\frac{1}{m}(\gamma+1)\sum\limits_{i=1}^{m} D_\lambda^n f_i(z)} \prec h(z), \tag{2.6}$$

since $f=\{f_1,f_2,\ldots,f_m\}\in\sum(m;n;\lambda;h)$. Now we can write for any $z_0\in E$,

$$\frac{-\frac{1}{m}z_0P_i'(z_0)}{\frac{1}{m}\sum\limits_{j=1}^{m} P_j(z_0)+\gamma+2} - \frac{1}{m}P_i(z_0) = \frac{1}{m}h(w_i),$$

for some $w_0\in E$. This is true for $i=\overline{1,m}$. Since h is convex, there exists a $w_i\in E$ such that

$$\frac{z_0Q'(z_0)}{-Q(z_0)+\gamma+2} + Q(z_0) = h(w_0),$$

where $Q(z) = -\frac{1}{m}\sum\limits_{i=1}^{m} P_i(z)$.

Hence

$$\frac{zQ'(z)}{-Q(z)+\gamma+2} + Q(z) \prec h(z).$$

Since $\operatorname{Re} h$ is bounded and $\operatorname{Re}(\lambda+2) > \max \operatorname{Re} h\,(z)$, it follows by Lemma 1.1 that $Q(z)\prec h(z)$, $z\in E$.

From (2.6) we have

$$\frac{z[-P_i(z)]'}{-Q(z)+\gamma+2} + [-P_i(z)] \prec h(z),$$

where $Q(z)\prec h(z)$, $i=\overline{1,m}$. An application of Lemma 1.2 gives $-P_i(z)\prec h(z)$, $z\in E$, $i=\overline{1,m}$.

That is

$$\frac{-z[D_\lambda^n F_i(z)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)} \prec h(z),$$

Now

$$F_i(z) = \frac{\gamma+1}{z^{\gamma+2}}\int_0^z t^{\gamma+1} f_i(t)\,dt,\ \gamma\in C,\ \operatorname{Re}\gamma>0.$$

It can be proved, easily, that, for every i, $1\le i\le m$

$$D_\lambda^n F_i(z) = \frac{\gamma+1}{z^{\gamma+2}}\int_0^z t^{\gamma+1} D_\lambda^n f_i(t)\,dt,$$

and hence

$$\sum_{i=1}^{m} D_\lambda^n F_i(z) = \frac{\gamma+1}{z^{\gamma+2}} \int_0^z t^{\gamma+1} \sum_{i=1}^{m} D_\lambda^n f_i(t)dt$$
$$= \frac{\gamma+1}{z^{\gamma+2}} \int_0^z t^{\gamma} g(t)dt,$$

where $g(t) = t \sum_{i=1}^{m} D_\lambda^n f_i(t) \neq 0$,for $t \in E$ by assumption.
Now define

$$\Omega(z) = \sum_{n=1}^{\infty} \frac{\gamma+1}{\gamma+n} z^{n-1}, \ \mathrm{Re}\,\gamma > 0,$$

Then an easy calculations shows that

$$z \sum_{i=1}^{m} D_\lambda^n F_i(z) = (\Omega * g)(z) \neq 0.$$

Thus $F = \{F_1, F_2, ..., F_m\} \in \sum(m; n; \lambda; h)$. □

Theorem 2.3. *if* $f = \{f_1, f_2, ..., f_m\} \in \sum(m; n+1; \lambda; h)$, $\lambda > 0$ *and* $\mathrm{Re}\,h$ *is bounded in* E*, then* $f = \{f_1, f_2, ..., f_m\} \in \sum(m; n; \lambda; h)$ *holds for* $(1 + \frac{1}{\lambda}) > d = \max_{z \in E} \mathrm{Re}\,h\,(z)$ *in* E.

Proof. Let

$$p_i(z) = \frac{mz[D_\lambda^n f_i(z)]'}{\sum_{j=1}^{m} D_\lambda^n f_j(z)}, \ z \in E. \tag{2.7}$$

From (1.1), we know that

$$z\ (D_\lambda^n f_i(z))' = \frac{1}{\lambda} D_\lambda^{n+1} f_i(z) - (1 + \frac{1}{\lambda})\, D_\lambda^n f_i(z). \tag{2.8}$$

From (2.7) and (2.8), we obtain

$$\frac{1}{m} p_i(z) \sum_{j=1}^{m} D_\lambda^n f_j(z) = \frac{1}{\lambda} D_\lambda^{n+1} f_i(z) - (1 + \frac{1}{\lambda})\, D_\lambda^n f_i(z).$$

Differentiating with respect to z, we get

$$\frac{z}{m}p_i'(z)\sum_{j=1}^{m} D_\lambda^n f_j(z) + \frac{z}{m}p_i(z)\sum_{j=1}^{m}(D_\lambda^n f_j(z))'$$
$$= \frac{z}{\lambda}\left(D_\lambda^{n+1} f_i(z)\right)' - z(1+\frac{1}{\lambda})\left(D_\lambda^n f_i(z)\right)'.$$

Using (2.7), we have

$$\frac{z}{m}p_i'(z)\sum_{j=1}^{m} D_\lambda^n f_j(z) + p_i(z)\left[\frac{z}{m}\sum_{j=1}^{m}(D_\lambda^n f_j(z))' + \frac{1}{m}(1+\frac{1}{\lambda})\sum_{j=1}^{m} D_\lambda^n f_j(z)\right]$$
$$= \frac{z}{\lambda}\left(D_\lambda^{n+1} f_i(z)\right)'.$$

Then

$$\frac{\frac{z}{m}p_i'(z)\sum\limits_{j=1}^{m} D_\lambda^n f_j(z)}{\frac{z}{m}\sum\limits_{j=1}^{m}\left(D_\lambda^n f_j(z)\right)' + \frac{1}{m}(1+\frac{1}{\lambda})\sum\limits_{j=1}^{m} D_\lambda^n f_j(z)} + p_i(z)$$
$$= \frac{\frac{z}{\lambda}\left(D_\lambda^{n+1} f_i(z)\right)'}{\frac{z}{m}\sum\limits_{j=1}^{m}\left(D_\lambda^n f_j(z)\right)' + \frac{1}{m}(1+\frac{1}{\lambda})\sum\limits_{j=1}^{m} D_\lambda^n f_j(z)}.$$

Using (2.8), we get

$$\frac{\frac{z}{m}p_i'(z)\sum\limits_{j=1}^{m} D_\lambda^n f_j(z)}{\frac{z}{m}\sum\limits_{j=1}^{m}\left(D_\lambda^n f_j(z)\right)' + \frac{1}{m}(1+\frac{1}{\lambda})\sum\limits_{j=1}^{m} D_\lambda^n f_j(z)} + p_i(z) = \frac{z\left(D_\lambda^{n+1} f_i(z)\right)'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^{n+1} f_j(z)}. \tag{2.9}$$

Using (2.7) in the left hand side of (2.9), we have

$$\frac{-zp_i'(z)}{\frac{1}{m}\sum\limits_{j=1}^{m} p_j(z) + 1 + \frac{1}{\lambda}} - p_i(z) = \frac{-z\left(D_\lambda^{n+1} f_i(z)\right)'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^{n+1} f_j(z)}.$$

Since $f = \{f_1, f_2, ..., f_m\} \in \sum(m; n+1; \lambda; h)$, then

$$\frac{-zp_i'(z)}{\frac{1}{m}\sum\limits_{j=1}^{m} p_j(z) + 1 + \frac{1}{\lambda}} - p_i(z) = \frac{-z\left(D_\lambda^{n+1} f_i(z)\right)'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^{n+1} f_j(z)} \prec h(z), \tag{2.10}$$

for $i = \overline{1,m}$.
Therefore for any $z_0 \in E$, we have

$$\frac{-\frac{z_0}{m}p_i'(z_0)}{\frac{1}{m}\sum\limits_{j=1}^{m} p_j(z_0) + (1+\frac{1}{\lambda})} - \frac{1}{m}p_i(z_0) = \frac{1}{m}h(w_i),$$

for some $w_0 \in E$. Since h is convex, there exists a $w_i \in E$, such that

$$\frac{-\frac{z_0}{m}\sum\limits_{i=1}^{m} p_i'(z_0)}{\frac{1}{m}\sum\limits_{j=1}^{m} p_j(z_0) + (1+\frac{1}{\lambda})} - \frac{1}{m}\sum_{i=1}^{m} p_i(z_0) = \frac{1}{m}\sum_{i=1}^{m} h(w_i) = h(w_0).$$

Setting $Q(z) = -\frac{1}{m}\sum\limits_{i=1}^{m} p_i(z)$, we have

$$\frac{zQ'(z)}{-Q(z) + (1+\frac{1}{\lambda})} + Q(z) \prec h(z).$$

Which by Lemma 1.1, implies that $Q(z) \prec h(z)$.
From (2.10), we have

$$\frac{-zp_i'(z)}{-Q(z) + (1+\frac{1}{\lambda})} - p_i(z) \prec h(z),$$

where $Q(z) \prec h(z)$. An application of Lemma 1.2 gives $-p_i(z) \prec h(z)$, which implies that $f = \{f_1, f_2, ..., f_m\} \in \sum(m; n; \lambda; h)$ □

3. THE CLASS $Q\,(m; n; \lambda; h)$

Definition 3.1. Let $Q\,(m; n; \lambda; h)$ denote the class of all functions $f \in \Sigma$ such that

$$\frac{-mz[D_\lambda^n f(z)]'}{\sum\limits_{j=1}^{m} D_\lambda^n g_j(z)} \prec h(z),\ z \in E.$$

where $g = \{g_1, g_2, ..., g_m\} \in \sum(m; n; \lambda; h)$.

Theorem 3.1. *Let $f \in Q(m; n; \lambda; h)$. If $\operatorname{Re} h$ is bounded in E and $\operatorname{Re}(c+2) > \max\limits_{z \in E} \operatorname{Re} h\,(z)$, then*

$$F(z) = \frac{c+1}{z^{c+2}} \int_0^z t^{c+1} f(t)dt,\ z \in E,\ c \in C,\ \operatorname{Re} c > 0,$$

also belongs to $Q(m;n;\lambda;h)$.

Proof. Since $f \in Q(m;n;\lambda;h)$, there exists a $g = \{g_1, g_2, ..., g_m\} \in \sum(m;n;\lambda;h)$ such that

$$\frac{-z[D_\lambda^n f(z)]'}{\frac{1}{m}\sum_{j=1}^{m} D_\lambda^n g_j(z)} \prec h(z), \ z \in E.$$

Let

$$G_i(z) = \frac{c+1}{z^{c+2}} \int_0^z t^{c+1} g_i(t)dt, \ c > 0.$$

Then by Theorem 2.2 $G = \{G_1, G_2, ..., G_m\} \in \sum(m;n;\lambda;h)$.

Let

$$P(z) = \frac{z[D_\lambda^n F(z)]'}{\frac{1}{m}\sum_{j=1}^{m} D_\lambda^n G_j(z)}. \tag{3.1}$$

Now, from the definition of G_i and F, we have

$$z[D_\lambda^n G_i(z)]' + (c+2)D_\lambda^n G_i(z) = (c+1)D_\lambda^n g_i(z), \tag{3.2}$$

and

$$z[D_\lambda^n F(z)]' + (c+2)D_\lambda^n F(z) = (c+1)D_\lambda^n f(z). \tag{3.3}$$

From (3.1) into (3.3), we obtain

$$\frac{1}{m}P(z)\sum_{j=1}^{m} D_\lambda^n G_j(z) + (c+2)D_\lambda^n F(z) = (c+1)D_\lambda^n f(z).$$

Differentiating with respect to z we obtain

$$\frac{1}{m}P'(z)\sum_{j=1}^{m} D_\lambda^n G_j(z) + \frac{1}{m}P(z)\sum_{j=1}^{m} [D_\lambda^n G_j(z)]' + (c+2)[D_\lambda^n F(z)]'$$
$$= (c+1)[D_\lambda^n f(z)]'.$$

Then

$$\frac{z}{m}P'(z)\sum_{j=1}^{m} D_\lambda^n G_j(z) + \frac{z}{m}P(z)\sum_{j=1}^{m} [D_\lambda^n G_j(z)]' + z(c+2)[D_\lambda^n F(z)]' \tag{3.4}$$
$$= z(c+1)[D_\lambda^n f(z)]'.$$

From (3.1) into (3.4) we have

$$\frac{z}{m}P'(z)\sum_{j=1}^{m} D_\lambda^n G_j(z) + \frac{z}{m}P(z)\sum_{j=1}^{m}[D_\lambda^n G_j(z)]' + (c+2)\frac{P(z)}{m}\sum_{j=1}^{m} D_\lambda^n G_j(z)$$
$$= z(c+1)[D_\lambda^n f(z)]'.$$

Hence

$$\frac{\frac{z}{m}P'(z)\sum\limits_{j=1}^{m} D_\lambda^n G_j(z)}{\frac{z}{m}\sum\limits_{j=1}^{m}[D_\lambda^n G_j(z)]' + \frac{c+2}{m}\sum\limits_{j=1}^{m} D_\lambda^n G_j(z)} + P(z)$$
$$= \frac{(c+1)z[D_\lambda^n f(z)]'}{\frac{z}{m}\sum\limits_{j=1}^{m}[D_\lambda^n G_j(z)]' + \frac{(c+2)}{m}\sum\limits_{j=1}^{m} D_\lambda^n G_j(z)}.$$

From (3.2) we get

$$\frac{-zP'(z)}{\frac{1}{m}\sum\limits_{j=1}^{m} Q_j(z) + c + 2} - P(z) = \frac{-z[D_\lambda^n f(z)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n g_j(z)} \prec h(z),$$

where $Q_j(z) = \frac{z[D_\lambda^n G_j(z)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n G_j(z)}$.

Now $(-Q_j(z)) \prec h(z), j = \overline{1,m}$, since $G = \{G_1, G_2, ..., G_m\} \in \sum(m;n;\lambda;h)$ and h is a convex function. Since $\mathrm{Re}\,(c+2) > \mathrm{Re}\,h$, an application of Lemma 1.2 implies that $-P(z) \prec h(z)$, hence $F \in Q(m;n;\lambda;h)$. □

Theorem 3.2. *If* $f \in Q(m;n+1;\lambda;h)$, $\lambda > 0$ *and* $\mathrm{Re}\,h$ *is bounded in* E, *then* $f \in Q(m;n;\lambda;h)$ *holds for* $(1+\frac{1}{\lambda}) > d = \max\limits_{z\in E}\mathrm{Re}\,h\,(z)$ *in* E.

Proof. The proof of this theorem is similar to that of Theorem 2.3 and is therefore omitted. □

4. THE CLASS $T(m;n;\lambda;\alpha;h)$

Definition 4.1. Let $T(m;n;\lambda;\alpha;h)$, $\alpha \geq 0$ denote the class of functions $f \in \sum$ satisfying the condition

$$S(\alpha; f, g_1, g_2, \ldots, g_m) = \left\{ \frac{-\alpha z[D_\lambda^{n+1} f(z)]'}{\frac{1}{m}\sum\limits_{i=1}^{m} D_\lambda^{n+1} g_i(z)} + \frac{-(1-\alpha) z[D_\lambda^{n} f(z)]'}{\frac{1}{m}\sum\limits_{i=1}^{m} D_\lambda^{n} g_i(z)} \right\} \prec h(z),$$

where $g = \{g_1, g_2, \ldots, g_m\} \in \sum(m;n;\lambda;h)$ and $z \sum\limits_{i=1}^{m} D_\lambda^{n+1} g_i(z) \neq 0$ in E.

Remark 4.1. If we put $\alpha = 0$, we get $T(m;n;\lambda;0;h) = Q(m;n;\lambda;h)$.

Theorem 4.1. *If $f \in T(m;n;\lambda;\alpha;h)$, $\lambda > 0$ and* Re h *is bounded in E, then $f \in T(m;n;\lambda;0;h)$ hold for $(\frac{1}{\lambda}+1) \geq d = \max\limits_{z\in E} \operatorname{Re} h(z)$.*

Proof. For $\alpha = 0$, the theorem is trivial and hence we can assume that $\alpha \neq 0$. Let

$$P(z) = \frac{z[D_\lambda^n f(z)]'}{\frac{1}{m}\sum\limits_{i=1}^{m} D_\lambda^n g_i(z)}.$$

Then an easy calculations shows that

$$\frac{zP'(z)}{\frac{1}{m}\sum\limits_{i=1}^{m} q_i(z) + \frac{1}{\lambda} + 1} + P(z) = \frac{z[D_\lambda^{n+1} f(z)]'}{\frac{1}{m}\sum\limits_{i=1}^{m} D_\lambda^{n+1} g_i(z)},$$

where $q_i(z) = \frac{z[D_\lambda^n g_i(z)]'}{\frac{1}{m}\sum\limits_{i=1}^{m} D_\lambda^n g_i(z)}$. Also $-\frac{1}{m}\sum\limits_{i=1}^{m} q_i(z) \prec h(z)$.

Hence

$$S(\alpha; f; g_1, g_2, \ldots, g_m) = \frac{-\alpha z P'(z)}{-\frac{1}{m}\sum\limits_{i=1}^{m} q_i(z) + \frac{1}{\lambda} + 1} - P(z) \prec h(z),$$

since $f \in T(m;n;\lambda;\alpha;h)$. Now an application of Lemma 1.2 gives $-P(z) \prec h(z)$ which complete the proof. □

Theorem 4.2. *For $\alpha > \beta \geq 0$, $\lambda > 0$,* Re h *is bounded in E and $(1+\frac{1}{\lambda}) \geq d = \max\limits_{z\in E} \operatorname{Re} h(z)$, $T(m;n;\lambda;\alpha;h) \subset T(m;n;\lambda;\beta;h)$.*

Proof. The case $\beta = 0$ was treated in the previous theorem. Hence we assume that $\beta \neq 0$, $f \in T(m;n;\lambda;\alpha;h)$ implies that

$$S(\alpha; f, g_1, g_2, ..., g_m) \prec h(z). \tag{4.1}$$

By Theorem 4.1, we have

$$\frac{-z[D_\lambda^n f(z)]'}{\frac{1}{m}\sum_{i=1}^{m} D_\lambda^n g_i(z)} \prec h(z). \tag{4.2}$$

Now

$$S(\beta; f, g_1, g_2, ..., g_m) = -(1-\frac{\beta}{\alpha})\frac{z[D_\lambda^n f(z)]'}{\frac{1}{m}\sum_{i=1}^{m} D_\lambda^n g_i(z)} + \frac{\beta}{\alpha}S(\alpha; f, g_1, g_2, ..., g_m).$$

From (4.1) and (4.2) it follows that

$$\frac{-z[D_\lambda^n f(z_1)]'}{\frac{1}{m}\sum_{i=1}^{m} D_\lambda^n g_i(z_1)} \in h(E)$$

and

$$\frac{-\alpha z_1[D_\lambda^{n+1} f(z_1)]'}{\frac{1}{m}\sum_{i=1}^{m} D_\lambda^{n+1} g_i(z_1)} - (1-\alpha)\frac{z_1[D_\lambda^n f(z_1)]'}{\frac{1}{m}\sum_{i=1}^{m} D_\lambda^n g_i(z_1)} \in h(E).$$

Now h is convex and $\frac{\beta}{\alpha} < 1$, hence we have $S(\beta; f, g_1, g_2, ..., g_m)(z_1) \in h(E)$, showing that $f \in T(m;n;\lambda;\beta;h)$. □

REFERENCES

1. Al-Oboudi, F.M., On univalent functions defined by a generalized Salagean operator, Int. J. Math. & Math. Sci.27(2004), 1429-1436.
2. Clunie, J., On meromorphic schlicht functions, J. London Math. Soc. 34(1959), 215-216.
3. Eeingenberg, P., Miller, S.S., Mocanu, P.T. and Reade, M.O., On a Briot-Boquet differential subordination, Proceeding of the Conference on General inequalities-IV, Birkhauser Verlag, Basel, I S N M 64(1983), 339-348.
4. Ganigi, M.D. and Uralegaddi, B.A., Subclasses of meromorphic close-to convex functions, Bull. Math. R.S. Roumanie (N.S.) 33(81) (1989), 105-109.

5. Padmanabhan, K.S. and Parvatham, R., Some applications of differential subordination, Bull. Austral. Math. Soc. 37(1985), 321-330.
6. Pommerenke, Ch., On meromorphic starlike functions, Pacific. J. Math. 13(1963), 221-235.
7. Salagean, G., Subclasses of univalent functions, Lecture Notes in Math. (Springer Verlag) 1013 (1983), 362-372.

Department of Mathematics
Girls College of Education, Science Sections
University of Riyadh for Girls
Siteen Street, Malaz, Riyadh,
Saudi Arabia
e-mail: fma34@yahoo.com

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