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EFFICIENCY CALCULATIONS OF SOME TESTS FOR EXPONENTIALITY BY USING TTT-TRANSFORMS

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ABSTRACT. The TTT-plot (TTT=Total Time on Test), an empirical and scale independent plot based on failure data, and the corresponding asymptotic curve, the scaled TTT-transform, were introduced as tools for model identification. They have later been used in several other reliability applications.

In this paper these tools are used as a source of inspiration for more theoretical discussions. We use them to illustrate and study the power of some test statistics aimed to be omnibus tests for exponentiality. Among these are Gnedenko's F-test, but also some generalizations of that statistic.

The TTT-plot is first used for graphical interpretation of the different test statistics. This gives an illustration why the statistics are inefficient under certain alternatives. Then we use scaled TTT-transforms when calculating optimal theoretical efficiency values for alternatives within the family of Weibull distributions and the family of Pareto distributions.

1. INTRODUCTION

The TTT-plot (TTT=Total Time on Test), an empirical and scale independent plot based on failure data, and the corresponding asymptotic curve, the scaled TTT-transform, were introduced by Barlow and Campo (1975) as tools for model identification. These tools have later proven to be very useful in several other applications within reliability, for instance for analysis of aging properties, maintenance optimization and burn-in optimization.

However, the TTT-concept has also been used for theoretical purposes within reliability. In Bergman and Klefsjö (1989) a family of test statistics was studied which was intended for testing exponentiality against the DMRL-property (DMRL= Decreasing Mean Residual Life). Here TTT-transforms were used as a source of inspiration when looking for suitable test statistics. Furthermore, the efficiency for different members of the family was analyzed by using TTT-transforms. In that way it was possible to suggest efficient statistics under different hypotheses alternative. Among the results were generalizations of a test statistic presented by Hollander and Proschan (1975) intended for detecting the DMRL (Decreasing Mean Residual Life) property.

In this paper the TTT-plot and the scaled TTT-transform are used to illustrate and study the power of some test statistics intended to be omnibus tests for exponentiality. Among these are Gnedenko's F-test, but also some generalizations of that statistic.

From the graphical interpretation of different statistics it is illustrated why they are inefficient under certain alternatives. This confirms, in a way, earlier published simulation studies. Furthermore, it is shown that the Pitman efficiency values can be studied as functions of TTT-transforms. This gives a possibility to calculate suitable values of parameters included in the test statistics in order to get as large efficiency as possible.

2. THE TTT-PLOT AND THE SCALED TTT-TRANSFORM

If we have a complete ordered sample $0 = t_{(0)} \leq t_{(1)} \leq \dots \leq t_{(n)}$ of times to failure from a certain type of unit, the TTT-plot, of these observations is obtained in the following way:

- Calculate the TTT-values $S_j = nt_{(1)} + (n-1)(t_{(2)} - t_{(1)}) + \dots + (n-j+1)(t_{(j)} - t_{(j-1)})$ for $j = 1, 2, \dots, n$ (for convenience we set $S_0 = 0$).
- Normalize these TTT-values by calculating $u_j = S_j/S_n$ for $j = 0, 1, \dots, n$.

- Plot $(j/n, u_j)$ for $j = 0, 1, \ldots, n$.
- Join the plotted points by line segments.

Accordingly, a TTT-plot lies within the unit square and consists of line segments starting at (0,0) and ending at (1,1). Figures 1 and 2 show examples of some TTT- plots.

The reason for the acronym "TTT", which means "Total Time on Test", is that if all the units are put into test at the same time then S_j is the total cumulated test time for all the units at time $t_{(j)}$.

When the sample size n increases to infinity the TTT-plot converges (with probability one and uniformly; see Langberg et al., 1980) to a certain curve called the *scaled TTT-transform* of the life distribution $F(t)$ from which our sample has come. This is illustrated in Figure 2. Mathematically the scaled TTT-transform is defined as

$$\varphi(u) = \frac{1}{\mu}\int_0^{F^{-1}(u)} R(t)dt \quad \text{for } 0 \leq u \leq 1$$

where $R(t) = 1 - F(t)$ is the survival function and μ is the mean, which is supposed to be finite. Some examples of scaled TTT-transforms are illustrated in Figures 2 and 3.

Note that this transform, as the name indicates, is independent of scale. For instance, for a Weibull distribution with survival function $R(t) = 1 - F(t) = \exp(-(t/\alpha)^\theta)$, $t \geq 0$, we get the same transform for a certain value of θ, independently of the value of α, and every exponential distribution $F(t) = 1 - \exp(-\lambda t), t \geq 0$, is transformed to the diagonal in the unit square independently of the value of the failure rate λ ; see Figure 3.

The scaled TTT-transform and the TTT-plot were first presented in a paper by Barlow & Campo (1975). They used these concepts for model identification purposes by comparing the TTT-plot to scaled TTT-transforms of different life distributions. Since then a lot of other applications have appeared, both theoretical and practical. Examples of such applications are age replacement problems with and without discounted costs, burn-in optimization, characterization of different aging properties and analysis of data from repairable systems. For details of these and other applications we refer to Bergman & Klefsjö (1982, 1984), Klefsjö (1991), Klefsjó & Kumar (1992) and Klefsjö & Westberg (1996). Discussion of TTT-plots for censored samples can be found in Westberg & Klefsjö (1994).

We want to mention here the correspondence between the hazard rate and the shape of the TTT-transform. The hazard rate (sometimes called the failure rate) is defined as $z(t) = f(t)/R(t)$ and can be interpreted as a measure of the probability that a non-repairable unit which has survived up to time t soon will fail, in the sense that the probability that the unit will fail during the next Δt units of time is roughly $z(t)\Delta t$.

Barlow & Campo (1975) proved that the exponential distribution, which has constant hazard rate, corresponds to the scaled TTT-transform $\varphi(u) = u, 0 \leq u \leq 1$, i.e. the diagonal in the unit square (see Figure 3). Furthermore, Barlow & Campo (1975) proved that increasing hazard rate corresponds to a concave TTT- transform (illustrated for instance by the TTT-transform in Figure 2). In the same way a decreasing hazard rate corresponds to a convex TTT-transform (illustrated by the TTT-transform of the Pareto distribution in Figure 3).

3. TESTS FOR EXPONENTIALITY

A lot of tests have been proposed over the years intended to see whether the exponential distribution might be a suitable model to describe different types of failure data. Some of these are suggested to detect a particular alternative, such as a certain type of aging. For a thorough treatment of goodness-of-fit tests, we refer to d'Agostino & Stephens (1986).

Here we will discuss and study three tests originally suggested as omnibus tests for exponentiality, i.e. independent of alternative hypotheses. The first one is Gnedenko's F-test, named after the Russian statistician

Boris Vladimoirovich Gnedenko (1912-1995). The other two are generalizations of that test statistic, presented by Harris (1976) and Vännman (1975), respectively.

4. GNEDENKO'S F-TEST

Let us use the same notations as in Section 2 but also, for convenience, introduce

$$D_j = (n-j+1)(t_{(j)} - t_{(j-1)}) \quad \text{for } j = 0, 1, \ldots, n.$$

Sometimes $D_j, j = 0, 1, \ldots, n$, are called the *normalized spacings*. It is well-known that if the sample is from an exponential distribution then $D_j, j = 0, 1, \ldots, n$, are independent random variables from the same exponential distribution, see e.g. Barlow & Proschan (1981, p. 60). With this notation we get $S_j = D_1 + D_2 + \ldots + D_j$.

Gnedenko et al. (1969, p. 236) discussed the test statistic

$$G(r) = \frac{\frac{1}{r}\sum_{j=1}^{r} D_j}{\frac{1}{n-r}\sum_{j=r+1}^{n} D_j}.$$

It was aimed for testing exponentiality, i.e. constant failure rate, against increasing or decreasing failure rate. The same test statistic $G(r)$ was in fact discussed already by Epstein (1960) as an omnibus test statistic for testing exponentiality. The statistic $G(r)$ is often called *Gnedenko's F-statistic*.

Power estimates for this statistic can be found in e.g. Fercho & Ringer (1972), for $[r/n] = 1/2$, Wang & Chang (1977), Lin & Mudholkar (1980) and d'Agostino & Stephens (1986, pp. 452-453).

Since

$$u_r = \frac{S_j}{S_n} = \frac{\sum_{j=1}^{r} D_j}{\sum_{j=1}^{n} D_j},$$

we can rewrite

$$G(r) = \frac{\frac{1}{r}u_r}{\frac{1}{n-r}(1-u_r)},$$

which means that the statistic $G(r)$ can easily be illustrated directly in the TTT-plot. With the notations α and β from Figure 4, we get that

$$\tan\alpha = \frac{1-u_r}{1-\frac{r}{n}} \qquad \text{and} \qquad \tan\beta = \frac{u_r}{\frac{r}{n}}.$$

Accordingly, we get that

$$G(r) = \frac{\tan\beta}{\tan\alpha},$$

i.e. $G(r)$ is the ratio between the slopes of the lines marked (1) and (2), respectively, in Figure 4.

The reason for the name "F-test" is that, if our sample is from an exponential distribution, then $G(r)$ is $F(2r, 2(n-r))$-distributed, see Johnson et al. (1995, p. 322). This result is based on the fact that under exponentiality $2\lambda D_j, j = 1, 2, \ldots, n$, are independent χ^2-distributed random variables with two degrees of freedom and that the sum of two independent χ^2-distributed random variables with r_1 and r_2 degrees of freedom, respectively, is itself χ^2-distributed with $r_1 + r_2$ degrees of freedom. Furthermore, we have that if X_1 is $\chi^2(f_1)$-distributed and X_2 is $\chi^2(f_2)$-distributed, and X_1 and X_2 are independent, then the ratio $\frac{X_1/f_1}{X_2/f_2}$ is $F(f_1, f_2)$-distributed. For later discussions we also remind of the fact that the mean μ and the variance σ^2 of an $F(m,n)$-distribution have the values

$$\mu = \frac{n}{n-2} \quad \text{and} \quad \sigma^2 = \frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}$$

see Johnson et al. (1995, p. 326).

If our sample is from an exponential distribution $G(r)$ is expected to be equal to one since the corresponding TTT-plot is expected to wriggle around the diagonal, which means that $\alpha \approx \beta$. A large deviation from one indicates, therefore, that our sample is from another life distribution than the exponential distribution. From Figure 4 and the fact that the TTT-plot based on exponential observations converges to the diagonal (which is the scaled TTT-transform of the exponential distribution) we can realize that in order for this statistic to be as efficient as possible the point A in Figure 4 should be as far away as possible from the diagonal. A consequence of this is that the statistic $G(r)$ can not be useful when testing against a distribution whose scaled TTT-transform crosses the diagonal near the value of r/n. One example of this is the lognormal distribution (see Figure 3). If we choose $r/n \approx 0.25$ it will be almost impossible to detect that our sample is from this lognormal distribution. This observation was done by Harris (1976) from another point of view as an argument for proposing a generalization of Gnedenko's statistic which will be discussed in Section 5.

The test statistic $G(r)$ is asymptotically normally distributed in the sense that if r and n increase in such a way that $r/n \to u$, $0 < u < 1$, then

$$\frac{\sqrt{n}(G(r) - \mu(\theta))}{\sigma(\theta)} \qquad \text{is asymptotically } N(0,1)$$

where

$$\mu(\theta) = \frac{\varphi_\theta(u)/u}{(1-\varphi_\theta(u))/(1-u)} \tag{4.1}$$

and

$$\sigma^2(\theta) = \left[\frac{\frac{1-\varphi_\theta(u)}{1-u}}{\frac{\varphi_\theta(u)}{u}}\right]^2 A(u).$$

Here $\varphi_\theta(u)$ denotes the scaled TTT-transform of the life distribution from which our sample is coming and which is supposed to depend on θ in such a way that when $\theta = \theta_0$ we get the exponential distribution. Examples of this, which will be used later, are

Weibull distribution: $R(t) = \exp(-(t/\alpha)^\theta)$, $t \geq 0$; here $\theta_0 = 1$

Pareto distribution: $R(t) = (1 + \theta t)^{-1/\theta}$, $t \geq 0$; here $\theta_0 = 0$

Furthermore

$$\begin{aligned} A(u) &= \frac{1}{\varphi_\theta^2(u)}\int_0^u \frac{\varphi_\theta^2(s)}{(1-s)^2}ds + K^2(u)\frac{u}{1-u} \\ &\quad -2\frac{K(u)}{\varphi_\theta(u)}\int_0^u \frac{\varphi_\theta(s)}{(1-s)^2}ds + \left(\frac{1}{1-\varphi_\theta(u)}\right)^2 \int_u^1 \left(\frac{1-\varphi_\theta(s)}{1-s}\right)^2 ds \end{aligned}$$

and

$$K(u) = \frac{(1-u)\varphi_\theta(u)}{\varphi_\theta(u)(1-\varphi_\theta(u))}$$

This asymptotic result follows, after some transformations and calculations, for instance from Chernoff et al. (1967); see also Vännman (1975).

For $\theta = \theta_0$, i.e. when we have the exponential distribution, the expression for the variance reduces to

$$\sigma^2(\theta_0) = \frac{1}{u(1-u)}. \tag{4.2}$$

In order to simplify the notations we, from now on, sometimes use φ instead of φ_θ and φ_{θ_0}.

When testing a simple hypothesis $\theta = \theta_0$ against an alternative $\theta > \theta_0$, different measures of asymptotic efficiencies can be applied, see e.g. Rao (1965, pp. 390-396). When the test statistic is asymptotically normally distributed with mean $\mu(\theta)$ and variance $\sigma^2(\theta)/n$ the most frequently used measure is the Pitman efficiency value, defined as

$$P = \left(\frac{\mu'(\theta_0)}{\sigma(\theta_0)}\right)^2 \tag{4.3}$$

where the dash means differentiation with respect to θ. Sometimes the Pitman value is defined without being squared.

A rough interpretation of the Pitmans efficiency value is as follows. Suppose that we have two test statistics T_1 and T_2, say, with Pitman values P_1 and P_2, respectively, defined according to (3). Then we need about nP_1/P_2 observations when using T_2 if we want the same power as when using T_1 and we need n observations using T_1. If e.g. $P_1 = 2P_2$ we need roughly twice as many observations when using T_2 as when using T_1 in order to get the same efficiency.

Combining (1), (2) and (3) above we get that the Pitman value, which here is a function of u and therefore is denoted $P(u)$, is given by

$$P(u) = \frac{\left(\left(\frac{d\varphi(u)}{d\theta}\right)^2_{\theta=\theta_0}\right)}{u(1-u)}.$$

Here

$$\begin{aligned}\frac{d\varphi_\theta(u)}{d\theta} &= \int_0^{F_\theta^{-1}(u)} \frac{d}{d\theta}\left[\frac{1}{\mu(\theta)}R_\theta(t)\right]dt + R_\theta(F_\theta^{-1}(u))\frac{dF_\theta^{-1}(u)}{d\theta} = \\ & \int_0^{F_\theta^{-1}(u)} \frac{d}{d\theta}\left[\frac{1}{\mu(\theta)}R_\theta(t)\right]dt + (1-u)\frac{dF_\theta^{-1}(u)}{d\theta}.\end{aligned}$$

For a Pareto distribution with survival function $R(t) = (1+\theta t)^{-1/\theta}$, $t \geq 0$, we get

$$\frac{d\varphi(u)}{d\theta} = (1-u)^{1-\theta}\ln(1-u)$$

and the Pitman value is therefore

$$P(u) = \frac{(1-u)(\ln(1-u))^2}{u}.$$

By studying $P'(u)$ and using numerical methods it is found that the function reaches its largest value for $u = 0.7968121 \approx 0.80$. The corresponding maximum value of $P(u)$ is $0.6475 \approx 0.65$. If we expect the alternative distribution to be a Pareto distribution we should therefore choose $r/n \approx 0.80$ if we use the Gnedenko's F-test in order to have as large possibility as possible to detect the alternative; see Figure 5. We can also see from Figure 5 that a value of r/n which is somewhat lower than 0.80 is better than a value higher than 0.80.

For a Weibull distribution with survival function $R(t) = \exp(-(t/\alpha)^\theta)$, $t \geq 0$, the expression is a little more complicated. We get, after some calculations, that

$$\begin{aligned}P(u) &= \frac{1}{u(1-u)}[(1-\gamma)u + (1-u)\ln(1-u)\ln(-\ln(1-u)) \\ &\quad - \int_0^{-\ln(1-u)} ye^{-y}\ln y dy]^2\end{aligned}$$

where $\gamma = 0.577215$ is Euler's constant. This function is illustrated in Figure 6.

Also here numerical methods can be used to determine that $u \approx 0.31$ gives the maximum value of $P(u)$ which is equal to 1.0. If the alternative is a Weibull distribution we should therefore choose $r/n \approx 0.30$ in order to get a test procedure which is as efficient as possible.

The same results was obtained using quite another technique by Vännman (1975) in an unpublished report discussing tests against heavy-tailed distributions; see also Section 6.

5. HARRIS' GENERALIZATION

Harris (1976) suggested a modified version of Gnedenko's F-test in which the middle part of the sum were set in the denominator and the "tail"-parts were added in the numerator. Harris (1976, p. 171) says that Gnedenko's F-test is "especially nonoptimal whenever the hazard rate is U-shaped (i.e. first increasing and then decreasing or vice versa), as it may be in the lognormal case". This fact is confirmed by our graphical interpretation of the statistic in Section 4. More precisely, Harris (1976) suggested the test statistic.

$$H(r) = \frac{\frac{1}{2r}\left(\sum_{j=1}^{r} D_j + \sum_{j=n-r+1}^{n} D_j\right)}{\frac{1}{n-2r}\sum_{j=r+1}^{n-r} D_j} = \frac{\frac{1}{2r}(u_r + 1 - u_{n-r})}{\frac{1}{n-2r}(u_{n-r} - u_r)} \text{ for } \tfrac{r}{n} < \tfrac{1}{2}.$$

Harris believed that this statistic should be more efficient against log-normal alternatives and suggested, based on minor simulation study, $r = [n/4]$ as a generic choice. If the sample is from an exponential distribution the Harris' statistic is $F(2 \cdot 2r, 2(n-2r))$-distributed. Also this statistic can easily be illustrated graphically by using the TTT-plot since

$$H(r) = \frac{1}{2}\frac{(\tan\beta + \tan\alpha)}{\tan\gamma}$$

with α, β and γ defined in Figure 7.

It is clear from this graphical interpretation that Harris' statistic is more efficient against lognormal alternatives than Gnedenko's statistic. Every lognormal alternative has first an increasing hazard rate and then a decreasing hazard rate (cf. Figures 3 and 7). This means that we expect the numerator to be larger and the denominator to be smaller than in the Gnedenko statistic. This gives a higher value of the ratio and accordingly a better possibility to detect lognormal alternatives. However, if the alternative instead is decreasing or increasing, corresponding to a convex or a concave scaled TTT-transform, respectively, we realize that Gnedenko's statistic should be better than Harris' statistic. This will be confirmed below when we calculate some Pitman values.

Also Harris' test statistic asymptotically normally distributed in the sense that

$$\frac{\sqrt{n}(H(r)-\mu(\theta))}{\sigma(\theta)} \qquad \text{is asymptotically } N(0,1)$$

when $n \to \infty$. Here we get

$$\mu(\theta) = \frac{[\varphi(u)+(1-\varphi(1-u))]/(2u)}{[\varphi(1-u)-\varphi(u)]/(1-2u)} \quad \text{for } 0 < u < 1/2.$$

The expression for $\sigma(\theta)$ can be obtained in a similar way as for Gnedenko's test statistic but it is rather complicated and of less value here. The only interesting fact is that the variance under exponentiality is

$$\sigma^2(\theta_0) = \frac{1}{2u(1-2u)} \quad \text{for } 0 < u < 1/2.$$

This means that the Pitman value is equal to

$$P(u) = \left(\frac{\varphi_0'(u)-\varphi_0'(1-u)}{\sqrt{2u(1-2u)}}\right)^2. \tag{5.1}$$

For a Pareto distribution the TTT-transform is equal to

$$\varphi_\theta(u) = 1-(1-u)^{1-\theta} \quad \text{and} \quad \varphi_0'(u) = (1-u)\ln(1-u).$$

This means that we have

$$P(u) = \frac{((1-u)\ln(1-u)-u\ln u)^2}{2u(1-2u)} \quad \text{for } 0 < u < 1/2.$$

Using Newton-Raphson's method we get that $P(u)$ has a maximum for $u = 0.074488 \approx 0.074$ and the corresponding P-value is equal to about 0.11; see Figure 8.

In the Weibull case

$$\varphi_0'(u) = (1-\gamma)u + (1-u)\ln(1-u)\ln(-\ln(1-u)) - \int_0^{-\ln(1-u)} ye^{-y}\ln y dy$$

where $\gamma = 0.577215$ is Euler's constant. This expression inserted in (4) gives the Pitman value. The maximum value of $P(u)$ is equal to 0.06 when $u \approx 0.67$; see Figure 9.

These Pitman values for the Harris' statistic strongly indicate that the statistic presented in Harris (1976) in general is less efficient that the Gnedenko statistic if the alternative has a decreasing or an increasing hazard rate, for instance a Pareto distribution and Weibull distribution.

6. VÄNNMAN'S F-TEST

Still another generalization was introduced by Vännman (1975) for testing exponentiality against heavy-tailedness, defined in that paper as a distribution for which the hazard rate $z(t) \to 0$ when $t \to \infty$. Vännman (1975) defined and studied a test statistic, a little more general than

$$V(r,m) = \frac{\frac{1}{m-r}\left(\sum_{j=r+1}^{m} D_j\right)}{\frac{1}{r}\sum_{j=1}^{r} D_j} = \frac{\frac{1}{m-r}(u_m - u_r)}{\frac{1}{r}u_r}$$

where $0 < r < m < n$. This means that the information from the last part of the observations is "thrown away". A graphical interpretation of Vännman's F- test can be found in Figure 10.

Also Vännman's test statistic is F-distributed if the sample is from an exponential distribution. More precisely $V(r,m)$ is $F(2(m-r), 2r)$-distributed. If $n \to \infty$ in such a way that $r/n \to u$ and $m/n \to t$, $0 < u < t < 1$, we get that

$$\frac{\sqrt{n}(V(r,m) - \mu(\theta))}{\sigma(\theta)} \quad \text{is asymptotically } N(0,1)$$

when $n \to \infty$, where

$$\mu(\theta) = \frac{\frac{\varphi_\theta(t)-\varphi_\theta(u)}{t-u}}{\frac{\varphi_\theta(u)}{u}} = \frac{u}{t-u}\frac{\varphi_\theta(t)-\varphi_\theta(u)}{\varphi_\theta(u)} \quad \text{for } 0 < u < t < 1$$

and

$$\sigma^2(\theta_0) = \frac{t}{u(t-u)} \quad \text{for } 0 < u < t < 1.$$

From this we get that the Pitman value $P(t,u)$ which here is dependent on both t and u, is equal to

$$P(t,u) = \frac{(u\varphi'_\theta(t) - t\varphi'_\theta(u))^2}{tu(t-u)} \quad \text{for } 0 < u < t < 1.$$

The problem now is how to choose u and t in order to maximize $P(t,u)$. For the Pareto distribution this means that we have to study

$$P(t,u) = \frac{(u(1-t)\ln(1-t) - t(1-u)\ln(1-u))^2}{tu(t-u)} \quad \text{for } 0 < u < t < 1.$$

We get that $P(t,u)$ has maximum when $u \approx 0.69$ and $t = 1$, see Figure 11.

For the Weibull distribution the maximum of $P(t,u)$ occurs for $u \approx 0.32$, $t = 1.0$; see Figure 12.

CONCLUSIONS

In this paper the TTT-plot and the scaled TTT-transform have been used to illustrate and discuss efficiency properties of some test statistics, suggested as omnibus tests for exponentiality. For instance, by illustrating the test statistics graphically it has been possible to indicate when and why the statistics are more or less efficient under different alternatives in a visible and clear way. This easily made visualization is in fact one of the strengths with the TTT-plot and the TTT- transform. Besides of the obtained results for the particular test statistics, this article shows that the TTT-concept is a powerful tool not only for practical reliability problems but also for more theoretical discussions.

Remark. All the numerical calculations have been done and all the figures have been produced by using the software Mathematica. For more information on this software, see e.g. Wolfram (1991).

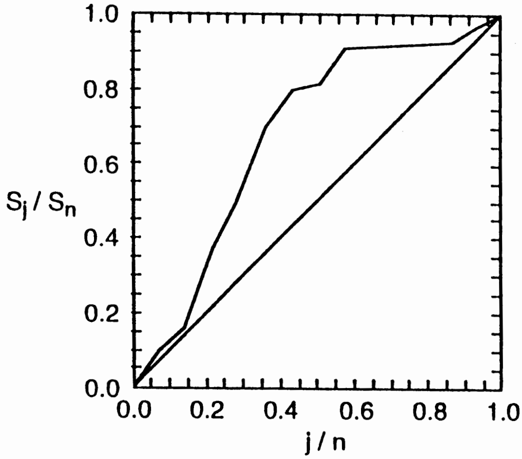

Figure 1 *A TTT-plot based on times to failure of the engine in a load haul dump machine working in a Swedish mine. (From Kumar et al., 1989.)*

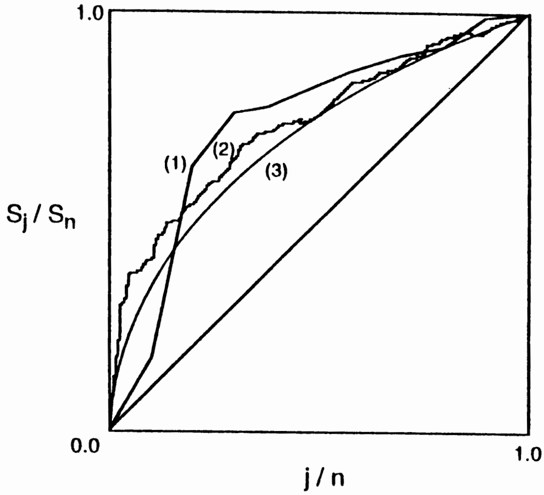

Figure 2 *TTT-plots based on simulated Weibull data with β = 2.0 and n = 10 in (1), β = 2.0 and n = 100 in (2) and the scaled TTT-transform of a Weibull distribution with β = 2.0 in (3). (From Bergman & Klefsjö, 1984.)*

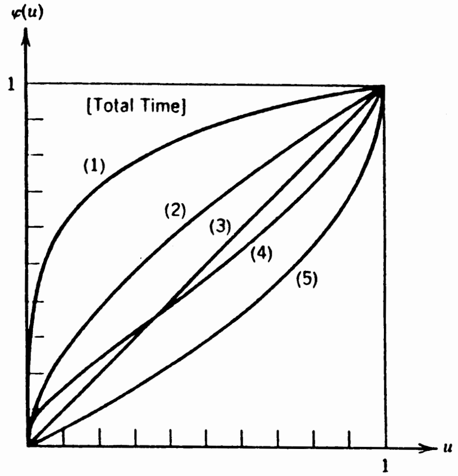

Figure 3 *Scaled TTT-transforms of five different life distibutions; (1) normal with $\mu = 1$ and $\sigma = 0.3$; (2) gamma distribution with shape parameter 2.0; (3) exponential distribution; (4) lognormal distribution with $\mu = 0$ and $\sigma = 1$; (5) Pareto distribution with $R(t) = (1+t)^{-2}$, $t \geq 0$. (From Bergman & Klefsjö, 1984.)*

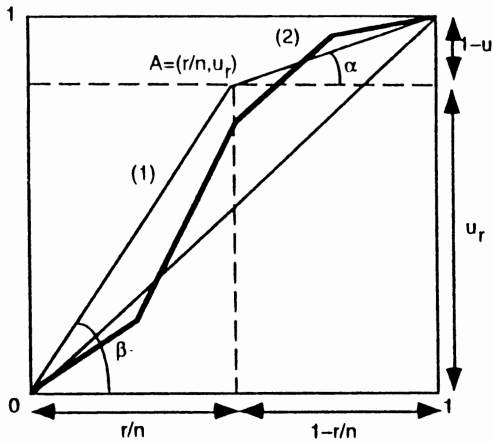

Figure 4 ***A graphical illustration of Gnedenko's F-test G(r) directly in the TTT-plot. Since tan $\alpha = (1-u_r)/(1-(r/n))$ and tan $\beta = u_r/(r/n)$ it is obvious that G(r) can be interpreted as the ratio between the slopes of the lines marked (1) and (2), respectively.***

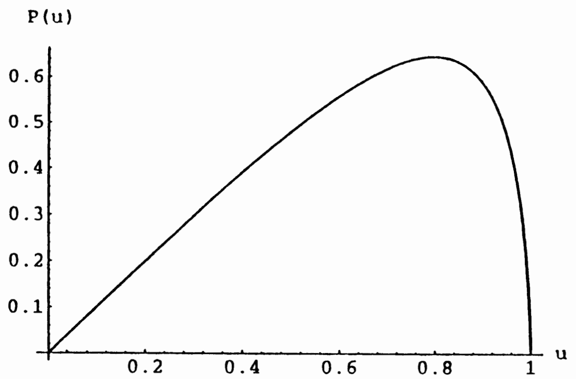

Figure 5 - *The curve illustrates the Pitman efficiency P(u) as a function of u, the limit of r/n, for Gnedenko's F-test when the sample is from a Pareto distribution.*

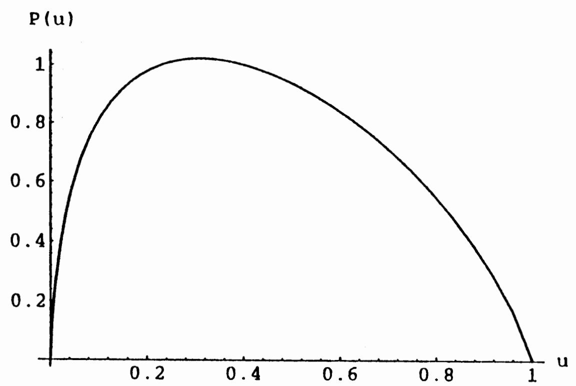

Figure 6 *The curve illustrates the Pitman efficiency P(u) as a function of u, the limit of r/n, for Gnedenko's F-test when the sample is from a Weibull distribution.*

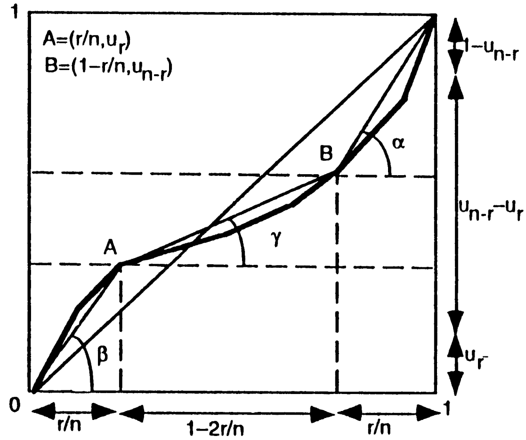

Figure 7 *Illustration of Harris' F-test statistic. The test statistic H(r) can be interpreted as* $H(r) = ((\tan \beta + \tan \alpha)/2)/\tan \gamma$.

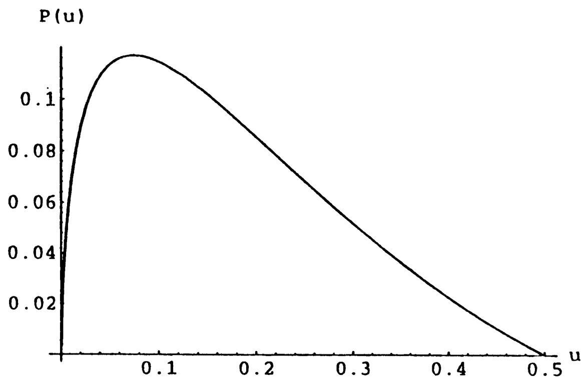

Figure 8 *Illustration of the Pitman efficiency function P(u) for Harris' test statistic when the alternative is a Pareto distribution.*

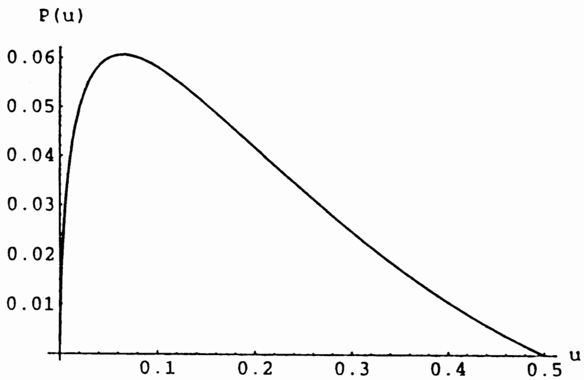

Figure 9 *Illustration of the Pitman efficiency function P(u) for Harris' test statistic when the alternative is a Weibull distribution.*

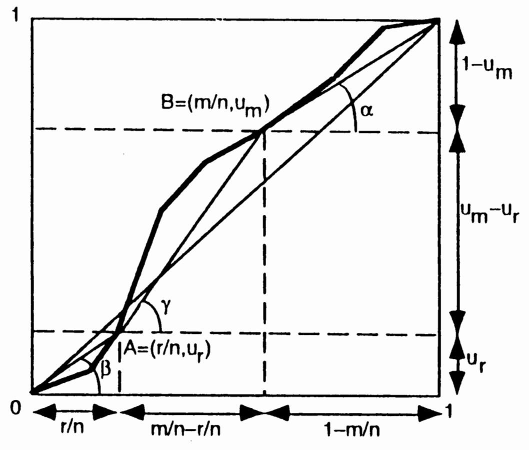

Figure 10 *A graphical interpretation of Vännman's F-test. The statistic V(r,m) can be interpreted as V(r,m) = (tan γ) / (tan β). In particular V(r,m) is expected to be about one if the sample is from an exponential distribution, since in that case both A and B roughly is on the diagonal independently of the values of r and m.*

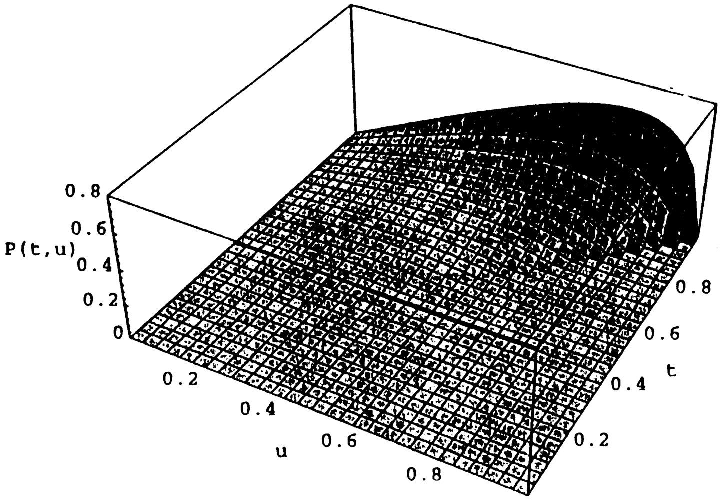

Figure 11 *The surface z = P(t,u), 0 < u < t < 1, illustrating the Pitman efficiency for Vännman's test statistic when the alternative is the Pareto distribution.*

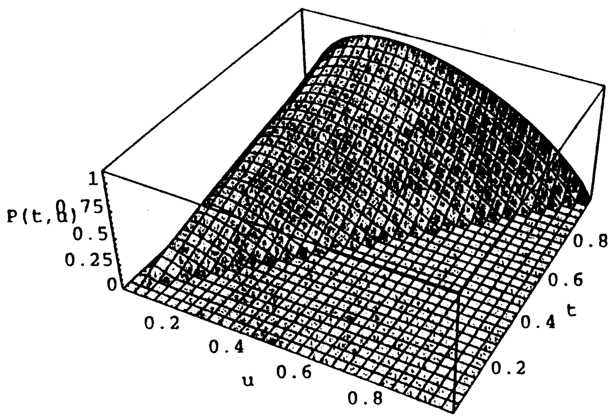

Figure 12 *The surface z = P(t,u), 0 < u < t < 1, illustrating the Pitman efficiency for Vännman's test statistic when the alternative is the Weibull distribution.*

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