Arab J. Math. Sc.
Volume 3, Number 1, June 1997
pp. 37-47

CHARACTERIZATION OF EXPONENTIAL MIXTURES BY MEANS OF A PREDICTOR OF THE FAILURE RATE AND EXPECTATIONS OF ORDER STATISTICS

A.N. AHMED

ABSTRACT. Discrete mixtures of exponential random variables X_1 and X_2 are identified in terms of (i) relations between the best predictor of max(X_1,X_2) given min (X_1 ,X_2) and the failure rate of the distribution, and (ii) relations between conditional expectation of some appropriate functions of order statistics on some others and the failure rate of the distribution.

1. INTRODUCTION

Let X_1 and X_2 be independent and identically distributed (i.i.d) non-degenerate random variables. Put Y_1 =min (X_1,X_2) and Y_2 =max (X_1,X_2). Consider the problem of predicting the value of Y_2 when the observed value of Y_1 is known. This problem is of some importance in life testing, and replacement policy situations where it is desired to predict the time of failure from times of the early failures in the same sample. As is well known, the almost surely (a.s.) unique best mean squared error predictor of Y_2 given $Y_1 = y$ is $E(Y_2|Y_1 = y)$. Kirmani and Alam (1980) proved that if the support of X_1 and X_2 is the set of all positive integers then $E(Y_2|Y_1 = y) = \alpha + y, y = 1, 2, \cdots$ if and only if (iff) X_1 and X_2 have the geometric distributions. Nagaraja (1988) has considered some generalizations of this result. Ahmed and Yehia (1993) have recently provided an interesting extension of this when X_1 and X_2 have been taken from a mixture of two geometric distributions.

1991 Mathematics Subject Classification. Primary 53C40, Secondary 53C42, 53C20

The analog of Kirmani and Alam (1980) result is proved for the exponential distribution in Dallas (1973).

Dallas (1987) characterized the geometric distribution in terms of $E[(Y_2 - Y_1)^2|Y_1 = y]$. This result was then extended to the mixture of two geometric distributions by Ahmed and Yehia (1993).

The object of this paper is to consider the prediction when X_1 and X_2 have been taken from a mixture of two exponential distributions.

We prove that if $\phi(x) = f(x)/P(X \geq x)$ is the failure rate of X, then

$$E(Y_2|Y_1 = y) = y + a + b - ab\phi(y) \quad \text{iff } X_1 \text{ has the p.d.f.}$$

$$f(y) = \frac{\lambda}{a}e^{-\frac{y}{a}} + (1-\lambda)(1/b)e^{-\frac{y}{b}},\ y > 0, a > 0, b > 0, \text{ and } 0 < \lambda < 1.$$

Thus, we provide an extension of Dallas' (1973) result for the exponential distribution. In section 3, we prove that

$$E[(Y_2 - Y_1)^2|Y_1 = y] = 2(a^2 + b^2) + 2ab - 2ab(a+b)\phi(y),$$

for all $y > 0$ iff there exists $\lambda \in (0,1), a > 0$ and $b > 0$ such that

$$Q(x) = \lambda e^{-\frac{x}{a}} + (1-\lambda)e^{-\frac{x}{b}}, \quad x > 0,$$

characterizes mixtures of two exponential distributions. This result is an extension of a result in Gupta (1984) for the exponential distribution.

The form of predictors of order statistics have also been used by Kaminsky and Nelson (1975) to characterize the geometric distribution and some continuous distributions. As for the various characterizations of the exponential distribution, reference may be made to Azlarov and Voldin (1986), Galambos and Kotz (1978), Nagaraja (1988), Nassar and Mahmoud (1985) and the references contained in Dallas (1987).

2. THE FIRST CHARACTERIZATION THEOREM

Let X_1 and X_2 be i.i.d random variables taking positive values with common probability density function (p.d.f) $f(x)$. Let $Q(x) = P(X \geq x), \phi(x) = f(x)/Q(x)$ be the failure rate of X_1 and suppose that $E(X_1) < \infty$. Writing $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$, we have the following result.

Theorem 2.1.

$$E(Y_2|Y_1 = y) = y(a+b) - ab\phi(y), \quad y > 0 \tag{2.1}$$

if and only if for $0 < \lambda < 1, X_1$ *has the mixture p.d.f.*

$$f(y) = \lambda/ae^{-\frac{y}{a}} + (1-\lambda)(1/b)e^{-\frac{y}{b}}, \quad y > 0 \tag{2.2}$$

We shall need the following lemma.

Lemma 2.2. *Let* X_1 *have the p.d.f. (2.2). Then*

$$\lambda(x+a)e^{-\frac{x}{a}} + (1-\lambda)(x+b)e^{-\frac{x}{b}} = (x+a+b)Q(x) - abf(x)$$

Proof. Observe that

$$\begin{aligned}
&\lambda(x+a)e^{-\frac{x}{a}} + (1-\lambda)(x+b)e^{-\frac{x}{b}} \\
&= \lambda ae^{-\frac{x}{a}} + (1-\lambda)be^{-\frac{x}{b}} + \lambda xe^{-\frac{x}{a}} + (1-\lambda)xe^{-\frac{x}{b}} \\
&= xQ(x) + (a+b)Q(x) - \lambda be^{-\frac{x}{a}} - (1-\lambda)ae^{-\frac{x}{b}} \\
&= xQ(x) + (a+b)Q(x) - abf(x) \\
&= (x+a+b)Q(x) - abf(x),
\end{aligned}$$

as desired.

Proof of Theorem 2.1. Suppose first that X_1 has the p.d.f. (2.2)

Then, for $x > 0$,

$$\begin{aligned} E(Y_2|Y_1 = y) &= \int_y^\infty y_2 f_{Y_1,Y_2}(y_1, y_2)\frac{dy_2}{f_{Y_1}}(y) \\ &= \int_y^\infty \frac{2y_2 f_{X_1}(y) f_{X_2}(y_2)}{2f_{X_2}(y)Q(y)} dy_2 \\ &= \int_y^\infty y_2 f_{X_1} f(y_2)\frac{dy_2}{Q(y)} \\ &= \frac{\lambda(y+a)e^{-\frac{y}{a}} + (y+b)(1-\lambda)e^{-\frac{y}{b}}}{Q(y)} \\ &= y + (a+b) - ab\frac{f(y)}{Q(y)}, \text{ by Lemma (2.2)} \\ &= y + (a+b) - ab\phi(y). \end{aligned}$$

To prove the converse, suppose that (2.1) holds.

Then

$$\int_x^\infty y f_{X_1}(y)dy = xQ(x) + (a+b)Q(x) - abf(x). \tag{2.3}$$

Differentiating (2.3) twice with respect to x gives

$$ab\frac{d^2 f(x)}{dx^2} + (a+b)\frac{df(x)}{dx} + f(x) = 0, \tag{2.4}$$

which is a homogeneous second order differential equation whose characteristic function is

$$abm^2 + (a+b)m + 1 = 0,$$

having roots $m = -1/a$ and $m = -1/b$.

Thus the solution of (2.4) is

$$f_{X_2}(x) = A\frac{1}{a}e^{-\frac{x}{a}} + B\frac{1}{b}e^{-\frac{x}{b}},$$

where A,B are two arbitrary constants and a,b>0.

Now, using the initial condition that

$$\int_0^\infty f_{X_1} f(x) dx = 1$$

we get that

$$A + B = 1.$$

Hence, the result follows.

Remark 2.3. Theorem 2.1 may be used to generalize the interesting results of Dallas (1973) and Nagaraja (1988). To this end, set $a = b = \gamma$ in (2.1) and (2.2) yielding that

$$E(Y_2|Y_1 = y) = y + \gamma.$$

3. THE SECOND CHARACTERIZATION THEOREM

Let X be a continuous non-negative random variable with p.d.f. $f(x)$ for $x > 0$ and let $Q(x) = P(X \geq x)$.

Consider a random sample of size $n = 2$ from this distribution and denote by Y_2 the maximum and by Y_1 the minimum of the sample. Gupta (1984) proved the following theorem.

Theorem 3.1 *(Gupta, 1984). Let Y_1 and Y_2 be as above and assume that $E(X^2) < \infty$ and $f(x) > 0$ for all $x \geq 0$. Then*

$$E[(Y_2 - Y_1)^2|Y_1 = y] = c, \qquad \text{for all } y > 0,$$

where $c \neq 0$ is a constant independent of y iff there exists $a > 0$ such that

$$Q(x) = e^{-\frac{x}{a}}, \; x > 0.$$

In our next result, we extend the above theorem to the case when X is a discrete mixture of two exponential distributions. We have been primarily motivated by practical situations in many life testing problems. Note that Y_1 and $Y_2 - Y_1$ are no longer independent.

Theorem 3.2. *Let Y_1, Y_2 be as above and assume that $E(X^2) < \infty$ and $f(x) > 0$ for all $x > 0$. Then*

$$E[(Y_2 - Y_1)^2 | Y_1 = y] = 2(a^2 + b^2) + 2ab - 2ab(a+b)\phi(y) \tag{3.1}$$

for all $y > 0$ iff there exists $\lambda \in (0,1), a > 0$ and $b > 0$ such that

$$f(y) = \frac{\lambda}{a} e^{-\frac{y}{a}} + \frac{1-\lambda}{b} e^{-\frac{y}{b}}, \; y > 0.$$

Proof. Assume that X_1 has the mixture density (2.2). Note that

$$\begin{aligned} &E[(Y_2 - Y_1)^2 | Y_1 = y] \\ &= E(Y_2^2 | Y_1 = y) + y^2 - 2yE[Y_2 | Y_1 = y] \end{aligned}$$

Also,

$$\begin{aligned} &E(Y_2^2 | Y_1 = y) \\ &= \frac{1}{Q(y)} \int_y^\infty z^2 \left[\frac{\lambda}{a} e^{-\frac{z}{a}} + \frac{1-\lambda}{b} e^{-\frac{z}{b}} \right] dz. \end{aligned} \tag{3.2}$$

Integrating the right side of (3.2) by parts, gives

$$\begin{aligned} E(Y_2^2 | Y_1 = y) &= y^2 + \tfrac{2y}{Q(y)} \left[a\lambda e^{-\frac{y}{a}} + b(1-\lambda) e^{-\frac{y}{b}} \right] \\ &= \frac{2}{Q(y)} \left[a^2 \lambda e^{-\frac{y}{a}} + b^2 (1-\lambda) e^{-\frac{y}{b}} \right]. \end{aligned}$$

Using Theorem 2.1, we have

$$\begin{aligned} &E[(Y_2 - Y_1)^2 | Y_1 = y] = \\ &= \frac{2y}{Q(y)} U + \frac{2}{Q(y)} V - 2y(a+b) + 2yab\phi(y) \end{aligned} \tag{3.3}$$

where

$$U = a\lambda e^{-\frac{y}{a}} + b(1-\lambda) e^{-\frac{y}{b}},$$

and

$$V = a^2 \lambda e^{-\frac{y}{a}} + b^2 (1-\lambda) e^{-\frac{y}{b}}$$

Substituting for U and V in (3.3) and rearranging terms cancelling out similar terms gives

$$E[(Y_2 - Y_1)^2 | Y_1 = y]$$

$$(3.4) = 2yab\phi(y) + \frac{2(a^2 - by)\lambda e^{-\frac{y}{a}} + 2(b^2 - ay)(1-\lambda)e^{-\frac{y}{b}}}{Q(y)}$$

$$= 2yab\phi(y) + 2a^2 + 2b^2 - 2ab\frac{\lambda(b+y)\frac{1}{a}e^{-\frac{y}{a}} + (1-\lambda)\frac{1}{b}(a+y)e^{-\frac{y}{b}}}{Q(y)}$$

$$= 2yab\phi(y) + 2a^2 + 2b^2 - 2aby\phi(y) - 2ab\frac{\frac{\lambda b}{a}e^{-\frac{y}{a}} + \frac{(1-\lambda)a}{b}\frac{1}{b}e^{-\frac{y}{b}}}{Q(y)}$$

$$= 2a^2 + 2b^2 - 2ab\frac{\frac{\lambda b}{a}e^{-\frac{y}{a}} + \frac{(1-\lambda)a}{b}\frac{1}{b}e^{-\frac{y}{b}}}{Q(y)}$$

Let

$$\frac{2ab\left[\frac{\lambda}{a}be^{-\frac{y}{a}} + \frac{(1-\lambda)a}{b}e^{-\frac{y}{b}}\right]}{Q(y)} = A + B\phi(y).$$

Multiplying both sides of (3.4) by $Q(y)$ and comparing the coefficients of λ and the coefficients of $1-\lambda$ in both sides, gives

$$(3.5) \qquad 2b^2 = A + \frac{B}{a},$$

and

$$(3.6) \qquad 2a^2 = A + \frac{B}{b}.$$

Solving (3.5) and (3.6) for A and B yields

$$B = 2ab(a+b)$$

and

$$A = -2ab$$

Therefore,

$$(3.7) \quad E[(Y_2 - Y_1)^2 | Y_1 = y) = 2a^2 + 2b^2 + 2ab - 2ab(a+b)\phi(y).$$

Conversely, assume that (3.1) holds, i.e.,

$$E[(Y_2 - Y_1)^2 | Y_1 = y) = 2(a^2 + b^2) + 2ab - 2ab(a+b)\phi(y). \tag{3.8}$$

Rewrite (3.8) in terms of $f(x)$ and $Q(x)$, we get

$$\begin{aligned} &\frac{1}{Q(y)} \int_y^\infty z^2 f_{X_1}(z)dz + y^2 - \frac{2y}{Q(y)} \int_y^\infty z f_{X_1}(z)dz \\ &= 2(a^2 + b^2) + 2ab - 2ab(a+b) f_{X_1}(y)/Q(y). \end{aligned} \tag{3.9}$$

Multiplying both sides of (3.9) by $Q(y)$ and then differentiating the resulting equation twice with respect to y, gives

$$ab(a+b)\frac{d^2 f(y)}{dy^2} + (a^2 + b^2 + ab)\frac{df(y)}{dy} + Q(y) = 0.$$

Differentiating once more with respect to y, gives

$$ab(a+b)\frac{d^3 f(y)}{dy^3} + (a^2 + b^2 + ab)\frac{d^2 f(y)}{dy^2} - f(y) = 0.$$

which is a third order differential equation whose characteristic equation is

$$ab(a+b)m^3 + (a^2 + b^2 + ab)m^2 - 1 = 0,$$

or

$$[(a+b)m - 1][abm^2 + (a+b)m + 1] = 0,$$

whose valid roots are $-1/a$ and $-1/b$. Thus, the general solution, using the initial condition

$$\int_0^\infty f_{X_1}(x)dx = 1, \text{ is}$$

$$f_{X_1}(x) = \frac{\lambda}{a} e^{-\frac{x}{a}} + \frac{(1-\lambda)}{b} e^{-\frac{x}{b}}, \; x > 0$$

and $0 < \lambda < 1$.

This completes the proof.

Remark 3.3. Theorem 3.2 may serve as a generalization of Gupta's (1984) and Nagaraja's (1988) characterization of the exponential distribution. This is easily seen by setting $a = b = \gamma$ in (3.1) yielding $\phi(y) = 1/\gamma$, hence the mentioned result.

4. EXTENSIONS AND POSSIBLE APPLICATIONS

Let $X_1, \ldots, X_n$ be independent and identically distributed. Denote by $Y_1 < Y_2 < \ldots < Y_n$ the ordered sample. In practice, $Y_1, \ldots, Y_n$ may represent the failure times of the components of a system composed of n units. If all of the components are connected in parallel, the system will continue to operate as long as at least one of its components has not yet failed. Thus at the instant of the failure of $(n-1)$st component the engineer might be wondering as to when the system will fail. On average, this time is given by $E(Y_n|Y_{n-1} = y)$.

To this end, we note that it can be shown that, for any integer $1 \leq m \leq n-1$,

$$\begin{aligned} E(Y_{m+1}|Y_m = x) &= \frac{n-m}{\{\bar{F}(x)\}^{n-m}} \int_x^\infty y f(y) \{\bar{F}(y)\}^{n-m-1} dy \\ &= x + \text{ the MRL of } \mathrm{Max}\{X_j : 1 \leq j \leq n-1\}, \end{aligned}$$

where MRL stands for the mean residual life.

Corollary 4.1. *If* $m = n-1$, *then*

$$\begin{aligned} E(Y_n|Y_{n-1} = x) &= \frac{1}{\bar{F}(x)} \int_x^\infty y f(y) dy = E(X_1|X_1 > x) \\ &= x + (a+b) - ab\phi(x) \text{ for all } x > 0 \text{ and } n = 2, 3 \ldots, \end{aligned}$$

if and only if X_1 *has the mixture p.d.f.*

$$f(y) = \frac{\lambda}{a} e^{-\frac{y}{a}} + \frac{1-\lambda}{b} e^{-\frac{y}{b}}, \quad y > 0, 0 \leq \lambda \leq 1, \qquad a > 0, b > 0.$$

Remark 4.2. The MRL of Max $\{X_j : 1 \leq j \leq n-i\}$ is the mean residual life of the maximum observation in a sample of size $n-i$. This is of course, very useful particularly in destructive tests when i is "Close" to n. The closer the value of i is to the value of n the smaller is the size of the burned sample used to predict the mean time to failure of the $(m+1)$st component when the failure time of the preceding components are known.

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THE INSTITUTE OF STATISTICAL STUDIES AND RESEARCH, CAIRO UNIVERSITY

Date received September 14, 1996.