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ON NORMAL FAMILIES

ABDULLAH ALOTAIBI

ABSTRACT. We prove that a family of functions meromorphic on the unit disc $B(0,1)$, such that $f(f')^m$ omits the value one, $m \geq 2$, is normal. Also, we prove that a family of functions meromorphic on $B(0,1)$, such that $f \neq 0$, $f(f^{(k)})^m$ omits the value one, $k, m \in \mathbb{N}$, is normal. Moreover, we generalize these results.

Keywords. Normal family, differential polynomial, Nevanlinna theory, meromorphic function.
MSC(2000): 30D35.

1. INTRODUCTION

In this paper, we are going to use the basic notations of Nevanlinna Theory [3], [7] such as $T(r,f)$, $\bar{N}(r,f)$, $m(r,f)$ and $S(r,f) = o(T(r,f))$.

Theorem 1.1. *Let n be a positive integer, and let $\mathcal{F}$ be a family of meromorphic functions such that $f'f^n$ omits the value one on $B(0,1)$, for each $f \in \mathcal{F}$. Then $\mathcal{F}$ is normal on $B(0,1)$.*

The proof of Theorem 1.1 is due to Yang and Chang [8] for $n \geq 5$, Ku [5] for $n = 3,4$, Pang [9] for $n = 2$, and Bergweiler and Eremenko [2] for $n = 1$. In 1979, Ku [5] proved the following theorem.

Theorem 1.2. *Let k be a positive integer, and let $\mathcal{F}$ be a family of meromorphic functions f on $B(0,1)$ such that $f \neq 0$, $f^{(k)}$ omits the value one on $B(0,1)$. Then $\mathcal{F}$ is normal on $B(0,1)$.*

For proving the normality, we will mainly use the following Zalcman Lemma [10].

Lemma 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc $B(0,1)$ such that all zeros of functions in $\mathcal{F}$ have multiplicities greater than or equal to l and all poles of functions in $\mathcal{F}$ have multiplicities greater than or equal to j. Let α be a real number satisfying $-l < \alpha < j$. Then $\mathcal{F}$ is not normal in any neighborhood of $z_0 \in B(0,1)$ if and only if there exist*

Lemma 1.2.
(1) *points $z_n \in B(0,1)$, $z_n \to z_0$,*
(2) *positive numbers ρ_n, $\rho_n \to 0$,*
(3) *functions $f_n \in \mathcal{F}$,*
(4) *a non-constant meromorphic function g,*

such that $g_n(z) = \rho_n^\alpha f_n(z_n + \rho_n z) \to g(z)$ locally uniformly on $\mathbb{C}$ with respect to the spherical metric.

Lemma 1.1 holds without any restrictions on the zeros and the poles of all $f \in \mathcal{F}$ for $-1 < \alpha < 1$. However, we can take $-1 < \alpha < \infty$ for a family of analytic functions. Also, we can take $-\infty < \alpha < 1$ for a family of meromorphic functions which do not vanish [10].

2. NECESSARY THEOREMS

A. Alotaibi [1] proved the following theorem.

Theorem 2.1. *Let m, k be two positive integers with $m \geq 2$, and let f be a transcendental meromorphic function. Then $f\left(f^{(k)}\right)^m - 1$ has infinitely many zeros in $\mathbb{C}$.*

J. Hinchliffe [4] proved the following theorem.

Theorem 2.2. *Let f be a transcendental meromorphic function, let $P[.]$ be a non-constant differential polynomial defined by*

$$P[f](z) = \sum_{k=1}^{n} a_k(z) \prod_{j=0}^{p} \left(f^{(j)}(z)\right)^{S_{kj}},$$

with $a_k \not\equiv 0$ *and*

$$\underline{d}(P[.]) = \min_{1\le k\le n}\left\{\sum_{j=0}^{p} S_{kj}\right\} \ge 2.$$

Let

$$Q = \max_{1\le k\le n}\left\{\sum_{j=1}^{p} jS_{kj}\right\}.$$

Then

$$T(r,f) \le \frac{Q+1}{\underline{d}(P[.])-1}\bar{N}(r,\frac{1}{f}) + \frac{1}{\underline{d}(P[.])-1}\bar{N}(r,\frac{1}{P[f]-1}) + S(r,f).$$

Corollary 2.1. *Let f be a transcendental meromorphic function with no zeros, and let $P[.]$ be a non-constant differential polynomial with $\underline{d}(P[.]) \ge 2$. Then $P[f]-1$ has infinitely many zeros in $\mathbb{C}$.*

We require the following result on rational functions.

Theorem 2.3. *Let k, m be two positive integers and let g be a non-constant rational function. Then either $g^{(k)} \equiv 0$ or $g(z)(g^{(k)}(z))^m = 1$ has at least one solution $z \in \mathbb{C}$.*

Proof. We may replace one by any non-zero complex number. Assume that $g^{(k)} \not\equiv 0$. Then $g \not\equiv 0$ and so $R(z) = g(z)(g^{(k)}(z))^m$ is not identically zero. We fiest show that

$$R(\infty) = 1 \tag{2.1}$$

is impossible. Assume that (2.1) holds. Thus $g(z)(g^{(k)}(z))^m \to 1$ as $z \to \infty$. Hence $g(\infty) = \infty$, since if not, we would have $g(\infty) \in \mathbb{C}$ and so $g'(\infty) = 0$ and then $(g^{(k)})^m(\infty) = 0$. This gives $g(g^{(k)})^m \to 0$ which is a contradiction. It is easy to see that moreover R cannot be constant. Since $g(\infty) = \infty$, we get $(g^{(k)})^m(\infty) = 0$. Using the Laurent expansion, we get

$$g(z) = c_n z^n + \cdots + c_0 + \frac{d_1}{z} + \cdots \quad \text{as } z \to \infty,$$

$$g^{(k)}(z) = n(n-1)\cdots(n-k+1)c_n z^{n-k} + \cdots + (-1)^k \frac{k!d_1}{z^{k+1}} + \cdots.$$

But $(g^{(k)})^m(\infty) = 0$, so $g^{(k)}(\infty) = 0$ and this gives

$$(g^{(k)})^m(z) = (-1)^{mk}\frac{(k!d_1)^m}{z^{m(k+1)}} + \cdots \quad \text{as } z \to \infty.$$

Since $g(z)(g^{(k)})^m(z) \to 1$ as $z \to \infty$, we have

$$g(z) = (-1)^{mk}\frac{z^{m(k+1)}}{(k!d_1)^m} + \cdots \quad \text{as } z \to \infty,$$

and this gives, since $m(k+1) > k$. $g^{(k)}(z) = \frac{(-1)^{mk}}{(k!d_1)^m}(mk+m)(mk+m-1)\cdots(mk+m-k+1)z^{m(k+1)-k} + \cdots$ as $z \to \infty$,

$$(g^{(k)})^m(z) = \left(\frac{(-1)^{mk}}{(k!d_1)^m}(mk+m)(mk+m-1)\cdots(mk+m-k+1)\right)^m \times z^{m^2(k+1)-mk} + \cdots \quad \text{as } z \to \infty.$$

Hence, $(g^{(k)})^m(\infty) = \infty$ which is a contradiction. So $R(z)$ cannot be a non-zero constant. Suppose now that $R(z)$ is non-constant, but never takes the value 1 in $\mathbb{C}$. Then $\frac{1}{R-1} \neq \infty$ on $\mathbb{C}$. So, $\frac{1}{R-1} = p$, where P is polynomial, $R - 1 = \frac{1}{P}$, $R = 1 + \frac{1}{P}$.

But this gives $R(\infty) = 1$, and we have already excluded this case. This completes the proof. □

3. THE NORMALITY WHEN $f(f' + af)^m$ OMITS THE VALUE ONE , $m \geq 2$

Theorem 3.1. *Let m be a positive integer with $m \geq 2$, and let a be an analytic function on $B(0,1)$. Suppose that $\mathcal{F}$ is a family of functions meromorphic on $B(0,1)$, such that for each $f \in \mathcal{F}$, $f(f'+af)^m$ omits the value one on $B(0,1)$. Then $\mathcal{F}$ is normal on $B(0,1)$.*

Proof. Suppose that $\mathcal{F}$ is not normal on $B(0,1)$. So, $\mathcal{F}$ is not normal at at least one point in $B(0,1)$, say z_0. Using the Zalcman Lemma, with $\alpha = \frac{-m}{m+1} \in (-1,1)$, there exist

(1) $z_n \in B(0,1)$, $z_n \to z_0$;
(2) positive numbers ρ_n, $\rho_n \to 0$;
(3) functions $f_n \in \mathcal{F}$;

(4) a non constant meromorphic function g;

such that

$$g_n(z) = \rho_n^{\alpha} f_n(z_n + \rho_n z) \to g(z) \tag{3.1}$$

locally uniformly on $\mathbb{C}$ with respect to the spherical metric. Let $P = g^{-1}(\{\infty\})$ be the set of all poles of g. So, $g_n' \to g'$ locally uniformly on $\mathbb{C}\backslash P$. Using (3.1), we have $g_n'(z) = \rho_n^{\alpha+1} f_n'(z_n + \rho_n z)$. Thus,

$$\begin{aligned}
&f_n(z_n + \rho_n z)\left(f_n'(z_n + \rho_n z) + a(z_n + \rho_n z) f_n(z_n + \rho_n z)\right)^m \\
&= \rho_n^{-\alpha} g_n(z)(\rho_n^{-\alpha-1} g_n'(z) + a(z_n + \rho_n z)\rho_n^{-\alpha} g_n(z))^m \\
&= \rho_n^{-\alpha} g_n(z) \rho_n^{-m(\alpha+1)}(g_n'(z) + \rho_n a(z_n + \rho_n z) g_n(z))^m \\
&= \rho_n^{-\alpha(m+1)-m} g_n(z)(g_n'(z) + \rho_n a(z_n + \rho_n z) g_n(z))^m \\
&= g_n(z)(g_n'(z) + \rho_n a(z_n + \rho_n z) g_n(z))^m \\
&\to g(z)(g'(z) + 0.a(z_0) g(z))^m \\
&= g(z)(g'(z))^m.
\end{aligned}$$

Using Theorem 2.1 and Theorem 2.3 , we have at least one $\zeta_0 \in \mathbb{C}$ with $g(\zeta_0)(g'(\zeta_0))^m = 1$, and $\zeta_0 \notin P$. Now let us prove that $g(g')^m \not\equiv 1$. Suppose that $g(g')^m \equiv 1$. Then $g^{\frac{1}{m}} g' = 1$. By integrating both sides, we get $g^{1+\frac{1}{m}} = z + \beta$, $\beta \in \mathbb{C}$. Hence $g^{m+1} = (z+\beta)^m$. This gives a contradiction since the zeros of the left hand side have multipicities greater than or equal to $m+1$ and the zero of the right hand side has multiplicity equals to m. Applying Hurwitz' theorem, there exist points $\zeta_n \to \zeta_0$ with

$$g_n(\zeta_n)(g_n'(\zeta_n) + \rho_n a(z_n + \rho_n \zeta_n) g_n(\zeta_n))^m = 1.$$

Thus,

$$f_n(z_n + \rho_n \zeta_n)(f_n'(z_n + \rho_n \zeta_n) + a(z_n + \rho_n \zeta_n) f_n(z_n + \rho_n \zeta_n))^m = 1.$$

But $z_n + \rho_n \zeta_n \in B(0,1)$ since $z_n \to z_0 \in B(0,1)$ and $\rho_n \to 0$. So for sufficiently large n, the functions $f_n(f_n' + a f_n)^m$ can take the value 1 in $B(0,1)$. This give us a contradiction. Hence, $\mathcal{F}$ is normal. □

Corollary 3.1. *Let m be a positive integer with $m \geq 2$. Suppose that $\mathcal{F}$ is a family of functions meromorphic on $B(0,1)$, such that for each $f \in \mathcal{F}$, $f(f')^m$ omits the value one on $B(0,1)$. Then $\mathcal{F}$ is normal on $B(0,1)$.*

We cannot let $a(z)$ be a meromorphic function in Theorem 3.1. The counter example is the following.

Example 3.1. Let n be a positive integer, and let $f(z) = \frac{1}{nz}$. This gives $f'(z) = \frac{-1}{nz^2}$. Let $a(z) = \frac{1}{z}$ which is a meromorphic function in $B(0,1)$. From all of these, we get $(f' + af)(z) = \frac{-1}{nz^2} + \frac{1}{z}\frac{1}{nz} = \frac{-1}{nz^2} + \frac{1}{nz^2} = 0$. Hence, $f(f' + af)^2 = 0 \neq 1$ on $B(0,1)$.However, $\mathcal{F} = \{\frac{1}{nz} : \ n \in \mathbb{N}\}$ is not normal on $B(0,1)$.

4. THE NORMALITY WHEN $P[f]$ OMITS THE VALUE ONE AND $f \neq 0$

Lemma 4.1. *Let g be a non constant rational function such that g is never 0 in the plane. Let*

$$Q(\zeta) = \prod_{j=0}^{p} g^{(j)}(\zeta)^{S_{\delta,j}}, \quad S_{\delta,j} \geq 0, \ \sum_{j=0}^{p} S_{\delta,j} \geq 1.$$

Let $\alpha \in \mathbb{C}\backslash\{0\}$. Then Q takes the value α at least once in $\mathbb{C}$.

Proof. We are given that g is never 0. This gives $g(\infty) = 0$, and so $g^{(j)}(\infty) = 0$. Thus $Q(\infty) = 0$. So provided Q is not constant, Q takes the value $\alpha \in \mathbb{C}\backslash\{0\}$. If Q is constant, $Q \equiv 0$. Hence, $g^{(j)} \equiv 0$ for some j. The Laurent expansion of g at ∞ shows that $g \equiv 0$. This is a contradiction and hence Lemma 4.1 is proved. □

Theorem 4.1. *Let m, p be two positive integers and let $T = \{1, 2, \cdots, m\}$. For each $k \in T$, and each $j \in \{0, 1, \cdots, p\}$ let $S_{k,j}$ be a non-negative integer and assume that*

$$\sum_{j=0}^{p} S_{k,j} \geq 2.$$

For each $k \in T$ let α_k be the solution of the following equation:

$$\sum_{j=0}^{p} S_{k,j}(-\alpha_k - j) = 0. \tag{4.1}$$

Assume that there is a unique $\delta \in T$ such that $\alpha_\delta < \alpha_k$ for all $k \in T\backslash\{\delta\}$, and assume that

$$\sum_{j=0}^{p} S_{k,j} \geq 1 \quad \forall\, k \in T\backslash\{\delta\}. \tag{4.2}$$

For each $k \in T$ let $a_k(z)$ be an entire function on, and assume further that $a_\delta(z)$ has no zeros in $B(0,1)$. Let $\mathcal{F}$ be a family of functions meromorphic on $B(0,1)$ such that, for each $f \in \mathcal{F}$, f has no zeros in $B(0,1)$ and the function $P[f]$ defined by

$$P[f](z) = \sum_{k=1}^{n} a_k(z) \prod_{j=0}^{p} (f^{(j)}(z))^{S_{k,j}}$$

does not take the value 1 in $B(0,1)$. Then $\mathcal{F}$ is normal on $B(0,1)$.

Proof. Suppose that $\mathcal{F}$ is not normal on $B(0,1)$. So, it is not normal at at least one point on $B(0,1)$, say z_0. We note that $\alpha_\delta \leq 0$ by (4.1). Applying the Zalcman Lemma, with $\alpha = \alpha_\delta$, we find that there exist

(1) $z_n \in B(0,1)$, $z_n \to z_0$,
(2) positive numbers ρ_n, $\rho_n \to 0$,
(3) functions $f_n \in \mathcal{F}$,
(4) a non constant function g meromorphic in $\mathbb{C}$,

such that

$$g_n(z) = \rho_n^\alpha f_n(z_n + \rho_n z) \to g(z) \tag{4.3}$$

locally uniformly on $\mathbb{C}$ with respect to the spherical metric. Since g is non-constant and each g_n omits 0 on $B(0,1)$, it follows from Hurwitz' theorem that g omits the value 0 on $\mathbb{C}$. Let $R = g^{-1}(\{\infty\})$ be the set of all poles of g. Thus, $g_n^{(j)} \to g^{(j)}$ locally uniformly on $\mathbb{C}\backslash R$, for $j = 1, 2, \cdots, p$. Using (4.3), we have

$$g_n^{(j)} = \rho_n^{\alpha+j} f_n^{(j)}(z_n + \rho_n z), \quad j = 1, 2, \cdots, p.$$

Hence, locally uniformly on $\mathbb{C}\backslash R$,

$$\begin{aligned}
&\sum_{k=1}^{m} a_k(z_n+\rho_n z)\prod_{j=0}^{p}(f_n^{(j)}(z_n+\rho_n z))^{S_{k,j}} \\
&\qquad = \sum_{k=1}^{m} a_k(z_n+\rho_n z)\prod_{j=0}^{p}\rho_n^{S_{k,j}(-\alpha-j)}(g_n^{(j)}(z))^{S_{k,j}} \\
&\qquad = \sum_{k=1}^{m} a_k(z_n+\rho_n z)\rho_n^{\sum_{j=0}^{p}S_{k,j}(-\alpha-j)}\prod_{j=0}^{p}(g_n^{(j)}(z))^{S_{k,j}} \\
&= a_\delta(z_n+\rho_n z)\prod_{j=0}^{p}(g_n^{(j)}(z))^{S_{\delta,j}} + \sum_{k\in T,\, k\neq\delta} a_k(z_n+\rho_n z)\rho_n^{\sum_{j=0}^{p}S_{k,j}(-\alpha_\delta-j)}\times \\
&\qquad\qquad \prod_{j=0}^{p}\left(g_n^{(j)}(z)\right)^{S_{k,j}} \to a_\delta(z_0)\prod_{j=0}^{p}\left(g^{(j)}(z)\right)^{S_{\delta,j}}
\end{aligned}$$

since $\rho_n \to 0$, $z_n \to z_0 \in B(0,1)$, each a_k is analytic on $B(0,1)$, and for $k \neq \delta$

$$\sum_{j=0}^{p} S_{k,j}(-\alpha_\delta - j) = \sum_{j=0}^{p} S_{k,j}(\alpha_k - \alpha_\delta) > 0,$$

using (4.1) and (4.2). Using Corollary 2.1 and Lemma 4.1, we have at least one $\zeta_0 \in \mathbb{C}$ with

$$a_\delta(z_0)\prod_{j=0}^{p}\left(g^{(j)}(\zeta_0)\right)^{S_{\delta,j}} = 1,$$

and $\zeta_0 \notin R$. Applying Hurwitz' theorem, we get points $\zeta_n \to \zeta_0$ with

$$\begin{aligned}
&a_\delta(z_n+\rho_n\zeta_n)\prod_{j=0}^{p}\left(g_n^{(j)}(\zeta_n)\right)^{S_{\delta,j}} + \\
&\qquad \sum_{k\in T,\, k\neq\delta} a_k(z_n+\rho_n\zeta_n)\rho_n^{\sum_{j=0}^{p}S_{k,j}(-\alpha_\delta-j)}\prod_{j=0}^{p}\left(g_n^{(j)}(\zeta_n)\right)^{S_{k,j}} = 1,
\end{aligned}$$

Thus, $\sum_{k=1}^{n} a_k(z_n+\rho_n\zeta_n)\prod_{j=0}^{p}\left(f_n^{(j)}(z_n+\rho_n\zeta_n)\right)^{S_{k,j}} = 1$.But $z_n+\rho_n\zeta_n \in B(0,1)$ since $z_n \to z_0 \in B(0,1)$, and $\rho_n \to 0$. So, we have functions $f_n \in \mathcal{F}$ such that $P[f_n] = \sum_{k=1}^{n} a_k \prod_{j=0}^{p}\left(f_n^{(j)}\right)^{S_{kj}}$ takes the value 1 on $B(0,1)$. This gives a contradiction and so $\mathcal{F}$ is normal. □

Corollary 4.1. *Let k, m be two positive integers, and let f be a meromorphic function. Let $\mathcal{F}$ be a family of functions meromorphic on $B(0,1)$ such that, for each $f \in \mathcal{F}$, $f \neq 0$ and $f(f^{(k)})^m$ omits the value one on $B(0,1)$. Then $\mathcal{F}$ is normal on $B(0,1)$.*

We cannot omit the condition that $a_\delta(z)$ has no zeros in Theorem 4.1. The counter example is the following.

Let n be a positive integer, and let $P[f](z) = a(z)f(z)f'(z)$, where $f(z) = \frac{1}{nz} \neq 0$ on $B(0,1)$. This gives $f'(z) = \frac{-1}{nz^2}$. Let $a(z) = \frac{z^3}{z^4+10}$. From all of these, we get

$$\begin{aligned} P[f](z) &= a(z)f(z)f'(z) \\ &= \frac{z^3}{z^4+10}\frac{1}{nz}\frac{-1}{nz^2} \\ &= \frac{-1}{n^2(z^4+10)} \\ &\neq 1 \quad \text{on} \quad B(0,1). \end{aligned}$$

However, $\mathcal{F} = \{\frac{1}{nz} : \quad n \in \mathbb{N}\}$ is not normal on $B(0,1)$.

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School of Mathematics, University of Nottingham, NG7 2RD, UK.

e-mail: pmxaa@maths.nottingham.ac.uk

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