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THE TRANSVERSAL VECTOR BUNDLE OF A LIGHTLIKE FINSLER SUBMANIFOLD

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ABSTRACT. Let $F^m = (M, g)$ be a Finsler submanifold of a pseudo-Finsler manifold $F^{m+n} = (\tilde{M}, \tilde{g})$. In case F^m is non-degenerate, the geometry of F^m in F^{m+n} is developed by using the normal Finsler bundle $VTM^{\perp}$, which is the complementary orthogonal vector bundle to VTM in $VT\tilde{M}_{|TM}$ (cf. Bejancu [2]). Contrary to this case, when F^m is lightlike (degenerate), the vector bundles VTM and $VTM^{\perp}$ are not complementary anymore. Therefore, in order to develop a theory of lightlike Finsler submanifolds we have to replace $VTM^{\perp}$ by a transversal vector bundle to VTM in $VT\tilde{M}_{|TM}$. In the present paper we construct a canonical transversal vector bundle of F^m in F^{m+n} with respect to some screen vector bundles over TM.

1. PRELIMINARIES

The theory of Finsler submanifolds is one of the most difficult theory in Finsler geometry. In case the ambient manifold is a Finsler manifold whose Finsler metric is positive definite, several important results have been obtained, some of them being brought together in separate chapters of monographs of Rund [6] and Bejancu [2]. However, in contrast to this case, until now only few results are known in case the Finsler metric on the ambient manifold is indefinite (see Bejancu [3]).

Let $F^{m+n} = (\tilde{M}, \tilde{F})$ be a Finsler manifold, where $\tilde{M}$ is a real $(m+n)$-dimensional manifold and $\tilde{F}$ is the fundamental function of F^{m+n} (cf. Matsumoto [4]). Usually, $\tilde{F}$ is not presumed to be smooth on the whole $T\tilde{M}$ but on an open subset of $T\tilde{M}$.

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Suppose that M is a real m-dimensional submanifold of $\tilde{M}$ locally given by the equations

$$x^A = x^A(u_1, \dots, u^m); \quad \text{rank } [X_\alpha^A] = m,$$

where $X_\alpha^A = \partial x^A / \partial u^\alpha$, $A \in \{1, \dots, m+n\}$ and $\alpha \in \{1, \dots, m\}$. Denote by i the immersion of M in $\tilde{M}$ and consider the differential mapping di of TM in $T\tilde{M}$. Locally, a point of TM with coordinates (u^α, v^α) is carried by di into a point of $T\tilde{M}$ with coordinates $(x^A(u), y^A(u,v))$, where

$$y^A(u,v) = X_\alpha^A v^\alpha.$$

Throughout the paper the following range for indices is used: $A, B, C, \dots \in \{1, ..., m+n\}; \alpha, \beta, \gamma, \dots \in \{1, ..., m\}; i, j, k, ... \in \{1, ..., r\}$. Also, we make use of Einstein convention, that is repeated indices with one upper and one lower index denotes summation over their range. By $\mathcal{F}(TM)$ and $\Gamma(E)$ we denote the algebra of smooth functions on TM and the $\mathcal{F}(TM)$-module of smooth sections of a vector bundle E over TM . By $\perp$ and $\oplus$ we denote the orthogonal subspace and the direct sum (but not orthogonal) of vector bundles, respectively.

The natural fields of frames $\{\partial/\partial u^\alpha, \partial/\partial v^\alpha\}$ and $\{\partial/\partial x^A, \partial/\partial y^A\}$ on manifolds TM and $T\tilde{M}$ respectively are related by

$$\frac{\partial}{\partial u^\alpha} = X_\alpha^A \frac{\partial}{\partial x^A} + X_{\alpha\beta}^A v^\beta \frac{\partial}{\partial y^A}; \quad X_{\alpha\beta}^A = \frac{\partial^2 x^A}{\partial u^\alpha \partial u^\beta},$$

and

$$\frac{\partial}{\partial v^\alpha} = X_\alpha^A \frac{\partial}{\partial y^A}.$$

As $\{\partial/\partial y^A\}$ and $\{\partial/\partial v^\alpha\}$ are local basis of the vertical vector bundles $VT\tilde{M}_{|TM}$ and VTM of $\tilde{M}$ and M respectively, from the last equalities we conclude that VTM is a vector subbundle of $VT\tilde{M}_{|TM}$.

Next we define

$$\tilde{g}_{AB}(x,y) = \frac{1}{2} \frac{\partial^2 \tilde{F}^2}{\partial y^A \partial y^B},$$

which can be thought of as the local components of a symmetric Finsler tensor field $\tilde{g}$ of type (0,2) on F^{m+n}, i.e., $\tilde{g}$ is a symmetric $\mathcal{F}(T\tilde{M})$-bilinear mapping

$$\tilde{g} : \Gamma(VT\tilde{M}) \times \Gamma(VT\tilde{M}) \to \mathcal{F}(T\tilde{M}),$$

such that

$$\tilde{g}\left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B}\right) = \tilde{g}_{AB}(x, y).$$

Suppose $\tilde{g}$ is non-degenerate on $T\tilde{M}$, that is, rank $[\tilde{g}_{AB}(x, y)] = m + n$ on any coordinate neighborhood of $T\tilde{M}$. Clearly at any point (x, y) of $T\tilde{M}$, $\tilde{g}(x, y)$ is a pseudo-Euclidean metric on the fiber $VT\tilde{M}_{(x,y)}$. Denote by q the index of $\tilde{g}(x, y)$, i.e., q is the dimension of the largest subspace of $VT\tilde{M}_{(x,y)}$ on which $\tilde{g}(x, y)$ is negative definite. We further suppose $\tilde{g}$ is of constant index q on $T\tilde{M}$. In this case g is said to be a *pseudo-Finsler metric* and $F^{m+n} = (\tilde{M}, \tilde{g})$ is called a *pseudo-Finsler manifold* or *indefinite Finsler manifold* (cf. Beem [1]). Actually, $\tilde{g}$ is nothing but a pseudo- Riemannian metric on the vertical vector bundle $VT\tilde{M}$. Hence $\tilde{g}$ induces a symmetric Finsler tensor field g of type (0,2) on F^m locally given by its components

$$g_{\alpha\beta}(u, v) = \tilde{g}\left(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}\right) = X^A_\alpha X^B_\beta \tilde{g}_{AB}(x(u), y(u, v)).$$

Now at any point $(u, v) \in TM$ we consider

$$\begin{aligned} VTM^\perp_{(u,v)} &= \{\tilde{X}_{(u,v)} \in VT\tilde{M}_{(u,v)}; \tilde{g}(u, v)(\tilde{X}_{(u,v)}, Y_{(u,v)}) = 0, \\ &\qquad \forall\, Y_{(u,v)} \in VTM_{(u,v)}\} \end{aligned}$$

In case the induced tensor g is non-degenerate on VTM at each point $(u, v) \in TM$, both $VTM_{(u,v)}$ and $VTM^\perp_{(u,v)}$ are complementary orthogonal non-degenerate subspaces of $VT\tilde{M}_{(u,v)}$.

Next, suppose g is degenerate at $(u, v) \in TM$, that is there exists a non-zero Finsler vector $\xi_{(u,v)} \in VTM_{(u,v)}$ such that

$$g_{(u,v)}(\xi_{(u,v)}, X_{(u,v)}) = 0, \quad \forall\, X_{(u,v)} \in VTM_{(u,v)}.$$

In this case $VTM_{(u,v)}$ and $VTM^{\perp}_{(u,v)}$ are degenerate orthogonal but not anymore complementary subspaces. Indeed $\xi_{(u,v)}$ lies in the subspace

$$Rad\, VTM_{(u,v)} = VTM_{(u,v)} \cap VTM^{\perp}_{(u,v)}.$$

As the dimension of $Rad\, VTM_{(u,v)}$ depends on the point (u, v), we give the following definition. The Finsler submanifold F^m of F^{m+n} is said to be *r-lightlike* (*r-degenerate*, *r-isotropic*) *Finsler submanifold* if the mapping

$$Rad\, VTM : (u, v) \in TM \rightarrow Rad\, VTM_{(u,v)}.$$

defines a vector subbundle of VTM of rank $r > 0$. Then we call $Rad\ VTM$ the *radical* (*lightlike*, *null*, *isotropic*) *vector subbundle* of VTM. Also, we say that g is a *r-lightlike* (*r-degenerate*, *r-isotropic*) *Finsler metric* on F^m. Clearly, lightlike Finsler submanifolds do not exist in Finsler manifolds with positive (negative) definite Finsler metric. For this reason, from now on, F^{m+n} *will represent a pseudo-Finsler manifold of index*

$$0 < q < m + n.$$

It is important to note that the condition from the definition of a lightlike Finsler submanifold has a local character. More precisely, it is easy to check the following result.

Theorem 1. *Let F^m be a Finsler submanifold of F^{m+n}. Then the following assertions are equivalent:*

(i) F^m is a r-lightlike Finsler submanifold of F^{m+n}.
(ii) On each coordinate neighborhood $\mathcal{U} \subset TM$ the induced tensor g by $\tilde{g}$ has a constant rank $m - r, 0 < r < m$.
(iii) On each coordinate neighborhood $\mathcal{U} \subset TM$ the mapping

$$Rad\, VTM_{|\mathcal{U}} : (u, v) \in \mathcal{U} \rightarrow Rad\, VTM_{(u,v)},$$

defines on $\mathcal{U}$ a vector subbundle of rank $r > 0$ of VTM.

As in case on Finsler non-degenerate submanifolds we consider the vector bundle

$$VTM^{\perp} = \bigcup_{(u,v)\in TM} VTM^{\perp}_{(u,v)}.$$

But contrary to that case, if F^m is r-lightlike, $VTM^\perp$ is not complementary to VTM in $V\tilde{TM}_{|TM}$ since

$$Rad\, VTM = VTM \cap VTM^\perp,$$

is now a vector subbundle of VTM of rank $r > 0$. That is why, we have to look for a certain complementary vector bundle to VTM in $V\tilde{TM}_{|TM}$ which is going to replace the Finsler normal bundle from theory of non-degenerate Finsler submanifolds. In the next section we shall construct such a vector bundle.

THE CONSTRUCTION OF THE TRANSVERSAL VECTOR BUNDLE OF A LIGHTLIKE FINSLER SUBMANIFOLD

In order to construct the transversal vector bundle of a lightlike Finsler submanifold we examine four cases with respect to the dimension and codimension of F^m and rank of the radical vector bundle.

Case I. $(0, < r < \min\{m, n\})$. In this case we consider the complementary vector bundles $SVTM$ and $sVTM$ to $Rad\, VTM$ in VTM and $VTM^\perp$ respectively. First we prove the following result.

Proposition 1. *The vector bundles $SVTM$ and $sVTM$ are orthogonal to $Rad\, VTM$ and non-degenerate.*

Proof. By the definition $Rad\, VTM$ is orthogonal to VTM and $VTM^\perp$. Thus in particular it follows that it is orthogonal to $SVTM$ and $sVTM$. Now suppose there exist a point $(u, v) \in TM$ and a non-zero vector $\xi_{(u,v)} \in SVTM_{(u,v)}$ such that

$$g_{(u,v)}(\xi_{(u,v)}, X_{(u,v)}) = 0, \quad \forall\, X_{(u,v)} \in SVTM_{(u,v)}.$$

As we also have

$$g_{(u,v)}(\xi_{(u,v)}, X_{(u,v)}) = 0, \quad \forall\, Y_{(u,v)} \in Rad\, VTM_{(u,v)},$$

we deduce that $\xi_{(u,v)}$ belongs to $Rad\, VTM_{(u,v)}$, which is a contradiction. Thus $SVTM$ is a non-degenerate vector subbundle of VTM. In a similar way it follows that $sVTM$ is a non-degenerate vector subbundle of $VTM^\perp$.

We call $SVTM$ and $sVTM$ the *screen vector bundle* and the *screen transversal vector bundle* of the r-lightlike Finsler submanifold F^m. According to Proposition 1 we have the orthogonal decompositions

$$\begin{aligned} VTM &= Rad\, VTM \perp SVTM \quad \text{and} \\ VTM^{\perp} &= Rad\, VTM \perp sVTM. \end{aligned} \tag{1}$$

As $SVTM$ is non-degenerate, we put

$$VT\tilde{M}_{|TM} = SVTM \perp SVTM^{\perp},$$

where $SVTM^{\perp}$ is the complementary orthogonal vector bundle of $SVTM$ in $VT\tilde{M}_{|TM}$. It is important to note that $sVTM$ is a vector subbundle of $SVTM^{\perp}$ and since both are non-degenerate we obtain

$$SVTM^{\perp} = sVTM \perp sVTM^{\perp}.$$

Moreover $sVTM^{\perp}$ is of rank $2r$.

Lemma 1. *Let $(F^m, g, SVTM, sVTM)$ be a 1-lightlike Finsler submanifold of F^{m+n}. Suppose $\mathcal{U}$ is a coordinate neighborhood of TM and ξ is a non-zero section of $Rad\, VTM_{|\mathcal{U}}$. Then there exists a unique section N of $sVTM^{\perp}_{|\mathcal{U}}$ such that*

$$\tilde{g}(N, \xi) = 1, \tag{2}$$

and

$$\tilde{g}(N, N) = 0. \tag{3}$$

Proof. As $sVTM^{\perp}$ is non-degenerate and $Rad\, VTM$ is a vector subbundle of $sVTM^{\perp}$ there exists a section V of $sVTM^{\perp}$ such that $g(V, \xi) \neq 0$ on $\mathcal{U}$. Thus any section N of $sVTM^{\perp}_{|\mathcal{U}}$, is expressed $N = \alpha V + \beta\xi$, since in this case rank $sVTM^{\perp} = 2$. Then by direct calculations one deduces that N satisfies (2) and (3) if and only if it is given by

$$N = \frac{1}{\tilde{g}(V, \xi)} \left\{ V - \frac{\tilde{g}(V, V)}{2\tilde{g}(V, \xi)} \xi \right\}. \tag{4}$$

Moreover, N does not depend on V. This completes the proof of the lemma.

Lemma 2. *Let* $(F^m, g, SVTM, sVTM)$ *be a* r*- lightlike Finsler submanifold of* F^{m+n} *with* $r > 1$. *Suppose* $\mathcal{U}$ *is a coordinate neighborhood of* TM *and* $\{\xi\}$, $i \in \{1, ..., r\}$ *is a basis of* $\Gamma(Rad\ VTM_{|\mathcal{U}})$. *Then there exist smooth sections* $\{N_i\}$ *of* $sVTM^{\perp}_{|\mathcal{U}}$ *such that*

$$\tilde{g}(N_i, \xi_j) = \delta_{ij}, \tag{5}$$

and

$$\tilde{g}(N_i, N_j) = 0 \tag{6}$$

for any $i, j \in \{1, ..., r\}$.

Proof. Consider a complementary vector bundle F to $Rad\ VTM$ in $sVTM^{\perp}$ and choose a basis $\{V_i\}$, $i \in \{1, ..., r\}$ of $\Gamma(F_{|\mathcal{U}})$. Thus the sections we look for are expressed as follows;

$$N_i = \sum_{k=1}^{r} \{a_{ik}\xi_k + b_{ik}V_k\}, \tag{7}$$

where a_{ik} and b_{ik} are smooth functions on $\mathcal{U}$. Then $\{N_i\}$ satisfy (5) if and only if

$$\sum_{k=1}^{r} \{b_{ik}\tilde{g}_{jk}\} = \delta_{ij},$$

where $\tilde{g}_{jk} = \tilde{g}(\xi_j, V_k)$. Note that $G = det[\tilde{g}_{jk}]$ vanishes nowhere on $\mathcal{U}$ otherwise $sVTM^{\perp}$ is degenerate. It follows that the above system has the unique solution

$$b_{ik} = \frac{(\tilde{g}_{ik})'}{G}, \tag{8}$$

where $(\tilde{g}_{ik})'$ are the algebraic complements of the elements $\tilde{g}_{ik}$ in G. Finally we obtain that (6) is equivalent with

$$a_{ij} + a_{ji} + \sum_{k,h=1}^{r} \{b_{ik}b_{jh}\tilde{g}(V_k, V_h)\} = 0, \tag{9}$$

which clearly proves the existence of a_{ij}.

By using (5) and (6) it is easy to see that $\{\xi_i, N_i\}, i \in \{1, \dots, r\}$ is a local lightlike basis of the non-degenerate vector bundle $sVTM^{\perp}$.

Theorem 2. *Let $(F^m, g, SVTM, sVTM)$ be a 1-lightlike Finsler submanifold of F^{m+n} and $\mathcal{U}$ a coordinate neighborhood of TM. Then there exists a unique vector subbundle $tVTM$ of $sVTM^{\perp}$ of rank 1 and for any $\xi \in \Gamma(Rad\ VTM_{|\mathcal{U}}), \xi \neq 0$ on $\mathcal{U}$, there exists a unique $N \in \Gamma(tVTM_{|\mathcal{U}})$ satisfying (2) and (3).*

Proof. Consider two coordinate neighborhoods $\mathcal{U}$ and $\mathcal{U}^*$ of TM such that $\mathcal{U} \cap \mathcal{U}^* \neq \phi$. Then apply Lemma 1 for sections ξ and ξ^* and obtain N and N^* satisfying (2) and (3). As $\xi^* = \alpha\xi$ and $V^* = \beta V + \gamma\xi$ on $\mathcal{U} \cap \mathcal{U}^*$ by direct calculations one can deduces $N^* = (1/\alpha)N$. Thus we obtain a line bundle $tVTM$ whose local sections N are given by (4). The uniqueness of $tVTM$ is a direct consequence of the uniqueness of N.

Corollary 1. *Let $(F^m, g, SVTM, sVTM)$ be a r- lightlike submanifold of a Lorentz Finsler manifold F^{m+n}. Then $r = 1$ and there exists a unique vector bundle $tVTM$ satisfying conditions from Theorem 2.*

Proof. Since the Finsler metric is of Lorentz type we have $q = 1$. As $0 < r \leq q$ we get $r = 1$ and then apply Theorem 2.

Theorem 3. *Let $(F^m, g, SVTM, sVTM)$ be a r- lightlike Finsler submanifold of a pseudo-Finsler manifold F^{m+n}. Then there exists a complementary vector bundle $tVTM$ of $Rad\ VTM$ in $sVTM^{\perp}$ such that $\{N_i\}$ from Lemma 2 be a basis of $\Gamma(tVTM_{|\mathcal{U}})$.*

Proof. Choose $a_{ij} = a_{ji}$ in (9) and obtain

$$a_{ij} = -\frac{1}{2} \sum_{k,h=1}^{r} \{b_{ik} b_{jh} \tilde{g}(V_k, V_h)\}, \tag{10}$$

where b_{ik} are uniquely given by (8). Now consider another coordinate neighborhood $\mathcal{U}^* \subset TM$ such that $\mathcal{U} \cap \mathcal{U}^* \neq \phi$. Then take $\{\xi_i^*\}$ and $\{V_i^*\}$ as basis of $\Gamma(Rad\ VTM_{|\mathcal{U}^*})$ and $\Gamma(F_{|\mathcal{U}^*})$ respectively, and by Lemma 2 obtain the local vector fields

$$N_i^* = \sum_{j=1}^{r} \{a_{ij}^* \xi_j^* + b_{ij}^* V_j^*\}, \tag{11}$$

$$(12) \qquad b_{ij} = \frac{(\tilde{g}^*_{ij})'}{G^*}; \quad \tilde{g}^*_{ij} = \tilde{g}(\xi^*_i, V^*_j); G^* = det[\tilde{g}^*_{ij}],$$

and

$$(13) \qquad a^*_{ij} = -\frac{1}{2}\sum_{k,h=1}^{r}\{b^*_{ik}b^*_{jh}\tilde{g}(V^*_k, V^*_h)\}.$$

On $\mathcal{U}\cap\mathcal{U}^*$ we set

$$(14) \qquad \xi^*_i = C^j_i\xi_j \quad \text{and} \quad V^*_i = E^j_i V_j,$$

where C^j_i and E^j_i are smooth functions on $\mathcal{U}\cap\mathcal{U}^*$. It is easy to verify the algebraic identities.

$$(15) \qquad (\tilde{g}^*_{ij})' = (\tilde{g}_{hk})'(C^h_i)'(E^k_j)'; G^* = GCE,$$

where we put C and E for the determinants of invertible matrices $[C^j_i]$ and $[E^j_i]$. Then using (12), (14), (15), (8) and usual properties of algebraic complements we obtain

$$(16) \qquad \begin{aligned} \sum_{j=1}^{r}\{b^*_{ij}V^*_j\} &= \frac{1}{G^*}\sum_{j=1}^{r}\{(\tilde{g}^*_{ij})'E^t_j\}V_t \\ &= \frac{1}{GCE}\sum_{j=1}^{r}\{(E^k_j)'E^t_j\}(\tilde{g}_{hk})'(C^h_i)'V_t \\ &= \frac{1}{C}(C^h_i)'\sum_{k=1}^{r}\left\{\frac{(\tilde{g}_{hk})'}{G}V_k\right\} = \frac{1}{C}(C^h_i)^p\sum_{k=1}^{r}\{b_{hk}V_k\}. \end{aligned}$$

On the other hand, using (13), (14), (16) and (10) we infer

$$(17) \qquad \begin{aligned} \sum_{j=1}^{r}\{a^*_{ij}\xi^*_j\} &= -\frac{1}{2}\sum_{j=1}^{r}\{\tilde{g}(\sum_{k=1}^{r} b^*_{ik}V^*_k, \sum_{h=1}^{r} b^*_{jh}V^*_h)\xi^*_j\} \\ &= -\frac{1}{2C^2}\sum_{j=1}^{r}\{\tilde{g}(\sum_{k=1}^{r} b_{sk}(C^s_i)'V_k, \sum_{h=1}^{r} b_{ph}(C^p_j)'V_h)C^t_j\}\xi_t \\ &= -\frac{1}{2C}\sum_{t,k,h=1}^{r}\{b_{sk}b_{th}\tilde{g}(V_k, V_h)(C^s_i)'\xi_t\} \\ &= \frac{1}{C}\sum_{t=1}^{r}\{a_{st}(C^s_i)'\xi_t\} \end{aligned}$$

Taking account of (16), (17) and (7) in (11) we deduce

$$N_i^* = \frac{1}{C}(C_i^h)' N_h.$$

Hence there exists a vector bundle $tVTM$ of rank r locally spanned on each $\mathcal{U} \subset TM$ by $\{N_i\}$ given by (7), (8) and (10). Moreover, $tVTM$ is complementary to $Rad\ VTM$ in $sVTM^\perp$. To show this assertion we suppose there exists a point $(u,v) \in TM$ and a vector $W_{(u,v)} \in \Gamma(Rad\ VTM_{(u,v)} \cap tVTM_{(u,v)})$. Thus

$$W_{(u,v)} = W^i_{(u,v)}(u,v)N_i = W^i(u,v)\xi_i(u,v),$$

which together with (5) imply

$$\tilde{g}_{(u,v)}(W_{(u,v)}, \xi_j(u,v)) = \tilde{g}_{(u,v)}(W^i(u,v)N_i(u,v), \xi_j(u,v)) = W^j(u,v),$$

and

$$\tilde{g}_{(u,v)}(W_{(u,v)}, \xi_j(u,v)) = \tilde{g}_{(u,v)}(W^i(u,v)\xi_i(u,v), \xi_j(u,v)) = 0.$$

Hence $W_{(u,v)} = 0_{(u,v)}$, and this completes the proof of the theorem.

We call $tVTM$ constructed in Theorem 3 the *canonical lightlike transversal vector bundle* of F^m with respect to the pair $(SVTM, sVTM)$. Any other complementary vector bundle of Rad VTM in $sVTM^\perp$ satisfying conditions as $tVTM$ from Theorem 3 is called a *lightlike transversal vector bundle* of F^m.

Next, consider the vector bundle

$$NVTM = tVTM \perp sVTM, \tag{18}$$

where $tVTM$ is one of the vector bundles in Theorem 3. Clearly $NVTM$ is of rank n and

$$NVTM_{(u,v)} \cap VTM_{(u,v)} = \{0_{(u,v)}\},$$

for any $(u,v) \in TM$. Thus $NVTM$ is a complementary (but not orthogonal) vector bundle to VTM in $V\tilde{TM}_{|TM}$. For this reason we call

$NVTM$ the *canonical transversal vector bundle* or the *transversal vector bundle*, according as $tVTM$ is canonical or not, respectively.

By using (1) and (18) we obtain

$$(19)\quad \begin{aligned} V T\tilde{M}_{|TM} &= VTM \oplus NVTM \\ &= SVTM \perp sVTM \perp (tVTM \oplus Rad\ VTM). \end{aligned}$$

Hence we obtain a local quasi-orthogonal field of frames on F^{m+n} along F^m, $\{\xi_1,\ldots,\xi_r, X_{r+1},\ldots,X_m, N_1,\ldots,N_r, W_{r+1},\ldots,W_n\}$, where $\{\xi_1,\ldots,\xi_r\}$ and $\{N_1,\ldots,N_r\}$ are lightlike basis of $\Gamma(Rad\ VTM_{|\mathcal{U}})$ and $\Gamma((tVTM_{|\mathcal{U}})$ respectively, related by (5), while $\{X_{r+1},\ldots,X_m\}$ and $\{W_{r+1},...,W_n\}$ are orthonormal basis of $\Gamma(SVTM_{|\mathcal{U}})$ and $\Gamma(sVTM_{|\mathcal{U}})$ respectively. As each pair $\{\xi_i, N_i\}, i \in \{1,...,r\}$ is a hyperbolic plane, we conclude that $RadVTM \oplus tVTM$ is nondegenerate vector bundle of constant index r on M. Hence by a result of O'Neill [5], p. 51, from (19) we derive

$$(20)\qquad q = r + \text{index}(SVTM_{(u,v)}) + \text{index}(sVTM_{(u,v)}),$$

at any point $(u,v) \in TM$. Therefore, if one of the vector bundles $SVTM$ and $sVTM$ is of constant index the other one is of constant index too. Moreover, from (20) it follows

Proposition 2. *Let F^m be a lightlike Finsler submanifold of a Lorents Finsler manifold F^{m+n}, i.e., $q = 1$. Then any screen vector bundle and any screen transversal vector bundle of F^m is a Finsler vector bundle, that is, the induced Finsler metrics on these vector bundles are positive definite.*

Case II. $(0 < r = n < m)$. In this case $Rad\ VTM = VTM^{\perp}$, and thus the former normal Finsler bundle from the theory of non-degenerate Finsler submanifolds becomes a vector subbundle of VTM. Consider as in Case I the screen vector bundle $SVTM$ over TM and obtain

$$VTM = SVTM \perp VTM^{\perp}.$$

As $sVTM = \{0\}$ we need to replace Lemma 2 by

Lemma 3. *Let $(F^m, g, SVTM)$ be a n-lightlike Finsler submanifold of F^{m+n}. Suppose $\mathcal{U}$ is a coordinate neighborhood of TM and $\{\xi_1, .., \xi_n\}$ is a basis of $\Gamma(VTM^{\perp}_{|\mathcal{U}})$. Then there exist smooth sections $\{N_1, ..., N_n\}$ of $VTM^{\perp}_{|\mathcal{U}}$ such that*

$$\tilde{g}(N_i, \xi_j) = \delta_{ij},$$

and

$$\tilde{g}(N_i, N_j) = 0; \quad \tilde{g}(N_i, X) = 0,$$

for any $i, j \in \{1, ..., n\}$ and $X \in \Gamma(SVTM_{|\mathcal{U}})$.

Proof. As both $SVTM$ and $VT\tilde{M}_{|TM}$ are non-degenerate we set

$$VT\tilde{M}_{|TM} = SVTM \perp SVTM^{\perp}.$$

Clearly, $SVTM^{\perp}$ is of rank $2n$ and $VTM^{\perp}$ is a vector subbundle of $SVTM^{\perp}$. Then consider a complementary vector bundle F to $VTM^{\perp}$ in $SVTM^{\perp}$. Define N_i by the same formula (7) and from this point the proof follows as in Lemma 2.

Moreover, the proof of the following theorem runs as the one of Theorem 3.

Theorem 4. *Let $(F^m, g, SVTM)$ be a n-lightlike Finsler submanifold of F^{m+n}. Then there exists a complementary vector bundle $tVTM$ of $VTM^{\perp}$ in $SVTM^{\perp}$ such that $\{N_1, ..., N_n\}$ from Lemma 3 be a basis of $\Gamma(tVTM_{|\mathcal{U}})$.*

In this case (19) becomes

$$VT\tilde{M}_{|TM} = VTM \oplus tVTM = SVTM \perp (tVTM \oplus VTM^{\perp}). \tag{21}$$

Therefore the local quasi-orthogonal field of frames on F^{m+n} along F^m is given by

$$\{\xi_1, \ldots, \xi_n, X_{n+1}, \ldots, X_m, N_1, ..., N_n\},$$

where $\{X_{n+1}, ..., X_m\}$ is an orthonormal basis of $\Gamma(SVTM_{|\mathcal{U}})$. Finally, from (21) we obtain

$$q = \text{index } (SVTM_{(u,v)}) + n,$$

at any point $(u, v) \in TM$. Hence in this case it follows

Proposition 3. *Let F^m and F^{m+n} as in Theorem 4. Then any screen vector bundle $SVTM$ has constant index $q - n$.*

Case III. $(0 < r = m < n)$. In this case $Rad\ VTM = VTM$ and therefore VTM becomes a vector subbundle of $VTM^{\perp}$. Then we set

$$VTM^{\perp} = VTM \perp sVTM,$$

where $sVTM$ is a transversal screen vector bundle. Replace Lemma 2 by

Lemma 4. *Let $(F^m, g, sVTM)$ be a m-lightlike Finsler submanifold of F^{m+n}. Suppose $\mathcal{U}$ is a coordinate neighborhood of TM and $\{\xi_1, \ldots, \xi_m\}$ be a lightlike basis of $\Gamma(VTM_{|\mathcal{U}})$. Then there exist smooth sections $\{N_1, ..., N_m\}$ of $(VTM_{|\mathcal{U}})$, such that*

$$\tilde{g}(N_i, \xi_j) = \delta_{ij}, \quad and \quad \tilde{g}(N_i, N_j) = 0; \ \tilde{g}(N_i, W) = 0,$$

for any $i, j \in \{1, ..., m\}$ and $W \in \Gamma(sVTM)$.

Proof. Here we set

$$VT\tilde{M}_{|TM} = sVTM \perp sVTM^{\perp},$$

and note that rank $sVTM^{\perp} = 2m$ and VTM is a vector subbundle of $sVTM^{\perp}$. Consider a complementary vector bundle F of VTM in $sVTM^{\perp}$ and the proof continues as in Lemma 2.

In this case Theorem 3 is replaced by

Theorem 5. *Let $(F^m, g, sVTM)$ be a m-lightlike submanifold of F^{m+n}. Then there exists a complementary vector bundle $tVTM$ of VTM in $sVTM^{\perp}$ such that $\{N_1, \ldots, N_m\}$ from Lemma 4 be a basis of $\Gamma(tVTM_{|\mathcal{U}})$.*

Therefore we have the decomposition

$$(22) \quad VT\tilde{M}_{|TM} = sVTM \perp (VTM \oplus tVTM) = VTM \oplus NVTM,$$

and the local quasi-orthogonal field of frames

$$\{\xi_1, \ldots, \xi_m, N_1, \ldots, N_m, W_{m+1}, \ldots, W_n\},$$

where $\{W_{m+1}, ..., W_n\}$ is an orthonormal basis of $t(sVTM_{|\mathcal{U}})$. From (22) we deduce

$$q = \text{index } (sVTM_{(u,v)}) + m, \; \forall \; (u, v) \in TM,$$

which enables us to state the following result.

Proposition 4. *Let F^m and F^{m+n} as in Theorem 5. Then any screen transversal vector bundle of F^m has a constant index $q - m$.*

Case IV. $(0 < r = m = n)$. In this case $Rad\ VTM = VTM = VTM^{\perp}$ and thus there exist neither screen vector bundles nor screen transversal vector bundles. However, we state

Lemma 5. *Let F^m be a m-lightlike Finsler submanifold F^{2m}. Suppose $\mathcal{U}$ is a coordinate neighborhood of TM and $\{\xi_1, \ldots, \xi_m\}$ be a basis of $\Gamma(VTM_{|\mathcal{U}})$. Then there exist smooth sections $\{N_i, ..., N_m\}$ of $\Gamma(VT\tilde{M}_{|\mathcal{U}})$ such that*

$$\tilde{g}(N_i, \xi_j) = \delta_{ij},$$

and

$$\tilde{g}(N_i, N_j) = 0,$$

for any $i, j \in \{1, ..., m\}$.

Proof. Consider a complementary vector bundle F of VTM in $VT\tilde{M}_{|TM}$, a basis $\{V_1, ..., V_m\}$ of $\Gamma(F_{|\mathcal{U}})$ and from this point the proof follows the same steps as in Lemma 2.

Therefore, as in the previous cases we obtain the following result.

Theorem 6. *Let F^m be a m-lightlike Finsler submanifold of F^{2m}. Then there exists a complementary vector bundle $tVTM$ of VTM in $VT\tilde{M}_{|TM}$ such that $\{N_1, ..., N_m\}$ from Lemma 5 is a basis of $\Gamma(tVTM_{|\mathcal{U}})$.*

Hence we have

$$VT\tilde{M}_{|TM} = VTM \oplus tVTM,$$

and the local quasi-orthogonal field of frames

$$\{\xi_1, \dots, \xi_m, N_1, \dots, N_m\}.$$

As $VTM_{(u,v)} \oplus tVTM_{(u,v)}$ is a hyperbolic space at any $(u, v) \in TM$, we conclude that $q = m = n = r$.

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