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ON A NEW OPERATOR BASED ON A GRILL AND ITS ASSOCIATED TOPOLOGY

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ABSTRACT. In one of our earlier papers a topology $\tau_\mathcal{G}$ and certain associated concepts were studied, where the topology $\tau_\mathcal{G}$ was introduced in terms of an operator $\Phi_\mathcal{G}$, constructed from a grill G on a topological space (X, τ). In this article we define a new operator Γ by using the operator $\Phi_\mathcal{G}$, and undertake an investigation in respect of the operator Γ via-a-vis the operator $\Phi_\mathcal{G}$. We ultimately show that Γ induces another topology $\tau_{\Gamma(\mathcal{B})}$, constructed out of any given base B of the topology τ or $\tau_\mathcal{G}$ and a grill G on the ambient space X.

1. INTRODUCTION AND PREREQUISITES

In 1947, Choquet [2] initiated the idea of grills. Thereafter, in course of the last sixty years, different topological investigations have revealed that grills can be used as a highly technical appliance for manoeuvring many-a-course of study in mathematics, like the problems concerning proximity spaces and certain theories of extensions [7, 1] and so on.

In an earlier paper [4], a topology $\tau_\mathcal{G}$ was introduced in terms of an operator $\Phi_\mathcal{G}$, constructed rather naturally from a grill $\mathcal{G}$ on a topological space (X, τ). In the said paper and also in [5, 6], a detailed description of the topology along with certain other associated ideas and many results thereof are laid down.

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In this paper we endeavour for an investigation along a similar course with the grill-associated topology, but with a new orientation. We introduce here a new operator Γ, defined in terms of the previously introduced operator Φ, as a kind of dual of Φ. We study some basic properties of this new operator, which helps us to derive a few equivalent expressions for the operator Γ and a characterizing condition, in terms of Γ, for the suitability of a topology τ on X for a given grill $\mathcal{G}$ on X. Some equivalent criteria for $\mathcal{G}$ to contain all the non-null members of τ are also established. Finally, we show that from a given grill $\mathcal{G}$ on a space (X, τ) and a given (open) base $\mathcal{B}$ for τ, we can arrive at a new topology $\tau_{\Gamma(\mathcal{B})}$ on X, which is weaker than the given topology τ on X. The deliberation culminates with the interesting result that in terms of this method of construction, all the bases of τ and of the grill-based topology $\tau_{\mathcal{G}}$ give rise to the same topology.

We now recall a few concepts and certain result from [4], to be used in course of the deliberations that follow. We start with the definition of grill, as given by Choquet [2].

Definition 1.1. A collection $\mathcal{G}$ of nonempty subsets of a space X is called a grill on X if
(i) $A \in \mathcal{G}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$,
and (ii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Henceforth by $\mathcal{G}$ and $\mathcal{P}(X)$ we shall denote a grill on a topological space (X, τ) (to be sometimes abbreviated as a 'space X') and the power set of X respectively. For any $x \in X$, the set of all open sets of (X, τ) containing x, will be denoted by $\tau(x)$. The interior and closure of a subset A in a space X are denoted, as usual, by int$A$ and cl$A$ respectively.

Definition 1.2. [4] Let $\mathcal{G}$ be a grill on a topological space (X, τ). A mapping $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$, denoted by $\Phi_{\mathcal{G}}(A, \tau)$ (for $A \in \mathcal{P}(X)$) or $\Phi_{\mathcal{G}}(A)$ or simply by $\Phi(A)$ (when there is no confusion regarding the topology τ and the grill $\mathcal{G}$, being used), is defined by $\Phi_{\mathcal{G}}(A) = \{x \in X : A \cap U \in \mathcal{G}, \text{ for all } U \in \tau(x)\}$.

Result 1.3. [4] Let (X, τ) be a topological space.
(a) If $\mathcal{G}$ is a grill on X, then
(i) $A \subseteq B \subseteq X \Rightarrow \Phi(A) \subseteq \Phi(B)$;
(ii) $A \subseteq X$ and $A \notin \mathcal{G} \Rightarrow \Phi(A) = \varnothing$;

(iii) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$, for any $A, B \subseteq X$;
(iv) $U \in \tau$ and $\tau \setminus \{\varnothing\} \subseteq \mathcal{G} \Rightarrow U \subseteq \Phi(U)$;
(v) $\Phi(A) \setminus \Phi(B) = \Phi(A \setminus B) \setminus \Phi(B)$, for any $A, B \subseteq X$;
(vi) $\Phi(\Phi(A)) \subseteq \Phi(A) = \text{cl}\Phi(A) \subseteq \text{cl}A$ for any $A \subseteq X$;
(vii) $\Phi(A \cup B) = \Phi(A) = \Phi(A \setminus B)$ for any $A, B \subseteq X$ with $B \notin \mathcal{G}$
(b) If $\mathcal{G}_1$, $\mathcal{G}_2$ be two grills on X and $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\Phi_{\mathcal{G}_1}(A) \subseteq \Phi_{\mathcal{G}_2}(A)$, for any $A \subseteq X$.

Result 1.4. [4] Given a grill on a space (X, τ), the map $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$, defined by $\Psi(A) = A \cup \Phi(A)$ (for $A \in \mathcal{P}(X)$), is a Kuratowski closure operator, giving rise to a topology $\tau_{\mathcal{G}}$ on X for which $\mathcal{B}(\mathcal{G}, \tau) = \{V \setminus A : V \in \tau$ and $A \notin \mathcal{G}\}$ is an open base. Moreover, $\tau \subseteq \mathcal{B}(\mathcal{G}, \tau) \subseteq \tau_{\mathcal{G}}$.

Result 1.5. [4] If $\mathcal{G}$ is a grill on a space (X, τ) and $A \subseteq X$ such that $A \subseteq \Phi(A)$, then $\text{cl}A = \tau_{\mathcal{G}}\text{-cl}A = \text{cl}(\Phi(A)) = \Phi(A)$.

2. THE OPERATOR Γ

Here we first define the proposed operator Γ and take up some basic associated results.

Definition 2.1. Let $\mathcal{G}$ be a grill on a topological space (X, τ). We define a map $\Gamma_{\mathcal{G}} : \mathcal{P}(X) \to \mathcal{P}(X)$, given by $\Gamma_{\mathcal{G}}(A) = X \setminus \Phi(X \setminus A)$ for any $A \subseteq X$. We shall simply write $\Gamma(A)$ for $\Gamma_{\mathcal{G}}(A)$, assuming that the grill $\mathcal{G}$ under consideration is understood.

Remark 2.2. It follows from Result 1.3(a)(vi) that $\Gamma(A)$ is open in (X, τ) for any subset A of X. Thus Γ can be treated as a mapping from $\mathcal{P}(X)$ to τ.

Note 2.3. In view of Result 1.3(b) it turns out that for two grills $\mathcal{G}_1$ and $\mathcal{G}_2$ on X, ($\mathcal{G}_1 \subseteq \mathcal{G}_2 \Rightarrow \Gamma_{\mathcal{G}_1}(A) \supseteq \Gamma_{\mathcal{G}_2}(A)$). But for a given grill $\mathcal{G}$ on X, $\Gamma_{\mathcal{G}}$ is increasing in the sense that whenever $A \subseteq B \subseteq X$, then $\Gamma(A) \subseteq \Gamma(B)$. This is again an immediate consequence of Note 1.3(a)(i); however it may so happen that $\Gamma(A) \subseteq \Gamma(B)$ even if $A \not\subseteq B$. The following is an example to justify our contention.

Example 2.4. Consider the topological space (X, τ), where $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{b, c\}\}$ and let $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then $\mathcal{G}$ is a grill on X. Now, $\Phi(\{a\}) = \{a\}$ and $\Phi(\{b\}) = \varnothing$. Then $\Gamma(\{b, c\}) =$

$X \setminus \Phi(\{a\}) = \{b, c\}$ and $\Gamma(\{a, c\}) = X \setminus \Phi(\{b\}) = X$. Thus $\Gamma(\{b, c\}) \subseteq \Gamma(\{a, c\})$ although $\{b, c\} \nsubseteq \{a, c\}$.

Some basic properties concerning the operator Γ are now obtained.

Theorem 2.5. Let $\mathcal{G}$ be a grill on a topological space (X, τ).
(a) If $U \in \tau_{\mathcal{G}}$, then $U \subseteq \Gamma(U)$.
(b) If $A, B \subseteq X$, then $\Gamma(A \cap B) = \Gamma(A) \cap \Gamma(B)$.
(c) If $A \subseteq X$ and $A \notin \mathcal{G}$, then $\Gamma(A) = X \setminus \Phi(X)$.
(d) If $A, B \subseteq X$ with $B \notin \mathcal{G}$, then $\Gamma(A) = \Gamma(A \setminus B) = \Gamma(A \cup B)$.
(e) If $A, B \subseteq X$ with $A \bigtriangleup B \notin \mathcal{G}$, then $\Gamma(A) = \Gamma(B)$ (where $A \bigtriangleup B$ denotes, as usual, the symmetric difference of A and B).

Proof. (a) In fact, $U \in \tau_{\mathcal{G}} \Rightarrow \Phi(X \setminus U) \subseteq X \setminus U$ (by Result 1.4) $\Rightarrow U \subseteq X \setminus \Phi(X \setminus U) = \Gamma(U)$.
(b) $\Gamma(A \cap B) = X \setminus \Phi(X \setminus (A \cap B)) = X \setminus \Phi[(X \setminus A) \cup \Phi(X \setminus B)]$(by Result 1.3(a)(iii)) $= [X \setminus \Phi(X \setminus A)] \cap [X \setminus \Phi(X \setminus B)] = \Gamma(A) \cap \Gamma(B)$.
(c) $\Gamma(A) = X \setminus \Phi(X \setminus A) = X \setminus [\Phi(X \setminus A) \setminus \Phi(A)]$ (by Result 1.3(a)(ii)) $= X \setminus [\Phi(X) \setminus \Phi(A)]$ (by Result 1.3(a)(v)) $= X \setminus \Phi(X)$ (by Result 1.3(a)(ii)).
(d) $\Gamma(A \setminus B) = X \setminus \Phi((X \setminus A) \cup B) = X \setminus [\Phi(X \setminus A) \cup \Phi(B)]$ (by Result 1.3(a)(iii)) $= X \setminus \Phi(X \setminus A)$ (by Result 1.3(a)(ii)) $= \Gamma(A)$.
Again, $\Gamma(A \cup B) = X \setminus \Phi(X \setminus (A \cup B)) = X \setminus \Phi((X \setminus A) \setminus B) = X \setminus \Phi(X \setminus A)$ (by Result 1.3(a)(vii)) $= \Gamma(A)$.
(e) Let $A \Delta B \notin \mathcal{G}$ so that $A \setminus B, B \setminus A \notin \mathcal{G}$. Then by using Result 1.3(a)(vii) we have, $\Gamma(A) = \Gamma((B \setminus (B \setminus A)) \cup (A \setminus B)) = \Gamma(B \setminus (B \setminus A)) = \Gamma(B)$. □

Remark 2.6. From (b) of the above theorem we see that the operator Γ is distributive over finite intersection. That this is not necessarily true for finite union is now shown below. Also, it is shown by the example following the next that for two sets A, B in X, $\Gamma(A) = \Gamma(B)$ may be true even if $A \Delta B \in \mathcal{G}$, i.e., the converse of Theorem 2.5(e) need not hold.

Example 2.7. Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a, b\}, X\}$. Consider $\mathcal{G} = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ which is clearly a grill on the topological space (X, τ). Now, $\Phi(\{a, c\}) = \{a, b, c\} = X = \Phi(\{b, c\})$ and $\Phi(\{c\}) = \varnothing$.

Then $\Gamma(\{a\}) = X \setminus \Phi(\{b,c\}) = \varnothing$, $\Gamma(\{b\}) = X \setminus \Phi(\{a,c\}) = \varnothing$ and $\Gamma(\{a,b\}) = X \setminus \Phi(\{c\}) = X$. Thus $\Gamma(\{a\}) \cup \Gamma(\{b\}) \neq \Gamma(\{a,b\})$.

Example 2.8. Let X be an infinite set with the discrete topology τ (say) on X. Let $\mathcal{G}$ be the grill on X, given by $\mathcal{G} = \{A \subseteq X : A \text{ is infinite}\}$. Then $\Phi(\varnothing) = \Phi(X) = \varnothing$. Thus $\Gamma(X) = \Gamma(\varnothing) = X$, but $X \Delta \varnothing = X \in \mathcal{G}$.

We now derive two equivalent expressions for $\Gamma(A)$, where A is any subset of a space X.

Theorem 2.9. Let $\mathcal{G}$ be a grill on a topological space (X, τ). Then for any $A \subseteq X$,
(a) $\Gamma(A) = \{x \in X : \exists\, U_x \in \tau(x) \text{ such that } U_x \setminus A \notin \mathcal{G}\}$; (b) $\Gamma(A) = \bigcup\{U \in \tau : U \setminus A \notin \mathcal{G}\}$.

Proof. (a) $x \in \Gamma(A) \Leftrightarrow x \notin \Phi(X \setminus A) \Leftrightarrow \exists\, U_x \in \tau(x)$ such that $U_x \setminus A$ $(= U_x \cap (X \setminus A)) \notin \mathcal{G} \Leftrightarrow x \in R.H.S.$
(b) Let $A^* = \bigcup\{U \in \tau : U \setminus A \notin \mathcal{G}\}$. Now, $x \in A^* \Rightarrow \exists\, U \in \tau$ with $x \in U$ such that $U \setminus A \notin \mathcal{G} \Rightarrow \exists\, U \in \tau(x)$ such that $U \setminus A \notin \mathcal{G}$. Thus by (a) above, $x \in \Gamma(A)$.
From the expression of $\Gamma(A)$ in (a), it is clear that $\Gamma(A) \subseteq A^*$. □

Remark 2.10. Let $\mathcal{G} = \mathcal{P}(X) \setminus \{\varnothing\}$, then by Theorem 2.9(b),
$\Gamma(A) = \bigcup\{U \in \tau : U \setminus A = \varnothing\} = \bigcup\{U \in \tau : U \subseteq A\} = \text{int}A$, for any space (X, τ).

Corollary 2.11. Let $\mathcal{G}$ be a grill on a topological space (X, τ). Then for any $A \subseteq X$, $A \cap \Gamma(A) = \tau_{\mathcal{G}}\text{-int}A$.

Proof. We have, $x \in A \cap \Gamma(A) \Rightarrow x \in A$ and $x \in \Gamma(A) \Rightarrow x \in A$ and $\exists\, U_x \in \tau(x)$ such that $U_x \setminus A \notin \mathcal{G}$ (by Theorem 2.9(a)) $\Rightarrow U_x \setminus (U_x \setminus A)$ is a basic $\tau_{\mathcal{G}}$-open neighbourhood of x such that $U_x \setminus (U_x \setminus A) \subseteq A \Rightarrow x \in \tau_{\mathcal{G}}\text{-int}A$. Now, $x \in \tau_{\mathcal{G}}\text{-int}A \Rightarrow x \in A$. Again, $x \in \tau_{\mathcal{G}}\text{-int } A \Rightarrow \exists$ a basic $\tau_{\mathcal{G}}$-open neighbourhood $V_x \setminus B$ of x, where $V_x \in \tau$ and $B \notin \mathcal{G}$, such that $x \in V_x \setminus B \subseteq A$ $\Rightarrow V_x \setminus A \subseteq B$ and $V_x \setminus A \notin \mathcal{G}$ (as $B \notin \mathcal{G}$). So by Theorem 2.9(a), $x \in \Gamma(A)$. Thus $x \in A \cap \Gamma(A)$. So $A \cap \Gamma(A) = \tau_{\mathcal{G}}\text{-int}A$. □

As a consequence of Theorem 2.9, we can have yet another expression for $\Gamma(A)$ when A is an open set.

Theorem 2.12. Let $\mathcal{G}$ be a grill on a space (X, τ). Then for any open set A of X, $\Gamma(A) = \bigcup\{U \in \tau : U \Delta A \notin \mathcal{G}\}$.

Proof. Let $A^* = \bigcup\{U \in \tau : U\Delta A \notin \mathcal{G}\}$. Then by Theorem 2.9(b), $A^* \subseteq \Gamma(A)$. Now, $x \in \Gamma(A) \Rightarrow \exists\ U \in \tau(x)$ such that $U \setminus A \notin \mathcal{G}$ (by Theorem 2.9(a)). Let $V = U \cup A \in \tau$. Then $V \Delta A = U \setminus A \notin \mathcal{G}$ and $x \in V \in \tau$. Thus $x \in A^*$. □

From the results so far, we arrive at the following simple and alternative description of the topology $\tau_{\mathcal{G}}$ in terms of our introduced operator.

Theorem 2.13. Let $\mathcal{G}$ be a grill on a topological space (X, τ). Then $\tau_{\mathcal{G}} = \{A \subseteq X : A \subseteq \Gamma(A)\}$.

Proof. Let $\sigma = \{A \subseteq X : A \subseteq \Gamma(A)\}$. We shall first show that σ is a topology on X. In fact, $\varnothing \subseteq \Gamma(\varnothing) \Rightarrow \varnothing \in \sigma$. $\Gamma(X) = X \setminus \Phi(X \setminus X) = X \setminus \Phi(\varnothing) = X \setminus \varnothing = X \Rightarrow X \in \sigma$.
Now, $A_1, A_2 \in \sigma \Rightarrow A_1 \subseteq \Gamma(A_1)$ and $A_2 \subseteq \Gamma(A_2) \Rightarrow A_1 \cap A_2 \subseteq \Gamma(A_1) \cap \Gamma(A_2)$ $= \Gamma(A_1 \cap A_2)$ (by Theorem 2.5(b)). Again, $\{A_\alpha : \alpha \in \Lambda\} \in \sigma \Rightarrow A_\alpha \subseteq \Gamma(A_\alpha)$ for each $\alpha \in \Lambda \Rightarrow A_\alpha \subseteq \Gamma(\cup_{\alpha\in\Lambda} A_\alpha)$ for each $\alpha \in \Lambda$ (by Note 2.3) $\Rightarrow \cup_{\alpha\in\Lambda} A_\alpha \subseteq \Gamma(\cup_{\alpha\in\Lambda} A_\alpha) \Rightarrow \cup_{\alpha\in\Lambda} A_\alpha \in \sigma$.
We shall now show that $\sigma = \tau_{\mathcal{G}}$. Indeed, $U \in \tau_{\mathcal{G}} \Rightarrow U \subseteq \Gamma(U)$ (by Theorem 2.5(a))$\Rightarrow U \in \sigma$.
Conversely, $A \in \sigma \Rightarrow A \subseteq \Gamma(A) \Rightarrow A = A \cap \Gamma(A) = \tau_{\mathcal{G}}\text{-int}A$ (by Corollary 2.11) $\Rightarrow A \in \tau_{\mathcal{G}}$. □

3. CERTAIN CONDITIONS IN TERMS OF THE OPERATOR Γ

In [4] a condition, in terms of the topology of a space X and a grill thereon, was formulated. It was observed that such a condition, when imposed on a grill $\mathcal{G}$, makes it in some sense more compatible with the topology of the space and the induced topology $\tau_{\mathcal{G}}$ more well behaved and suitable for application. In fact, an important topological result was also achieved by application of this so called suitability condition. The said condition, as proposed in [4], goes as follows.

Definition 3.1. Let $\mathcal{G}$ be a grill on a topological space (X,τ). Then τ is said to be suitable for the grill $\mathcal{G}$ if for all $A \subseteq X$, $A \setminus \Phi(A) \notin \mathcal{G}$.

To facilitate our intended deliberation, let us recall the following equivalent descriptions of the above concept as found in [4]:

Theorem 3.2. For a grill $\mathcal{G}$ on a space (X,τ), the following are equivalent:
(a) τ is suitable for the grill $\mathcal{G}$.
(b) For any $\tau_{\mathcal{G}}$-closed subset A of X, $A \setminus \Phi(A) \notin \mathcal{G}$.
(c) Whenever for any $A \subseteq X$ and each $x \in A$ there corresponds some $U_x \in \tau(x)$ with $U_x \cap A \notin \mathcal{G}$, it follows that $A \notin \mathcal{G}$.
(d) $A \subseteq X$ and $A \cap \Phi(A) = \varnothing \;\Rightarrow\; A \notin \mathcal{G}$.

It is now our turn to derive, in terms of the operator Γ, a characterizing condition for a topology τ to be suitable for a grill $\mathcal{G}$ on a space X.

Theorem 3.3. Let (X,τ) be a topological space and $\mathcal{G}$ a grill on X. Then τ is suitable for $\mathcal{G}$ iff $\Gamma(A) \setminus A \notin \mathcal{G}$ for any $A \subseteq X$.

Proof. Let τ be suitable for $\mathcal{G}$ and $A \subseteq X$. We first observe that $x \in \Gamma(A) \setminus A$ iff $x \in \Gamma(A)$ and $x \notin A$ iff there exists $U_x \in \tau(x)$ such that $x \in U_x \setminus A \notin \mathcal{G}$. Thus to each $x \in \Gamma(A) \setminus A$, $\exists\, U_x \in \tau(x)$ such that $U_x \cap (\Gamma(A) \setminus A) \notin \mathcal{G}$ (as $\Gamma(A) \setminus A \subseteq X \setminus A$). As τ is suitable for $\mathcal{G}$, we have $\Gamma(A) \setminus A \notin \mathcal{G}$ (by Theorem 3.2).Conversely, let $A \subseteq X$ and further suppose that to each $x \in A$ there corresponds some $U_x \in \tau(x)$ with $U_x \cap A \notin \mathcal{G}$. We need to show by virtue of Theorem 3.2 that $A \notin \mathcal{G}$. Now, by Theorem 2.9(a) we have, $\Gamma(X \setminus A) = \{x \in X : \exists\, U_x \in \tau(x) \text{ such that } U_x \setminus (X \setminus A) \notin \mathcal{G}\} = \{x \in X : \exists\, U_x \in \tau(x)$ such that $U_x \cap A \notin \mathcal{G}\}$. Thus $A \subseteq \Gamma(X \setminus A)$ and hence $A = \Gamma(X \setminus A) \cap A = \Gamma(X \setminus A) \setminus (X \setminus A) \notin \mathcal{G}$ (by hypothesis). □

Corollary 3.4. Let $\mathcal{G}$ be a grill on a topological space (X,τ) such that τ is suitable for $\mathcal{G}$. Then Γ is an idempotent operator i.e., for any $A \subseteq X$, $\Gamma(\Gamma(A)) = \Gamma(A)$.

Proof. By Remark 2.2, $\Gamma(A) \in \tau$ for any $A \subseteq X$, and so $\Gamma(A) \in \tau_{\mathcal{G}}$ (by Theorem 1.4). Hence by Theorem 2.5(a), $\Gamma(A) \subseteq \Gamma(\Gamma(A))$ for any $A \subseteq X$. Again τ is suitable for $\mathcal{G}$, so $\Gamma(A) \subseteq (A \cup B)$ for some $B \notin \mathcal{G}$ (by Theorem 3.3). Thus $\Gamma(\Gamma(A)) \subseteq \Gamma(A \cup B)$ (by Note 2.3) $= \Gamma(A)$ (by Theorem 2.5(d)). □

The converse of the above corollary is false as is shown by the next example.

Example 3.5. Consider X to be an infinite set with the discrete topology τ (say). Let $\mathcal{G} = \{A \subseteq X : A \text{ is infinite}\}$. Then for any $A \subseteq X$, $\Phi(A) = \varnothing$ and hence $\Gamma(A) = X \setminus \Phi(X \setminus A) = X$. Thus $\Gamma(\Gamma(A)) = \Gamma(A)$, for all $A \subseteq X$. But τ is not suitable for $\mathcal{G}$. For, $X \cap \Phi(X) = X \cap \varnothing = \varnothing$ but $X \in \mathcal{G}$ (see Theorem 3.2).

Corollary 3.6. Let $\mathcal{G}$ be a grill on a space on a space (X, τ) such that τ is suitable for $\mathcal{G}$. Let $A \subseteq X$ and U be a non-null open set such that $U \subseteq \Phi(A) \cap \Gamma(A)$. Then $U \setminus A \notin \mathcal{G}$ and $U \cap A \in \mathcal{G}$.

Proof. $U \subseteq \Phi(A) \cap \Gamma(A) \Rightarrow U \subseteq \Gamma(A) \Rightarrow U \setminus A \subseteq \Gamma(A) \setminus A \notin \mathcal{G}$ (by Theorem 3.3) $\Rightarrow U \setminus A \notin \mathcal{G}$.
Again, $U \subseteq \Phi(A)$ and $U \neq \varnothing \Rightarrow U \cap A \in \mathcal{G}$ (using the definition of $\Phi(A)$). □

In certain results in [4, 5, 6] (and also in Theorem 4.2 later), it is assumed that the condition that the given grill contains all the non-null open sets. We now obtain several necessary and sufficient conditions for such an assertion to hold.

Theorem 3.7. Let $\mathcal{G}$ be a grill on a topological space (X, τ). Then the following are equivalent:
(a) $\tau \setminus \{\varnothing\} \subseteq \mathcal{G}$.
(b) $\Gamma(\varnothing) = \varnothing$.
(c) If $A (\subseteq X)$ is closed, then $\Gamma(A) \setminus A = \varnothing$.
(d) If $A \subseteq X$, then $\text{intcl}A = \Gamma(\text{intcl}A)$.
(e) If A is regular open in X, then $A = \Gamma(A)$.
(f) If $U \in \tau$, then $\Gamma(U) \subseteq \text{intcl}U \subseteq \Phi(U)$.

Proof. **(a) ⇒ (b):** $\Gamma(\varnothing) = \bigcup\{U \in \tau : U \setminus \varnothing \notin \mathcal{G}\}$ (by Theorem 2.9(b)) $= \bigcup\{U \in \tau : U \notin \mathcal{G}\} = \varnothing$ (by hypothesis).

(b) $\Rightarrow$ (c): $x \in \Gamma(A)\backslash A \Rightarrow \exists\, U_x \in \tau(x)$ such that $x \in U_x \backslash A \notin \mathcal{G}$. Thus noting that A is closed, we obtain $x \in U_x \setminus A \in \{U \in \tau : U \notin \mathcal{G}\}$, a contradiction to $\Gamma(\varnothing) = \varnothing$.

(c) $\Rightarrow$ (d): Since (intcl$A$) is open, we have by Theorem 2.5(a) that intcl$A \subseteq \Gamma($intcl$A)$. Again, $\Gamma($cl$A) \subseteq$ cl$A$ (by (c)) $\Rightarrow \Gamma($cl$A) =$ int$(\Gamma($cl$A))$(by Remark 2.2) $\subseteq$ intcl$A \Rightarrow \Gamma$ (intcl$A$) $\subseteq \Gamma$ (cl$A$) (see Note 2.3) $\subseteq$ intcl$A$.
Thus intcl$A = \Gamma($intcl$A)$.

(d) $\Rightarrow$ (e): It is trivial.

(e) $\Rightarrow$ (f): Let $U \in \tau$. Now $\varnothing = \Gamma(\varnothing)$ (by (e)) $= \bigcup\{V \in \tau : V \notin \mathcal{G}\}$ (by Theorem 2.9(b)) and we obtain, $\tau \setminus \{\varnothing\} \subseteq \mathcal{G}$. Then by Result 1.3(a)(iv), $U \subseteq \Phi(U)$ and hence by Result 1.5 we have, $\Phi(U) =$ cl$U$. Now, $U \subseteq$ intcl$U \subseteq$ cl$U = \Phi(U) \Rightarrow \Gamma(U) \subseteq \Gamma($intcl$U) =$ intcl$U$ (by (e), as intcl$U$ is regular open) $\subseteq \Phi(U)$.

(f) $\Rightarrow$ (a): If $U \in \tau \setminus \mathcal{G}$, then by Theorem 2.5(a), $U \subseteq \Gamma(U) \subseteq \Phi(U)$ (by (f)) $= \varnothing$ (by Result 1.3(a)(ii)), i.e., $U = \varnothing$. □

Theorem 3.8. Let $\mathcal{G}$ be a grill on a topological space (X, τ) and $\mathcal{G}_\delta$ be the grill on X, given by $\mathcal{G}_\delta = \{A \subseteq X :$ intcl$A \neq \varnothing\}$.
(a) Let $\mathcal{G}_\delta \supseteq \mathcal{G}$. Then $A = \Gamma(A) \Rightarrow A$ is regular open.
(b) Let $\mathcal{G}_\delta \supseteq \mathcal{G}$ and $\tau \setminus \{\varnothing\} \subseteq \mathcal{G}$. Then $A \subseteq X \Rightarrow \Gamma(A)$ is regular open.
(c) Let $\mathcal{G}_\delta \supseteq \mathcal{G}$ and τ be suitable for $\mathcal{G}$. Then $A \subseteq X \Rightarrow \Gamma(A)$ is regular open.

Proof. (a) Let $A \subseteq X$ be such that $A = \Gamma(A)$. Then A is open (by Remark 2.2) and so $A \subseteq$ intcl$A$. Again, $x \in$ intcl$A \Rightarrow \exists\, U_x \in \tau(x)$ such that $U_x \subseteq$ cl$A \Rightarrow U_x \backslash A \subseteq$ cl$A \backslash A \notin \mathcal{G}_\delta$ (as intcl(cl$A\backslash A$) = int(cl$A\backslash A$) $= \varnothing \Rightarrow U_x \backslash A \notin \mathcal{G}$ where $U_x \in \tau(x) \Rightarrow x \in \Gamma(A)$ (by Theorem 2.9(a)). So intcl$A \subseteq \Gamma(A) = A$. Thus A is regular open.
(b) Follows from (a) and Theorem 3.7(e).

(c) Follows from (a) and Corollary 3.4. □

4. TOPOLOGY INDUCED BY Γ

We show now that the operator Γ induces yet another topology $\tau_{\Gamma(\mathcal{B})}$, constructed rather naturally out of any given base $\mathcal{B}$ of the topology τ or $\tau_{\mathcal{G}}$ and a grill $\mathcal{G}$ on the ambient space X. Also it is seen that the topology $\tau_{\Gamma(\mathcal{B})}$ is weaker than the topology τ on the given space X, whereas the topology $\tau_{\mathcal{G}}$ is known [4] to be finer than τ.

We observe in view of in view of Theorem 2.5 and Remark 2.2 that if $\mathcal{B}$ is base for some topology τ on X, then $\Gamma(\mathcal{B}) = \{\Gamma(B) : B \in \mathcal{B}\}$ is also base for some topology on X, which is weaker than the given topology. Let us denote this topology by $\tau_{\Gamma(\mathcal{B})}$ and call it the Γ-topology determined by $\mathcal{B}$.

We first show that starting from any base $\mathcal{B}$ of the topology τ on X (and hence from τ, in particular) we arrive at the same topology $\tau_{\Gamma(\mathcal{B})}$, i.e., the latter topology is unique irrespective of the chosen base of a topology τ.

Theorem 4.1. Let $\mathcal{G}$ be a grill on a topological space (X, τ). Suppose that $\mathcal{B}$ is a base for τ. Then $\tau_{\Gamma(\tau)} = \tau_{\Gamma(\mathcal{B})}$.

Proof. We shall first show that $\tau_{\Gamma(\tau)} \subseteq \tau_{\Gamma(\mathcal{B})}$. Let $U \in \tau$ and $x \in \Gamma(U)$. Then $U = \cup_\alpha B_\alpha$ where $B_\alpha \in \mathcal{B}$ for each α.
Case-(i): $x \in U$. Then $\exists\ B_\beta \in \mathcal{B}$ such that $x \in B_\beta \subseteq U$. Hence $x \in \Gamma(B_\beta) \subseteq \Gamma(U)$.
Case-(ii): $x \notin U$. Then $\exists$ some $B_x \in \mathcal{B}$ such that $x \in B_x \setminus U \notin \mathcal{G}$ (as $x \in \Gamma(U)$). Also, $x \in B_x \subseteq \Gamma(U)$. In fact, $(U \cup B_x)\ \Delta\ U \notin \mathcal{G}$, so by Theorem 2.5((a),(e)) and Note 2.3, $x \in B_x \subseteq \Gamma(B_x) \subseteq \Gamma(U \cup B_x) = \Gamma(U)$. Thus in either of the cases, there exists some $B \in \mathcal{B}$ such that $x \in \Gamma(B) \subseteq \Gamma(U)$, and hence $\tau_{\Gamma(\tau)} \subseteq \tau_{\Gamma(\mathcal{B})}$.
Now, $\tau_{\Gamma(\mathcal{B})} \subseteq \tau_{\Gamma(\tau)}$ follows from the fact that $\mathcal{B} \subseteq \tau$. $\square$

In is known [3] that the set of all regular open sets in a space (X, τ) forms a base for a topology τ_s, called the semiregularization topology on X, such that $\tau_s \subseteq \tau$. Noting the fact that $\tau_{\Gamma(\tau)} \subseteq \tau$ also holds, we now try to associate the topologies τ_s and $\tau_{\Gamma(\tau)}$ by choosing the grill $\mathcal{G}$ suitably.

Theorem 4.2. Let $\mathcal{G}$ be a grill on a topological space (X, τ) and $\mathcal{G}_\delta$ be the grill on X given by $\mathcal{G}_\delta = \{\ A \subseteq X : \text{intcl}\ A \neq \varnothing\}$. Then
(a) $\tau \setminus \{\varnothing\} \subseteq \mathcal{G} \Rightarrow \tau_s \subseteq \tau_{\Gamma(\tau)} \subseteq \tau$;

(b) $\tau \setminus \{\varnothing\} \subseteq \mathcal{G}$ and $\mathcal{G} \subseteq \mathcal{G}_\delta \Rightarrow \tau_s = \tau_{\Gamma(\tau)}$;
(c) τ is suitable for $\mathcal{G}$ and $\mathcal{G} \subseteq \mathcal{G}_\delta \Rightarrow \tau_{\Gamma(\mathcal{P}(X))} \subseteq \tau_s$.

Proof. (a) Follows from Theorem 3.7(e).

(b) $\tau_s \subseteq \tau_{\Gamma(\tau)}$ follows from (a).Conversely, let $U \in \tau$. Then by Theorems 3.7(f) and 2.5(a), it follows that $U \subseteq \Gamma(U) \subseteq \text{intcl } U \subseteq \Phi(U) = \text{cl}U$. Then $\Gamma(U) \setminus U \subseteq \text{cl}U \setminus U \notin \mathcal{G}_\delta$. Thus $\Gamma(U) \setminus U \notin \mathcal{G}$. Let $V = \Gamma(U) \setminus U$, then $\Gamma(U) = V \cup U$ (as $U \subseteq \Gamma(U)$), where $V \notin \mathcal{G}$. So $\Gamma(\Gamma(U)) = \Gamma(V \cup U) = \Gamma(U)$ (by Theorem 2.5(c)). Thus $\Gamma(U)$ is regular open (By Theorem 3.8(c)). Hence $\tau_{\Gamma(\tau)} \subseteq \tau_s$.

(c) Follows from Theorem 3.8(c). □

As the final result of the present discussion, we show (in Corollary 4.4) that even any base of the larger topology $\tau_\mathcal{G}$ induces the same topology as induced by any base of τ. The following theorem leads us half the way towards our contention.

Theorem 4.3. Let $\mathcal{G}$ be a grill on a topological space (X, τ), and let $\mathcal{B}^* = \{U \setminus A\colon U \in \tau \text{ and } A \notin \mathcal{G}\}$. Then $\tau_{\Gamma(\mathcal{B}^*)} = \tau_{\Gamma(\tau)}$.

Proof. We first note that $\mathcal{B}^*$ is a base for $\tau_\mathcal{G}$ (see Result 1.4). Now, $\tau \subseteq \mathcal{B}^* \Rightarrow \tau_{\Gamma(\tau)} \subseteq \tau_{\Gamma(\mathcal{B}^*)}$. Let $U \setminus A \in \mathcal{B}^*$, where $U \in \tau$ and $A \notin \mathcal{G}$. then $\Gamma(U \setminus A) = \Gamma(U)$ (by Theorem 2.5(d)), where $\Gamma(U)$ is a basic open set of $\tau_{\Gamma(\tau)}$. Thus $\tau_{\Gamma(\mathcal{B}^*)} \subseteq \tau_{\Gamma(\tau)}$. □

In view of Theorems 4.1 and 4.3, it ultimately follows

Corollary 4.4. For any grill $\mathcal{G}$ on a space (X, τ), $\tau_{\Gamma(\tau)} = \tau_{\Gamma(\tau_\mathcal{G})} = \tau_{\Gamma(\mathcal{B})}$, where $\mathcal{B}$ is any base for τ or $\tau_\mathcal{G}$.

Remark 4.5. We have seen above that although $\mathcal{B} \subseteq \tau \subseteq \tau_\mathcal{G}$ (where $\mathcal{B}$ is any base of τ), the corresponding Γ-topologies are the same. As $\tau_\mathcal{G} \subseteq (\tau_\mathcal{G})_\mathcal{G} \subseteq ((\tau_\mathcal{G})_\mathcal{G})_\mathcal{G}$ etc., it may be asked whether the corresponding Γ-topologies coincide. That this is indeed the case is evident from the following theorem.

Theorem 4.6. For any grill $\mathcal{G}$ on a topological space (X, τ), $(\tau_{\mathcal{G}})_{\mathcal{G}} = (\tau_{\mathcal{G}})$.

Proof. In view of Result 1.4 we have $\tau_{\mathcal{G}} \subseteq (\tau_{\mathcal{G}})_{\mathcal{G}}$.
Conversely, let $V \setminus A$ be a $(\tau_{\mathcal{G}})_{\mathcal{G}}$ -basic open set, where $V \in \tau_{\mathcal{G}}$ and $A \notin \mathcal{G}$, and $x \in V \setminus A$. Again, $x \in V \in \tau_{\mathcal{G}} \Rightarrow$ there exist $U \in \tau$ and $B \notin \mathcal{G}$ such that $x \in U \setminus B \subseteq V$. Then $x \in (U \setminus B) \setminus A = U \setminus (B \cup A) \subseteq V \setminus A$. As $A \cup B \notin \mathcal{G}$ and $U \in \tau$, $U \setminus (A \cup B)$ is a basic open set of $\tau_{\mathcal{G}}$ (by Result 1.4). Thus $V \setminus A \in \tau_{\mathcal{G}}$. □

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