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SINGULARITIES OF THE SOLUTION OF A 2m-PARABOLIC PROBLEM IN A POLYGONAL DOMAIN

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ABSTRACT. We consider in a polygonal domain, described by the time variable t and one space variable x, a boundary value problem associated with the operator $\partial_t + (-1)^m \partial_x^{2m}$, where m is a positive integer. We prove that some singularities appear in the solution and we determine the number of these singularities as well as their smoothness properties.

1. INTRODUCTION

Let Ω be an open set of $\mathbb{R}^2$ described by the variables (t, x) and m be a positive integer. This work deals with the linear operator $\partial_t + (-1)^m \partial_x^{2m}$, which we will denote by L. More precisely, we consider the equation $Lu = f \in L^2(\Omega)$ where $L^2(\Omega)$ stands for the usual Lebesgue space whose elements are square-integrable functions on Ω. When Ω is a rectangular domain, say $\Omega =]0, T[\times]0, a[$, we know that this equation admits a (unique) solution $u \in L^2(\Omega)$ such that

$$\partial_t u \in L^2(\Omega), \partial_x^j u \in L^2(\Omega), \qquad j = 1, \ldots, 2m,$$

and

$$u(0, x) = 0, \partial_x^{2j} u(t, 0) = \partial_x^{2j} u(t, a) = 0, \quad j = 0, \ldots, m-1,$$
$$\text{for } x \in]0, a[\text{ and } t \in]0, T[,$$

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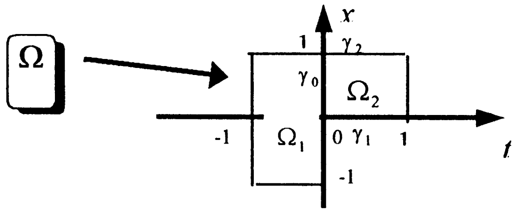
(see, for example, [6]). Observe, in this case, that the smoothness of the solution is optimal when $f \in L^2(\Omega)$. However, if the domain Ω is not rectangular, the solution u has not necessarily the optimal smoothness, for it may contain a "singular" part i.e. a part which does not belong to the natural space of smooth solutions (for the cases m = 1,2, see [9], [10], [11], [12] and [13]).

Here, we choose the simplest polygonal domain Ω which guarantees the appearance of a singular part in the solution u, and then we study the smoothness of the singularities which generate the singular part. Hereafter, Ω stands for the non convex polygonal domain

$$\Omega =]-1,0[\times]-1,1[\cup[0,1[\times]0,1[.$$

We also introduce the following notation:
$\partial\Omega$ = boundary of Ω,
$\Omega_1 =]-1,0[\times]-1,1[$,0
$\Omega_2 =]0,1[\times]0,1[$,
$\gamma_0 = \{0\}\times]0,1[$,
$\gamma_1 =]0,1[\times\{0\}$,
$\gamma_2 =]0,1[\times\{1\}$.



The problem we wish to study is

$$(\mathrm{P})\qquad \begin{cases} Lu = f \\ \partial_x^{2j} u_{|\partial\Omega\setminus\{0\}\times]-1,0[\cup\{1\}\times]0,1[} = 0, \quad j = 0,\dots,m-1. \end{cases}$$

The treatment of this problem needs some anisotropic Sobolev spaces, which we recall in the following definition

$$H_{t,x}^{1,2m}(\Omega) = \{u \in L^2(\Omega) : \partial_t u \in L^2(\Omega), \partial_x^j u \in L^2(\Omega), \ j = 0,\dots,2m\}.$$

For the sake of simplicity, we will write $H^{1,2m}$ instead of $H_{t,x}^{1,2m}$. Let $r \in]0,1[$, then

$$H^{r,2mr}(\Omega_2) = H^r(0,1;L^2(0,1)) \cap L^2(0,1;H^{2mr}(0,1)).$$

Note that all Sobolev spaces used in this paper are those defined in [6] with the same notations.

Because of the measure of the vertex (0,0) of Ω whose interior opening lies in $[3\pi/2, 2\pi[$, the solutions u of Problem (P) may be written as follows $u = u_R + u_S$. Here u_R (the regular part) belongs to $H^{1,2m}(\Omega)$ while u_S is a singular part (i.e. $\not\in H^{1,2m}(\Omega)$); more precisely: $u_{s|\Omega_1} \in H^{1,2m}(\Omega_1)$ and $u_{s|\Omega_2} \not\in H^{1,2m}(\Omega_2)$. Then we prove the existence of some singular functions $(v_k)_{k=0,\dots,\alpha_m-1}$, depending only on the operator L and on the domain Ω, which generate u_S. The number α_m, of the singularities $(v_k)_{k=0,\dots,\alpha_m-1}$ is equal to $\dfrac{m}{2}$ if m is even and to $\frac{m+1}{2}$ if m is odd.

The main result (see Theorem 3.1) of this work is the determination of the optimal smoothness of each singularity v_k in the spaces $H^{r,2mr}(\Omega_2)$. The proof is based on the Fourier transform as well as on some properties of interpolation theory and the fractional powers of operators.

This kind of problem, set in non-regular domains, has been treated in particular by Grisvard (see [4], [5] and [2] and the references given there). We may also mention other authors such as [1], [7] and [8].

Note that this work may be extended at least in the following directions:

1. The polygonal domain Ω may be replaced by a polyhedral domain (the case $m = 1$ has been treated in [12]).

2. The function f on the right-hand side of the equation of Problem (P) may be taken in $L^p(\Omega)$, where $p \in]1, \infty[$. The method used here does not seem appropriate for the space $L^p(\Omega)$ when $p \neq 2$.

3. The function f may be chosen more regular, e.g. $f \in H^{1,2m}(\Omega)$ or $f \in H^1(\Omega)$, etc. The case $f \in H^{1,2m}(\Omega)$ for $m = 2$ has been studied in [13].

4. The operator L may be replaced by some other operators with variable coefficients.

5. We can change the boundary conditions. Some of these conditions need a spectral study of the operator L which does not appear in this

work.

2. STUDY OF PROBLEM (P)

In the sequel, f stands for an arbitrary fixed element of $L^2(\Omega)$ and $f_i = f_{|\Omega_i}, i = 1, 2$. Recall the following result (cf. [6])

Theorem 2.1. *The problem*

$$(\mathrm{P}_1) \qquad \begin{cases} Lu_1 = f_1 \in L^2(\Omega_1) \\ \partial_x^{2j} u_{1|\partial\Omega_1 \setminus \{0\}\times]-1,1[} = 0, \; j = 0, \dots, m-1. \end{cases}$$

admits a (unique) solution $u_1 \in H^{1,2m}(\Omega_1)$.

Hereafter, we denote the trace $u_{1|\gamma_0}$ by φ, which is in the Sobolev space $H^m(\gamma_0)$ because $u_1 \in H^{1,2m}(\Omega_1)$ (cf. [3]). Now, we consider the following problem on Ω_2

$$(\mathrm{P}_2) \qquad \begin{cases} Lu_2 = f_2 \in L^2(\Omega_2) \\ u_{2|\gamma_0} = \varphi \in H^m(\gamma_0) \\ \partial_x^{2j} u_{2|\gamma_1 \cup \gamma_2} = 0, \; j = 0, \dots, m-1. \end{cases}$$

It is known that this problem admits a unique solution u_2 in $H^{1,2m}(\Omega_2)$ if and only if some compatibility conditions are fulfilled, that is (cf. [6]):

(a) $\varphi^{(2j)}(0) = 0, \quad 0 \le 2j \le m-1$

(b) $\varphi^{(2j)}(1) = 0, \quad 0 \le 2j \le m-1$.

Remark 2.1. We can observe that the boundary conditions of Problem (P_1) yield $\varphi^{(2j)}(1) = \partial_x^{2j} u_1(0,1) = 0$ for all integers j such that $0 \le 2j \le m-1$. So the compatibility conditions (b) are automatically satisfied. On the other hand, we have $\varphi^{(2j)}(0) = \partial_x^{2j} u_1(0,0)$ but we do not know whether $\partial_x^{2j} u_1(0,0)$ vanishes for all (or for some) j such that $0 \le 2j \le m-1$. This is the reason why singularities may arise in the solution u_2 of Problem (P_2), and consequently, in the solution u of Problem (P). Recall that the integer α_m is defined as follows

$$\alpha_m = \begin{cases} \frac{m}{2} & \text{if } m \text{ is even} \\ \frac{m+1}{2} & \text{if } m \text{ is odd} \end{cases}$$

Since the compatibility conditions (b) are always fulfilled, we can write conditions (a) as

$$\varphi^{(2j)}(0) = 0 \text{ for } j = 0, \dots, \alpha_m - 1. \tag{1}$$

Remark 2.2. The solution u of Problem (P) will be defined by

$$u = \begin{cases} u_1 & \text{in } \Omega_1 \\ u_2 & \text{in } \Omega_2 \end{cases}$$

where u_1 and u_2 are the solutions of (P_1) and (P_2) respectively. Observe that $u_{1|\gamma_0} = u_{2|\gamma_0} = \varphi$ and $u_1 \in H^{1,2m}(\Omega_1)$. Consequently, if $u_2 \in H^{1,2m}(\Omega_2)$ then $u \in H^{1,2m}(\Omega)$ (see [3]). On the other hand, u is regular in Ω_1 because $u_{|\Omega_1} = u_1 \in H^{1,2m}(\Omega_1)$, and this means that the singularities which we seek are contained in $u_{|\Omega_2}$, i.e. in u_2. So, in the sequel, we will restrict ourselves to u_2.

In the following result we will introduce some polynomials $(P_k)_{k=0,\dots,\alpha_m-1}$ which enable us to study the singular part of u_2.

LEMMA 2.1. *There exist α_m polynomials $(P_k)_{k=0,\dots,\alpha_m-1}$ of degree $2m-1$ such that*

$$\begin{aligned} P_k^{(2j)}(1) &= 0, \\ P_k^{(2j)}(0) &= \delta_{jk} \;\; \textit{for} \;\; k = 0, \dots, \alpha_m - 1 \;\; \textit{and} \;\; j = 0, \dots, m-1. \end{aligned}$$

Indeed, for each polynomial P_k, there are $2m$ conditions which allow us to determine its $2m$ coefficients, because P_k is of degree $2m-1$.

The determination of the smoothness of the singularities arising in u_2 needs the study of the following problem, set on Ω_2, for $k = 0, 1, \dots, \alpha_m - 1$

$$(P_3) \qquad \begin{cases} Lv_k = 0 \\ v_{k|\gamma_0} = P_k \in H^m(\gamma_0) \\ \partial_x^{2j} v_{k|\gamma_1 \cup \gamma_2} = 0, \;\; j = 0, \dots, m-1. \end{cases}$$

where $(P_k)_{k=0,\dots,\alpha_m-1}$ are the polynomials defined in Lemma 2.1. Problem (P_3) admits a unique solution $v_k \in L^2(\Omega_2)$. So we can define the function

v on Ω_2 by

$$v = u_2 - \sum_{k=0}^{\alpha_m - 1} \varphi^{(2k)}(0) v_k,$$

where u_2 is the solution of Problem (P_2). It is easy to verify that v is the (unique) solution of the following problem

$$\begin{cases} Lv = f_2 \in L^2(\Omega_2) \\ v_{|\gamma_0} = \varphi - \sum_{k=0}^{\alpha_m-1} \varphi^{(2k)}(0) P_k \in H^m(\gamma_0) \\ \partial_x^{2j} v_{|\gamma_1 \cup \gamma_2} = 0, \; j = 0, \dots, m-1, \end{cases}$$

and that the compatibility conditions

$$\partial_x^{2j} v(0,0) = \partial_x^{2j} v(0,1) = 0, \quad 0 \le 2j \le m-1$$

are fulfilled by virtue of conditions on the polynomials $(P_k)_{k=0,\dots,\alpha_m-1}$ in Lemma 2.1. Therefore $v \in H^{1,2m}(\Omega_2)$ and we have proved the following result

Proposition 2.1. *The solution u_2 of Problem (P_2) may be written as*

$$u_2 = v + \sum_{k=0}^{\alpha_m - 1} \varphi^{(2k)}(0) v_k,$$

where $v \in H^{1,2m}(\Omega_2)$ and v_k stands for the solution of Problem (P_3) for $k = 0, 1, \dots, \alpha_m - 1$.

Remark 2.3. Observe that v is the regular part of the solution u_2 while $\sum_{k=0}^{\alpha_m-1} \varphi^{(2k)}(0) v_k$ denotes the singular part which is generated by the singularities $(v_k)_{k=0,\dots,\alpha_m-1}$. We also note that these singularities are solutions of Problem (P_3) and are independent of the data for f in Problem (P). However, the singular part $\sum_{k=0}^{\alpha_m-1} \varphi^{(2k)}(0) v_k$ depends on f via the coefficients $\varphi^{(2k)}(0)$.

3. SMOOTHNESS OF THE SINGULAR SOLUTIONS $(v_k)_{k=0,\dots,\alpha_m-1}$

The sequence of functions $\varphi_n(x) = \sqrt{2}\sin n\pi x$ on $]0,1[$ is an orthonormal basis of $L^2(0,1)$. Let $A : D(A) \to L^2(0,1)$ be the differential operator defined by $Au = (-1)^m u^{(2m)}$ with

$$D(A) = \{u \in H^{2m}(0,1) : u^{(2j)}(0) = u^{(2j)}(1) = 0, \quad j = 0,\dots,m-1\}.$$

Observe that $\varphi_n \in D(A)$ and $A\varphi_n = (n\pi)^{2m}\varphi_n$. This means that each φ_n is an eigenfunction of A and $\lambda_n = (n\pi)^{2m}$ is the corresponding elgenvalue. Since the polynomials $(P_k)_{k=0,\dots,\alpha_m-1}$ defined in Lemma 2.1 are in $L^2(0,1)$, we may write

$$P_k = \sum_{n\geq 1} a_{nk}\varphi_n, \quad k = 0,\dots,\alpha_m - 1. \tag{2}$$

Integrating by parts, we arrive at

$$\begin{aligned} a_{nk} &= \int_0^1 \varphi_n(x)P_k(x)dx \\ &= \frac{(-1)^k\sqrt{2}}{(\pi n)^{2k+1}} + \frac{(-1)^k\sqrt{2}}{(\pi n)^{2k+1}} \int_0^1 P_k^{(2k+1)}(x)\cos n\pi x dx. \end{aligned}$$

But $\int_0^1 P_k^{2k+1}(x)\cos n\pi x dx = 0$ because P_k is a polynomial of degree $2m-1$ and

$$P_k^{(2j)}(0) = P_k^{(2j)}(1) = 0 \quad \text{for } j \geq k+1.$$

Hence

$$|a_{nk}| = \frac{C}{n^{2k+1}} \quad \text{for } k = 0,\dots,\alpha_m - 1 \text{ and } n \geq 1, \tag{3}$$

where C is a constant.

On the other hand, by (2), we deduce that the solution v_k of Problem (P_3) may be written as

$$v_k(t,x) = \sum_{n\geq 1} a_{nk}e^{-\lambda_n t}\varphi_n(x) \quad \text{for } k = 0,\dots,\alpha_m - 1, \tag{4}$$

where $\lambda_n = (n\pi)^{2m}$.

Now, our aim is to determine the largest real number $r \in]0,1[$ such that $v_k \in H^{r,2mr}(\Omega_2)$. This question will be treated in two steps.

Step 1. When does v_k lie in $H^r(0,1;L^2(0,1))$?

To answer this question we begin by extending v_k with respect to t and we set

$$\tilde{v}_k(t,x) = \sum_{n\geq 1} a_{nk} e^{-\lambda_n |t|} \varphi_n(x) \quad \text{for } k = 0, \dots, \alpha_m - 1 \ .$$

It is known that $\tilde{v}_k \in H^r(0,1;L^2(0,1))$ if and only if $(1+t^2)^{\frac{r}{2}} \hat{\tilde{v}}_k \in L^2(\mathbb{R} \times (0,1))$ where $\hat{\tilde{v}}_k$ denotes the Fourier transform of $\tilde{v}_k$ with respect to t. So, (4) gives $v_k \in H^r(0,1;L^2(0,1))$. It is easy to see that

$$\hat{\tilde{v}}(t,x) = C' \sum_n a_{nk} \lambda_n \frac{\varphi_n(x)}{t^2 + \lambda_n^2},$$

where C' is a constant. Therefore,

$$\begin{aligned} \left\|(1+t^2)^{\frac{r}{2}} \hat{\tilde{v}}_k\right\|_{L^2}^2 &= C'^2 \sum_n a_{nk}^2 \lambda_n^2 \int_{-\infty}^{+\infty} \frac{(1+t^2)^r}{(t^2+\lambda_n^2)^2} dt \\ &\leq C'^2 \sum_n a_{nk}^2 \lambda_n^{2r-1} \int_{-\infty}^{+\infty} \frac{1}{(1+y^2)^{2-r}} dy. \end{aligned}$$

We can observe that the convergence of $\int_{-\infty}^{+\infty} \frac{1}{(1+y^2)^{2-r}} dy$ holds when $r < \frac{3}{2}$. Then, if this condition is satisfied, the above-mentioned expression will be bounded if and only if the series $\sum_n a_{nk}^2 \lambda_n^{2r-1}$ is convergent, i.e., by (3), if and only if $\sum_n \frac{1}{n^{4k+2-2m(2r-1)}}$ is convergent. This holds if and only if $r < \frac{1}{2} + \frac{k}{m} + \frac{1}{4m}$ (note that the condition $r < \frac{3}{2}$ is then satisfied). So the following result is proved.

Proposition 3.1. *For each $k = 0, ..., \alpha_m - l$, if $r < \dfrac{1}{2} + \dfrac{k}{m} + \dfrac{1}{4m}$ then $v_k \in H^r(0,1;L^2(0,1))$.*

Step 2. When does v_k lie in $L^2(0,1;H^{2mr}(0,1))$?

Using the fractional powers A^r of the operator A (cf. [14]), we obtain, by (4),

(5) $$A^r v_k = \sum_n a_{nk}\lambda_n^r e^{-\lambda_n t}\varphi_n \quad \text{for } r \in]0,1[.$$

On the other hand, we have

(6) $$\|v_k\|^2_{L^2(0,1;H^{2mr}(0,1))} = \int_0^1 \|v_k(t,\cdot)\|^2_{H^{2mr}(0,1)}dt.$$

But the domain $D(A^r)$ of the operator A^r may be obtained by interpolation (we use here the notations of [6])

$$D(A^r) = [D(A), L^2(0,1)]_{1-r} \subset H^{2mr}(0,1)$$

with the equivalence of the norms of $D(A^r)$ and $H^{2mr}(0,1)$. Then we deduce, by (6), the equivalence of the two norms $\|v_k\|_{L^2(0,1;H^{2mr}(0,1))}$ and $\left[\int_0^1 \|v_k(t,\cdot)\|^2_{D(A^r)}dt\right]^{\frac{1}{2}}$. So the equality (5) shows that $v_k \in L^2(0,1;H^{2mr}(0,1))$ if and only if the series $\sum_n a^2_{nk}\lambda_n^{2r}\int_0^1 e^{-2\lambda_n t}dt$ is convergent. It is easy to see that this convergence holds if and only if

$$r < \frac{1}{2} + \frac{k}{m} + \frac{1}{4m}.$$

Observe that this condition is the same as the sufficient condition obtained in Proposition 3.1. So the following result is proved

Proposition 3.2. *For each $k = 0,\dots,\alpha_m-1$, v_k lies in $L^2(0,1;H^{2mr}(0,1))$ if and only if $r < \frac{1}{2} + \frac{k}{m} + \frac{1}{4m}$.*

Now, our main result follows easily from Proposition 3.1 and Proposition 3.2. That is,

Theorem 3.1. *For each $f \in L^2(\Omega)$, the solution u of (P) is such that*

1) $u_{|\Omega_1} \in H^{1,2m}(\Omega_1)$.

2) There exist α_m singularities $v_0,\dots,v_{\alpha_m-1}$ independent of f and a function $v \in H^{1,2m}(\Omega_2)$ such that

$$u_{|\Omega_2} = v + \sum_{k=0}^{\alpha_m-1}\beta_k v_k$$

where $(\beta_k)_{k=0,\dots,\alpha_m-1}$ *are real coefficients depending on* f *(in fact, we have* $\beta_k = \partial_x^{2k}u(0,0)$*).*

3) For each $k = 0,\dots,\alpha_m - 1$*, the smoothness of the singularities* v_k *mentioned above is as follows*

$$v_k \in H^{r_k,2mr_k}(\Omega_2) \quad \textit{iff} \quad r_k < \frac{1}{2} + \frac{k}{m} + \frac{1}{4m}.$$

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