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SEMI-COMMUTATIVE MODULES AND ARMENDARIZ MODULES

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ABSTRACT. A ring R is semi-commutative if whenever elements a, b in R satisfy $ab = 0$, then $acb = 0$ for each element c of R. Armendariz rings are defined through polynomial rings over them. The two concepts are closely related and have straight-forward extensions to modules. We prove a number of results concerning semi-commutative modules and Armendariz modules. Motivated by a characteristic property of commutative Gaussian rings, a ring R is called left Gaussian if all cyclic left R-modules are Armendariz. Basic properties of Gaussian rings are given.

1. INTRODUCTION

A ring R (associative with identity) is *abelian* if each idempotent in R is central; it is *reduced* if it has no nonzero nilpotent elements; it is *semi-commutative* if whenever elements a, b in R satisfy $ab = 0$, then $acb = 0$ for each element c of R. It is an elementary exercise to show that reduced rings are abelian. It was recorded in [**5**,§4] that the class of semi-commutative rings lies between the classes of reduced rings and abelian rings.

Closely related to semi-commutative rings are Armendariz rings. The notion of an Armendariz ring was introduced by Rege and Chhawchharia in [**9**]. A ring R is *Armendariz* if given polynomials $f(X) = \Sigma a_i X^i$ and $g(X) = \Sigma b_j X^j$ with coefficients in R, the condition $f(X)g(X) = 0$ implies $a_i b_j = 0$ for every i and

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j. A number of examples of Armendariz rings were recorded by Anderson and Camillo [**1**] and Kim and Lee [**7**]; they also extended several results of [**9**]. Huh, Lee and Smoktunowicz [**6**] made a comparative study of Armendariz rings and semi-commutative rings. The class of Armendariz rings lies between the classes of reduced rings and abelian rings by [**3**, Lemma 1] and [**6**, Corollary 8]. Semi-commutative rings need not be Armendariz by [**9**, 3.2] and Armendariz rings need not be semi-commutative by [**6**, Example 14]. Both concepts are defined through 'vanishing conditions' on elements and have straight-forward extensions to modules. In §2 of this paper we initiate the study of basic properties of semi-commutative modules and Armendariz modules. All left ideals of the ring R are two-sided if and only if all (cyclic) left R-modules are semi-commutative (Proposition 2.11). A consequence of Theorems 2.15 and 2.16 is the result that all flat modules over a reduced ring are both Armendariz and semi-commutative. In §3 we study rings over which all cyclic left modules are Armendariz; these rings have been called 'left Gaussian rings'. We also collect (in 3.12) a number of questions which arise in the course of our study.

2. BASIC PROPERTIES AND EXAMPLES

This section is devoted to developing basic properties and furnishing a number of examples of the concepts under consideration. Easier proofs are omitted. It is hoped that the omission of 'respectively' in parentheses does not cause confusion.

By a ring we mean an associative ring with identity. Ring homomorphisms and modules are unitary left modules. Throughout R denotes a ring and D a commutative domain. All our left-sided concepts and results have right-sided counterparts.

Definition 2.1. A module M over a ring R is *semi-commutative* if it satisfies the following condition: whenever elements $a \in R$ and $m \in M$ satisfy $am = 0$, then $acm = 0$ for each element c of R.

The concept of an Armendariz module was mentioned in [**9**, 4.7].

Definition 2.2 Let M be a module over a ring R. Let $M[X]$ be the corresponding polynomial module over $R[X]$. The module M is an *Armendariz module* if, whenever polynomials $f(X) = \Sigma a_i X^i \in R[X]$, $g(X) = \Sigma m_j X^j \in M[X]$ satisfy $f(X)g(X) = 0$, we have $a_i m_j = 0$ for every i and j.

For ease of reference we recall the following result (see [**1**, Theorem 12]).

Proposition 2.3 *If M is a torsion free module over a commutative domain D, then M is Armendariz as a D-module.*

Remark 2.4 A ring R is semi-commutative (Armendariz) if and only if the module ${}_{R}R$ is semi-commutative (Armendariz).

We next record a 'change of rings' result.

Proposition 2.5 *Let $\theta: R \longrightarrow A$ be a ring homomorphism and let M be an A-module. Regard M as a left R-module via θ. Then we have:*
(i) if ${}_{A}M$ is semi-commutative (Armendariz), then ${}_{R}M$ is semi-commutative (Armendariz);
(ii) if θ is onto, then the converses of the statements in (i) hold;
(iii) if A is a semi-commutative (Armendariz) ring , then A is semi-commutative (Armendariz) as a left R-module.

Proof. For the sake of notational simplicity we prove the result in the semi-commutative case. The proof in the Armendariz case is similar.
(i) Assume that $r, s \in R$, $m \in M$, and $rm = 0$. Then $\theta(r)m = 0$ implies

$$(rs)m = \theta(rs)m = \theta(r)\theta(s)m = 0$$

since the left A-module M is semi-commutative.
(ii) The proof is similar to that of part (i).
(iii) This is a consequence of 2.4 and part (i). □

The following assertion is immediate.

Proposition 2.6 *The class of semi-commutative (Armendariz) R-modules is closed under direct sums, direct products and submodules.*

An R-module is *torsionless* if it is a submodule of a direct product of copies of R. If M is a faithful R-module then R is a submodule of a direct product of copies of M. An application of 2.4 and 2.6 yields the following theorem.

Theorem 2.7 *The following conditions are equivalent.*
(1) R is a semi-commutative (Armendariz) ring.
(2) Every torsionless R-module is semi-commutative (Armendariz).
(3) Every submodule of a free R-module is semi-commutative (Armendariz).
(4) There exists a faithful R-module which is semi-commutative (Armendariz).

Remark 2.8 If M is a left R-module $\overline{R}$ denotes the ring $R/ann(M)$ and $E(M) = End_{\mathrm{R}}(M)$ the ring of endomorphisms of M. With this notation consider the following conditions.
(1) The left R-module M is semi-commutative (Armendariz).
(2) The left $\overline{R}$-module M is semi-commutative (Armendariz).
(3) $\overline{R}$ is a semi-commutative (Armendariz) ring.
(4) The right $E(M)$-module M is semi-commutative (Armendariz).
(5) The ring $E(M)$ is semi-commutative (Armendariz).

An application of Proposition 2.5 yields the equivalence of conditions (1) and (2); since the left R-, right $E(M)$-bimodule M is faithful as a left $\overline{\mathrm{R}}$-module and also faithful as a right $E(M)$-module, applying (4)$\Rightarrow$(1) of Theorem 2.7 we get (2)$\Rightarrow$(3) and (4)$\Rightarrow$(5).

Since the definition of a semi-commutative (Armendariz) module involves a single element (finitely many elements) of the module the following result is obvious.

Proposition 2.9 *A module M is semi-commutative (Armendariz) if and only if every cyclic (finitely generated) submodule of M is semi-commutative (Armendariz).*

A ring R is *left invariant* if every left ideal of R is two-sided. In Proposition 2.11 we characterize left invariant rings as rings over which all left modules are semi-commutative.

Proposition 2.10 *If the cyclic left R-module R/J is semi-commutative, then J is an ideal of R.*

Proof. Let $x \in J$, $r \in R$. Then we have (for $\bar{1} = 1 + J \in R/J$), $x\bar{1} = 0$ which implies $xr\bar{1} = 0$, yielding $xr \in J$. □

Proposition 2.11 *The following conditions are equivalent.*
(1) R is left invariant.
(2) Every left R-module is semi-commutative.
(3) Every cyclic left R-module is semi-commutative.

Proof. (1) ⇒ (2). Let M be a left R-module. Suppose that $a \in R$ and $m \in M$ satisfy $am = 0$. Let $c \in R$. Since the left ideal Ra is two-sided we have $aR \subseteq Ra$. It follows that $acm \in Ram = 0$. The implication (2) ⇒ (3) is trivial, while (3) ⇒ (1) follows from 2.10. □

Examples 2.12 For a field K and an integer $n \geq 2$, let $A = M_n(K)$, the full matrix ring over K. The ring A is non-abelian, non-semi-commutative and non-Armendariz. Indeed the zero module is the only A-module which is semi-commutative (Armendariz). Notice also that the example of the zero ideal in A shows that the converse of Proposition 2.10 does not hold. The ring A is the homomorphic image of a domain, namely, the ring R of polynomials in sufficiently many non-commuting indeterminates over K. Let $\theta : R \longrightarrow A$ be an onto ring homomorphism. By 2.5(ii) A is neither semi-commutative nor Armendariz as an R-module. However, it is a factor module of ${}_RR$ which is semi-commutative and Armendariz. Further, the left R-module $M = R \oplus A$ is R-faithful, but M (which has A as a submodule) is neither semi-commutative nor Armendariz as a left R-module; this shows that (3)⇒ (1) does not hold in (either case of) Remark 2.8.

In the rest of this section we present a few results related to Proposition 2.3. Proposition 2.13 is a straight-forward extension. Theorem 2.15 asserts that flat modules over Armendariz rings are Armendariz, and Theorem 2.16 is a semi-commutative analogue of this result. (Flat modules are torsion free, even over arbitrary rings [**10**, Example 1, §10, Chapter 1].) In Proposition 2.20 we extend 2.3 in another direction. Later we show in Proposition 3.8 that over Prüfer domains all modules are Armendariz.

Proposition 2.13 *Let D be a commutative domain and M a D-module. The module M is Armendariz if and only if its torsion submodule $T(M)$ is Armendariz.*

Proof. Let $f(X) = \Sigma a_i X^i \in D[X]$, $g(X) = \Sigma m_j X^j \in M[X]$ satisfy $f(X)g(X) = 0$. We have the system of equations $a_0 m_0 = 0, a_0 m_1 + a_1 m_0 = 0, a_0 m_2 + a_1 m_1 + a_2 m_0 = 0, \ldots, a_s m_t = 0$ (for some s, t). We can assume $a_0 \neq 0$. Now the second of these equations yields (on multiplying by a_0 and using $a_0 m_0 = 0$) $a_0^2 m_1 = 0$. Thus a_0^2 annihilates both m_0 and m_1. The third equation now implies $a_0^3 m_2 = 0$. Continuing we get $g(X) \in T(M)[X]$. Since $T(M)$ is Armendariz as a D-module, we conclude that $a_i m_j = 0$ for all i, j, proving the 'if' part. The other implication is trivial. □.

Next we consider flat modules. We first recall a well-known result [**10**, Corollary 11.4, Chapter 1]; module homomorphisms are written on the right.

Proposition 2.14 *Let M be a flat R-module and $0 \to K \to F \to M \to 0$ be an exact sequence with F free over R. For each finite family $u_1, u_2, \ldots, u_m$ of elements of K there exists an R-homomorphism v: $F \to K$ such that $(u_i)v = u_i$ for each i.*

Theorem 2.15 *Flat modules over Armendariz rings are Armendariz.*

Proof. Let R be an Armendariz ring and M a flat R-module. Let $0 \to K \to F \to M \to 0$ be an exact sequence with F free over R. (In what follows for an element y of F, we denote $\overline{y} = y + K$ in M) Let $f(X) = \Sigma_{i=0}^{s} a_i X^i \in R[X]$ and $g(X) = \Sigma_{j=0}^{t} \overline{y_j} X^j \in M[X]$ satisfy $f(X)g(X) = 0$. Then we have

$$a_0 \overline{y_0} = 0$$

$$a_0 \overline{y_1} + a_1 \overline{y_0} = 0$$

$$\vdots$$

$$a_s \overline{y_t} = 0$$

Therefore the elements $a_0 y_0, a_0 y_1 + a_1 y_0, \ldots, a_s y_t$ all belong to K. Proposition 2.14 ensures the existence of an R-module homomorphism $v: F \to K$ such that $(a_0 y_0)v = a_0 y_0, \ldots, (a_s y_t)v = a_s y_t$. Write $w_j = (y_j)v - y_j$ for $j = 0, \ldots, t$. Each w_j is an element of F and therefore the polynomial $h(X) = \Sigma_{j=0}^{t} w_j X^j \in F[X]$. Now F is free, and therefore Armendariz as an R-module. Since $f(X)h(X) = 0$

we have $a_i w_j = 0$ for each i and j. It follows that $a_i y_j \in K$ for each i and j. Hence $a_i \overline{y_j} = 0$ in M proving that the module M is Armendariz. □

Theorem 2.16 *Flat modules over semi-commutative rings are semi-commutative.*

Proof. The proof is similar to (and notationally simpler than) the proof of Theorem 2.15. We omit it.

By a *regular* ring we mean a von Neumann regular ring. For later reference we record the following immediate consequence of 2.15, 2.16 and the well-known result that over a regular ring all modules are flat.

Remark 2.17 If R is a reduced, regular ring (in particular, a commutative regular ring) all R-modules are Armendariz as well as semi-commutative.

Next we study localizations. Let M be an R-module and let C be the centre of R. If S is a multiplicatively closed subset of C then $S^{-1}M$ has an $S^{-1}R$-module structure. Let Spec(C) (respectively, $Max(C)$) denote the set of all prime (respectively, maximal) ideals of C. The module M is *S-torsion free* if whenever s is an element of S and m is a nonzero element of M, we have $sm \neq 0$. Applying standard localization techniques we can prove Propositions 2.18 and 2.19.(See [**1**] and [**6**] for related results.)

Proposition 2.18 *Let M be S-torsion free. The R-module M is semi-commutative (Armendariz) if and only if the $S^{-1}R$-module $S^{-1}M$ is semi-commutative (Armendariz).*

Proposition 2.19 *Let M be an R-module and let $C = Centre(R)$. Then the following conditions are equivalent.*
(1) M is semi-commutative (Armendariz).
(2) $S^{-1}M$ is a semi-commutative (Armendariz) $S^{-1}R$-module for each multiplicatively closed subset S of C.
(3) M_{P} is a semi-commutative (Armendariz) R_{P}-module for each $P \in Spec(C)$.
(4) M_{Q} is a semi-commutative (Armendariz) R_{Q}-module for each $Q \in Max(C)$.

Let S_0 denote the set of all non-zero-divisors of a commutative ring R. The ring R is *quasi-regular* if its classical quotient ring $S_0^{-1}R$ is regular. It is well-known that commutative pp-rings (i.e., rings in which every principal ideal is projective) are quasi-regular. Since commutative domains are trivially quasi-regular, the following result extends Proposition 2.3.

Proposition 2.20 *If R is a commutative quasi-regular ring, and M is an S_0-torsion free R-module, then M is Armendariz.*

Proof. Since $A = S_0^{-1}R$ is a commutative regular ring, $S_0^{-1}M$ is Armendariz as an A-module by 2.17. It follows, by 2.18, that the R-module M is Armendariz. □

3. GAUSSIAN RINGS

Let R be a ring and let M be a left R-module. For $g \in M[X]$, its *content* A_g is the R-submodule of M generated by the coefficients of g; a commutative ring R is *Gaussian* [**11**] if $A_{fg} = A_f A_g$ for all $f, g \in R[X]$.

We have the following classical result due to Tsang ([**11**], [**4**, Exercise 7 on page 351] and Theorem 1.3 in [**2**].)

Theorem 3.1 *A commutative domain D is Gaussian if and only if it is Prüfer (i.e. every finitely generated ideal of D is invertible).*

If 'P' is a term applicable to rings, a ring R is *completely P* if every homomorphic image of R is P. We have the following reformulation of Theorem 8 of [**1**].

Proposition 3.2 *A commutative ring is Gaussian if and only if it is completely Armendariz.*

Gaussian/Prüfer domains play a significant role in multiplicative ideal theory [**4**]. We have not come across a non-commutative generalization of the concept of a Gaussian ring. We attempt to bridge this gap (see Definitions 3.4). Our motivation comes from the following result.

Proposition 3.3 *Let R be a commutative ring. The ring R is Gaussian if and only if every cyclic R-module is Armendariz.*

Proof. By 2.4 and 2.5 the cyclic R-module R/J is Armendariz if and only if the ring R/J is Armendariz. Now apply 3.2. □

Definition 3.4 A ring R is *left Gaussian* if every cyclic left R-module is Armendariz; it is *left strongly Gaussian* if every left R-module is Armendariz.

We can use 2.5 to prove the following remarks.

Remarks 3.5(a) The class of left (strongly) Gaussian rings is closed under homomorphic images and the formation of finite direct products; consequently left Gaussian rings are completely Armendariz.
(b) Left invariant, completely Armendariz rings are left Gaussian.

Example 3.6 shows: (1) the Gaussian property is not left-right symmetric, and (2) completely Armendariz rings need not be left Gaussian.

Example 3.6 Let L be a field and let $K = L(Y, T)$ be the field of rational functions in two commuting indeterminates Y and T. Let h be the endomorphism of the field K defined via $h(\mathrm{Y}) = \mathrm{Y}^2$ and $h(\mathrm{T}) = \mathrm{T}^2$. With notation as in [**9**], the ring $R = K(+)_h K$ has the following properties:
(i) Apart from R and 0 the only right ideal of R is $0 \oplus K$, which is also a left ideal. Since the ring $R_0 = R/(0 \oplus K)$ is a field, R_0 is Armendariz as a right R-module. By Proposition 2.8 of [**9**] R itself is an Armendariz ring. The ring R is thus a right Gaussian and completely Armendariz ring.
(ii) Let $W = h(K) = L(Y^2, T^2)$, a subfield of K. Then $B = 0 \oplus W$ is a left ideal of R. We verify that the cyclic left R-module R/B is not Armendariz, and thus the ring R is not left Gaussian.

Write $f(X) = (0, Y) + (0, T)X \in R[X]$, and $g(X) = \overline{(\mathrm{Y}, 0)} + \overline{(-\mathrm{T}, 0)}X \in (R/B)[X]$. Then $f(X)g(X) = 0$; however, we have $(0, Y)\overline{(-\mathrm{T}, 0)} \neq 0$ in R/B.

A commutative ring is *arithmetical* if its lattice of ideals is distributive, equivalently, if every finitely generated ideal is locally principal. Prüfer domains, principal ideal rings and regular rings are arithmetical. Finite products of arithmetical rings are arithmetical. It was pointed out in [**1**] (and follows from Theorem 1.3 of [**2**]) that arithmetical rings are Gaussian. In Proposition 3.8 we show that a commutative ring is arithmetical if and only if it is strongly Gaussian. We first recall a result implicit in Remark 1.5 of [**2**].

Lemma 3.7 *Let R be a commutative ring and let $f(X) \in R[X]$. The ideal A_f is locally principal if and only if for each R-module M and each $g(X) \in M[X]$ we have $A_f A_g = A_{fg}$.*

Proposition 3.8 *The commutative ring R is strongly Gaussian if and only if R is arithmetical.*

Proof. Let R be an arithmetical ring, let M be an R-module and let $f(X) \in R[X]$ and $g(X) \in M[X]$ satisfy $f(X)g(X) = 0$. Since A_f is locally principal, by Lemma 3.7 we have $A_f A_g = A_{fg} = 0$ implying that M is Armendariz. This proves the 'if' part. Next, let R be a strongly Gaussian ring and let B be a finitely generated ideal of R. Clearly $B = A_f$ for some polynomial $f(X) \in R[X]$. Let M be an R-module, let $g(X) \in M[X]$, let $N = A_{fg}$, let $\overline{\mathrm{M}}$ be the factor module M/N and let $\overline{g}(X)$ denote the residue class of $g(X)$ in $\overline{\mathrm{M}}[X] = M[X]/N[X]$. We have $A_{f\overline{g}} = 0$ implying (since $\overline{\mathrm{M}}$ is Armendariz) $0 = A_f A_{\overline{g}} = A_f A_g / A_{fg}$. Hence $A_f A_g = A_{fg}$ implying (by 3.7 again) that $B = A_f$ is locally principal. □

Remarks 3.9 (a) As a special case of Proposition 3.8 all abelian groups are Armendariz as **Z**-modules.
(b) Using 3.1, 3.8 and Theorem 12 of [**1**] we can deduce:if D is a domain then D is Prüfer $\Leftrightarrow$ D is Gaussian $\Leftrightarrow$ D is strongly Gaussian $\Leftrightarrow$ for each D-module M the idealisation $D(+)M$ is an Armendariz ring. (cf. Theorems 2.2, 2.3 and Corollary 2.9 of [**9**])
(c) It is easy to give examples of commutative Gaussian rings which are not arithmetical and therefore not strongly Gaussian. Let K be a field and V a K-vector space of dimension ≥ 2. Then the idealisation $K(+)$V is a local ring which is Gaussian [**1**] with a non-distributive lattice of ideals. Choosing K to be the field of two elements and V a vector space of dimension 2 we thus get a Gaussian ring of 8 elements which is not strongly Gaussian.

Remarks 3.10 Some background to the example studied in 3.11 is provided by the following remarks.

(a) Commutative P.I.R.s are completely Armendariz ($\equiv$ Gaussian); this follows from 3.8 and was recorded in the remarks after Corollary 9 of [**1**]. Any semi-simple (artinian) non-reduced ring is a left and right P.I.R. which is not Armendariz. Let $H = R+Ri+Rj+Rk$ be the division ring of real quaternions. In Example 3.11 we verify that the ring $H[X]$ (which is a left and right P.I.D.) is not completely abelian, and therefore is neither completely Armendariz nor completely semi-commutative.
(b) Theorem 16 of [**1**] asserts that a commutative ring R is regular if and only if $R[X]$ is completely Armendariz. Example 3.11 shows that this result does not extend to non-commutative rings.
(c) It was shown by Anderson and Camillo [**1**, Theorem 5] that if R is a reduced ring then for each positive integer n the ring $R[X]/(X^n)$ is an Armendariz ring. The example below also shows that a conjectured generalization of this result, namely, 'if R is a reduced ring then for each polynomial $f(X)$ in the centre of $R[X]$ the ring $R[X]/(f(X))$ is an Armendariz ring' is false.

Example 3.11 We show that the ring $T = H[X]/(X^2+1)$ has infinitely many non-central idempotents, and is, in particular, non-abelian. Thus $H[X]$ is not completely abelian.
(i) **Claim.** *Let α, β be real quaternions. The residue class of $\beta\ X + \alpha$ in T is an idempotent element if and only if conditions (1) below hold.*

Proof. The equality
$$\beta\ \mathrm{X} + \alpha = (\beta\mathrm{X} + \alpha)^2 \ (\mathrm{mod}\ \mathrm{X}^2 + 1)$$
holds if and only if
$$\beta\mathrm{X} + \alpha = \beta^2\mathrm{X}^2 + \beta\alpha\mathrm{X} + \alpha\beta\mathrm{X} + \alpha^2 \ (\mathrm{mod}\ \mathrm{X}^2 + 1)$$
$$\Leftrightarrow \quad 0 = -\beta^2\mathrm{X}^2 + (\beta - \beta\alpha - \alpha\beta)\mathrm{X} + \alpha - \alpha^2 \ (\mathrm{mod}\ \mathrm{X}^2 + 1)$$
$$\Leftrightarrow \quad 0 = -\beta^2(\mathrm{X}^2 + 1) + (\beta - \beta\alpha - \alpha\beta)\mathrm{X} + \alpha - \alpha^2 + \beta^2 \ (\mathrm{mod}\ \mathrm{X}^2 + 1)$$
$$\Leftrightarrow \quad 0 = (\beta - \beta\alpha - \alpha\beta)\mathrm{X} + (\alpha - \alpha^2 + \beta^2) \ (\mathrm{mod}\ \mathrm{X}^2{+}1).$$
Necessary and sufficient conditions for the last equality to hold are

$$(1) \qquad \beta - \beta\alpha - \alpha\beta = 0 \quad \text{and} \quad \alpha - \alpha^2 + \beta^2 = 0$$

(ii) **Claim.** *There are infinitely many idempotents in T of the form the residue class of $kX + \alpha$.*

Proof. Assume $\beta = k$. Then the equations in (1) reduce to

$$k - k\alpha - \alpha k = 0 \quad \text{and} \quad \alpha - \alpha^2 - 1 = 0 \tag{2}$$

Let $\alpha = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$, where the α_m $(0 \leq m \leq 3)$ are real numbers. Then easy computations show that conditions (2) hold if and only if $\alpha_0 = 1/2$, $\alpha_3 = 0$ and α_1 and α_2 satisfy $\alpha_1^2 + \alpha_2^2 = 3/4$. Since there are infinitely many pairs (α_1, α_2) of real numbers satisfying this last equation our claim is proved.

(iii) Since $j(kX + \alpha) - (kX + \alpha)j$ is never a multiple of $X^2 + 1$ it follows that for each α the residue class of $kX + \alpha$ does not belong to the centre of T.

In the next paragraph we collect some questions which arise naturally in the context of our study. A ring R is *left quasi-invariant* if every maximal left ideal of R is two-sided; it is *semi-primitive* if its Jacobson radical vanishes.

Remark 3.12 It would be of interest to get examples (if they exist) of (i) a left (strongly)Gaussian ring which is not left (quasi-)invariant (ii) a semi-primitive left Gaussian ring which is not reduced and (iii) a reduced, left Gaussian ring which is not left strongly Gaussian.

Remarks 3.13 (a) Clearly an example satisfying the conditions in (ii) will also satisfy those of (i) since quasi-invariant, semi-primitive rings are reduced by [**8**, Corollary 4.5].
(b) While in 3.12(i) we ask for an example of a left Gaussian ring which is not left quasi-invariant, the domain R considered in 2.12 is an Armendariz ring which is neither left nor right quasi-invariant (since it has the ring $M_n(K)$ as a homomorphic image).
(c) Left Gaussian rings are Armendariz, and therefore abelian; abelian rings are Dedekind finite (i.e., for elements a, b of the ring $ab = 1$ implies $ba = 1$). Left quasi-invariant rings are also Dedekind finite. It follows that left Gaussian rings and left quasi-invariant rings are sub-classes of the class of completely Dedekind finite rings. Thus we have some (albeit meagre) evidence for suspecting that(under fairly general conditions) the implication left Gaussian $\Rightarrow$ left quasi-invariant may hold.

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