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ENDOMORPHISMS OF A MULTIPLICATION MODULE

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ABSTRACT. Let R be a commutative ring and M be a faithful multiplication $R-$module. The ideal $\theta(M) = \sum_{x \in M} (Rx : M)$ of R has proved to be useful in studying multiplication modules. We prove that if $I, J \subseteq \theta(M)$ are ideals of R, then $Hom_R(M/IM, M/JM) \cong Hom_R(\theta(M)/I, \theta(M)/J)$ and $Hom_R(IM, JM) \cong Hom_R(I, J)$. As applications of these results, we prove the following: (i) M/IM is quasi-projective if and only if $\theta(M)/I$ is quasi-projective. (ii) If R/I is self-injective then any endomorphism of M/IM is trivial.

1. INTRODUCTION

Throughout this paper all rings are commutative rings with identity $1 \neq 0$ and all modules are unital left modules. Let R be a ring. An $R-$module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N : M) = \{r \in R | \ rM \subseteq N\}$. As defined by Anderson [1], $\theta(M) = \sum_{x \in M} (Rx : M)$. The ideal $\theta(M)$ of R has proved to be useful in studying multiplication modules. In [5] it is proved that $Hom_R(M, M) \cong Hom_R(\theta(M), \theta(M))$. In this paper we give some generalizations of this result. Let M be a faithful multiplication $R-$module and I, $J \subseteq \theta(M)$ be ideals of R. In Section 2, Theorem 2.2, it is proved that there exists a canonical isomorphism λ from $Hom_R(\theta(M)/I, \theta(M)/J)$ onto $Hom_R(M/IM, M/JM)$ such that for any $\eta : \theta(M)/I \to \theta(M)/J$, $x \in M, a \in \theta(M), \lambda(\eta)(\overline{a}\,\overline{x}) = \eta(\overline{a})\overline{x}$. In Theorem 2.3, it is proved that there

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exists a canonical isomorphism λ from $Hom_R(I,J)$ onto $Hom_R(IM,JM)$ such that for any $\eta : I \longrightarrow J$, $x \in M$, $a \in I$, $\lambda(\eta)(ax) = \eta(a)x$. In Section 3, we give some applications. In Theorem 3.1, it is proved that M/IM is quasi-projective if and only if $\theta(M)/I$ is quasi-projective. In Theorem 3.2, it is proved that if R/I is self-injective, then any endomorphism of M/IM is trivial and if R is self-injective, then any endomorphism of IM is trivial.

2. ENDOMORPHISM RINGS

Lemma 2.1. *Let M be a faithful multiplication $R-$module.*

(i) *$N = \theta(M)N$ for any submodule N of M.*
(ii) *$A = \theta(M)A$ for any ideal $A \subseteq \theta(M)$.*
(iii) *aM is a finitely generated multiplication $R-$module for all $a \in \theta(M)$.*

Proof. (i) follows from [1, Theorem 2.1, (3)]. (ii) follows from (i) as $\theta(M)$ is a faithful multiplication ideal and $\theta(\theta(M)) = \theta(M)$ by [2, Theorems 2.3 and 2.6]. (iii) follows from [2, Lemma 2.1]. □

Lemma 2.2. *Let M be a faithful multiplication $R-$module and $J \subseteq \theta(M)$ be an ideal of R.*

(i) *$(JM : M) \cap \theta(M) = J$.*
(ii) *For $\overline{M} = M/JM$, $\overline{\theta(M)} = \theta(M)/J$, $[\bar{0} :_{\bar{R}} \bar{M}] \cap \overline{\theta(M)} = \bar{0}$, where $\overline{R} = R/J$*
(iii) *$(0 :_M J) = (0 : J)M$*

Proof. (i) Since M is a faithful multiplication $R-$module, by [3, Theorem 1.6], we have $((JM : M) \cap \theta(M))M = (JM : M)M \cap \theta(M)M = JM \cap \theta(M)M = JM$. Hence, by cancellation law [5, Theorem 1.4], $(JM : M) \cap \theta(M) = J$. (ii) follows from (i). (iii) As M is faithful, $(0 : JM) = (0 : J)$ and as M is a multiplication $R-$module, $(0 :_M J) = ((0 :_M J) : M)M = (0 : JM)M = (0 : J)M$. □

Theorem 2.1 ([5, Theorem 2.5]). *Let M be a faithful multiplication $R-$module. Then $Hom_R(M,M) \cong Hom_R(\theta(M), \theta(M))$.*

An $R-$module N is said to be quasi-projective, if for any submodule L of N and any $R-$homomorphism $f : N \rightarrow N/L$ there exists an $R-$endomorphism g of N such that $\pi \circ g = f$ where π is the natural projection of N onto N/L.

Lemma 2.3. *Let M be a faithful multiplication $R-$module, $I, J \subseteq \theta(M)$ be two ideals of R and let $\sigma \in Hom_R(M/IM, M/JM)$. Then for any $a \in \theta(M)$ there exists $r_a \in R$ such that $\sigma(\overline{ax}) = r_a\overline{ax}$ for all $x \in M$.*

Proof. Let $a \in \theta(M)$. Then, by Lemma 2.1 (iii), aM is a finitely generated multiplication $R-$module. Hence, by [6, Theorem 11], aM is a quasi-projective $R-$module. So there exists an $R-$endomorphism $\sigma_a : aM \rightarrow aM$ such that $\sigma(\overline{ax}) = \overline{\sigma_a(ax)}$. However aM is a finitely generated faithful multiplication $\overline{R}-$module where $\overline{R} = R/ann_R aM$ and $\sigma_a \in Hom_R(aM, aM)$, so it follows, from Theorem 2.1, that there exists an $r_a \in R$ such that $\overline{\sigma_a(ax)} = \overline{r_a ax} = r_a\overline{ax}$. □

Remark 2.1. If $x \in M$, then by Lemma 2.1 (i), there exists an $a \in \theta(M)$ such that $x = ax$. Hence $\sigma(\overline{x}) = \sigma(\overline{ax}) = r_a\overline{ax} = r_a\overline{x}$.

Lemma 2.4. *Let M be a faithful multiplication $R-$module, $I, J \subseteq \theta(M)$ be ideals of R and let $\eta \in Hom_R(\theta(M)/I, \theta(M)/J)$. Then $\eta(\overline{a})\overline{x} = \eta(\overline{b})\overline{x}$ for all $x \in M, a, b \in \theta(M)$ such that $ax = x = bx$.*

Proof. Since $ax = x = bx, (a - b) \in (0 : x) \cap \theta(M)$. Hence $\overline{a} - \overline{b} = \overline{r}$ for some $r \in (0 : x) \cap \theta(M)$. Therefore $\eta(\overline{a})\overline{x} - \eta(\overline{b})\overline{x} = \eta(\overline{r})\overline{x} = \eta(\overline{r})\overline{ax} = \eta(\overline{a})\overline{rx} = 0$. □

Corollary 2.1. *Let M be a faithful multiplication $R-$module and $I, J \subseteq \theta(M)$ be ideals of R. Let $\sigma \in Hom_R(M/IM, M/JM)$. Then there exists a unique $\eta \in Hom_R(\theta(M)/I, \theta(M)/J)$ such that $\sigma(\overline{a}\,\overline{x}) = \eta(\overline{a})\overline{x}$ for any $a \in \theta(M), x \in M$.*

Proof. Let $a \in \theta(M)$. Then, by Lemma 2.3, there exists an $r_a \in R$ such that $\sigma(\overline{a}\,\overline{x}) = r_a\overline{a}\,\overline{x}$ for all $x \in M$. Define $\eta : \theta(M)/I \rightarrow \theta(M)/J$ such that $\eta(\overline{a}) = r_a\overline{a}$ for all $a \in \theta(M)$. Suppose $\overline{a} = \overline{b}$ in $\theta(M)/I$ for some $a, b \in \theta(M)$. Then, for any $x \in M$, $\overline{a}\,\overline{x} = \overline{b}\,\overline{x}$ in M/IM. Thus $\sigma(\overline{a}\,\overline{x}) = \sigma(\overline{b}\,\overline{x})$ which implies that $r_a\overline{a}\,\overline{x} = r_b\overline{b}\,\overline{x}$ in M/JM. So in $\overline{R} = R/J$, $r_a\overline{a} - r_b\overline{b} \in$

$(\overline{0} :_{\overline{R}} M/JM) \cap \overline{\theta(M)} = \overline{0}$ by Lemma 2.2 (ii). Thus $r_a\overline{a} = r_b\overline{b}$ in $\theta(M)/J$, η is well defined and $\sigma(\overline{a}\,\overline{x}) = \eta(\overline{a})\overline{x}$ for any $a \in \theta(M), x \in M$. And if $a, b \in \theta(M)$, $x \in M$ then, $\eta(\overline{a}+\overline{b})\overline{x} = \sigma((\overline{a}+\overline{b})\overline{x}) = \sigma(\overline{a}\,\overline{x}) + \sigma(\overline{b}\,\overline{x}) = \eta(\overline{a})\overline{x} + \eta(\overline{b})\overline{x} = (\eta(\overline{a}) + \eta(\overline{b}))\overline{x}$. So, $\eta(\overline{a}+\overline{b}) - (\eta(\overline{a}) + \eta(\overline{b})) \in (\overline{0} :_{\overline{R}} \overline{M}) \cap \overline{\theta(M)} = \overline{0}$. Hence, $\eta(\overline{a}+\overline{b}) = \eta(\overline{a}) + \eta(\overline{b})$. Similarly if $r \in R$, $a \in \theta(M)$, $\eta(r\overline{a}) = r\eta(\overline{a})$. Therefore η is an R-homomorphism such that $\sigma(\overline{a}\,\overline{x}) = \eta(\overline{a})\overline{x}$ for any $a \in \theta(M)$, $x \in M$. That η is uniquely determined by σ follows from Lemma 2.2 (ii). □

Corollary 2.2. *Let M be a faithful multiplication $R-$module, $I, J \subseteq \theta(M)$ be ideals of R. Let $\eta \in Hom_R(\theta(M)/I, \theta(M)/J)$. Then there exists a unique $\sigma \in Hom_R(M/IM, M/JM)$ such that $\sigma(\overline{ax}) = \eta(\overline{a})\overline{x}$ for any $a \in \theta(M), x \in M$.*

Proof. Define $\sigma : M/IM \to M/JM$ such that for any $x \in M, \sigma(\overline{x}) = \eta(\overline{b})\overline{x}$ whenever $x = bx$ for some $b \in \theta(M)$. If $x \in M$, then there exists $b \in \theta(M)$ such that $x = bx$. Hence, by Lemma 2.4, σ is a well-defined map. Now if $x, y \in M$ then, by Lemma 2.1 (ii), there exist $s, t \in \theta(M)$ such that $sx = x$, $ty = y$. This gives that $(s+t-st)(x+y) = x+y$. Hence $\sigma(\overline{x}+\overline{y}) = \eta(\overline{s+t-st})(\overline{x}+\overline{y}) = (\eta(\overline{s}) + \eta(\overline{t}) - \eta(\overline{st}))(\overline{x}+\overline{y}) = \eta(\overline{s})\overline{x} + \eta(\overline{t})\overline{y} + \eta(\overline{s})\overline{y} + \eta(\overline{t})\overline{x} - \eta(\overline{t})\overline{s}\,\overline{x} - \eta(\overline{s})\overline{t}\,\overline{y} = \sigma(\overline{x}) + \sigma(\overline{y})$. And for any $r \in R$, $rx = srx$ gives $\sigma(r\overline{x}) = \eta(\overline{s})\overline{rx} = r\eta(\overline{s})\overline{x} = r\sigma(\overline{x})$. Hence σ is an $R-$homomorphism. The uniqueness of σ follows from its definition. □

Theorem 2.2. *Let M be a faithful multiplication $R-$module and $I, J \subseteq \theta(M)$ be ideals of R. Then there exists an isomorphism λ from $Hom_R(\theta(M)/I,$ $\theta(M)/J)$ onto $Hom_R(M/IM, M/JM)$ with the following properties:*
For any $\eta \in Hom_R(\theta(M)/I, \theta(M)/J)$, we have,

(i) *$\lambda(\eta)(\overline{a}\,\overline{x}) = \eta(\overline{a})\overline{x}$ for all $a \in \theta(M), x \in M$.*

(ii) *range $(\lambda(\eta)) = (range\ \eta)M/JM$ and $Ker\ \lambda(\eta) = (Ker\ \eta)M/IM$.*

(iii) *η is one-to-one (onto) if and only if $\lambda(\eta)$ is.*

Proof. (i) follows from Corollaries 2.1 and 2.2. For (ii), we have *range* $\eta = K/J$, $Ker\ \eta = L/I$ where K, L are ideals of R contained in $\theta(M)$. Now let $x \in M$. As $\theta(M)M = M$, $x = \Sigma a_i x_i$, $a_i \in \theta(M)$. So, $\lambda(\eta)(\overline{x}) = \Sigma\, \eta(\overline{a_i})\overline{x_i} \in (range\ \eta)M/JM$. Hence *range* $\lambda(\eta) \subseteq (range\ \eta)M/JM$. And for any $\overline{c} \in range\ \eta$, there exists an element $b \in \theta(M)$ such that $\eta(\overline{b}) = \overline{c}$. So,

$\overline{c}\,\overline{x} = \eta\left(\overline{b}\right)\overline{x} = \lambda\left(\eta\right)\left(\overline{b}\,\overline{x}\right) \in range\ \lambda\left(\eta\right), (range\ \eta)\,\overline{M} \subseteq range\ \lambda\left(\eta\right)$. Hence, $range\ \lambda\left(\eta\right) = (range\ \eta)\,\overline{M}$.

To prove that $Ker\ \lambda\left(\eta\right) = (Ker\ \eta)\,\overline{M}$, let $x \in M$. For any $a \in L, \lambda\left(\eta\right)\left(\overline{a}\,\overline{x}\right) = \eta\left(\overline{a}\right)\overline{x} = \overline{0}$. So, $(Ker\ \eta)\,\overline{M} \subseteq Ker\ \lambda\left(\eta\right)$. Now $Ker\ \lambda\left(\eta\right) = TM/IM$ for some ideal T of R such that $I \subseteq T \subseteq \theta\left(M\right)$. For any $b \in T$, $\overline{0} = \lambda\left(\eta\right)\left(\overline{b}\,\overline{x}\right) = \eta\left(\overline{b}\right)\overline{x}$. But $\eta\left(\overline{b}\right) = \overline{c}$ for some $c \in K$. So, $\overline{cM} = \overline{0}$, $cM \subseteq JM$. Hence, by cancellation law [5, Theorem 1.4], $c \in J$. So, $\eta\left(\overline{b}\right) = \overline{0}$, $\overline{b} \in Ker\ \eta$ and $Ker\ \lambda\left(\eta\right) \subseteq (Ker\ \eta)\,\overline{M}$. Therefore, $Ker\ \lambda\left(\eta\right) = (Ker\ \eta)\,\overline{M}$. (iii) is immediate from (ii) and the cancellation law. □

Corollary 2.3. *Let M be a faithful multiplication $R-$module, $I, J, K \subseteq \theta\left(M\right)$ be ideals of R. For any $\eta_1 \in Hom_R\left(\theta\left(M\right)/I, \theta\left(M\right)/J\right), \eta_2 \in Hom_R(\theta\left(M\right)/J, \theta\left(M\right)/K)$, $\lambda\left(\eta_2 \circ \eta_1\right) = \lambda\left(\eta_2\right) \circ \lambda\left(\eta_1\right)$.*

Proof. It follows from Theorem 2.2 (i). □

Lemma 2.5. *Let M be a faithful multiplication $R-$module and $I, J \subseteq \theta\left(M\right)$ be ideals of R.*

(i) *For any $\eta \in Hom_R\left(I, J\right)$ there exists a unique $\sigma \in Hom_R\left(IM, JM\right)$ such that $\sigma\left(ax\right) = \eta\left(a\right)x$ for all $a \in I, x \in M$.*
(ii) *For any $\sigma \in Hom_R\left(IM, JM\right)$ there exists a unique $\eta \in Hom_R\left(I, J\right)$ such that $\sigma\left(ax\right) = \eta\left(a\right)x$ for all $a \in I, x \in M$.*

Proof. (i) Let $a_1, a_2, \cdots, a_n$ be any n elements of $I, A = \sum_{i=1}^{n} Ra_i$. Define $\sigma_A : AM \rightarrow JM$ such that $\sigma_A\left(\sum a_i x_i\right) = \sum \eta\left(a_i\right)x_i$. We will prove by induction on n that σ_A is an $R-$homomorphism such that for any $a \in A, x \in M, \sigma_A\left(ax\right) = \eta\left(a\right)x$. Let $n = 1$. By Lemma 2.2 (iii), $(0 :_M a_1) = (0 : a_1)\,M$, $(0 :_M \eta\left(a_1\right)) = (0 : \eta\left(a_1\right))\,M$. If $a_1 x = 0$, then, $x \in (0 :_M a_1) = (0 : a_1)\,M \subseteq (0 : \eta\left(a_1\right))\,M = (0 :_M \eta\left(a_1\right))$. Hence $\eta\left(a_1\right)x = 0$ and hence σ_A is a well defined $R-$homomorphism. Let $x \in M$, $b \in A$. Then $b = ra_1$, $bx = ra_1 x = a_1 rx$ gives $\sigma_A\left(bx\right) = \sigma_A\left(a_1 rx\right) = \eta\left(a_1\right)rx = \eta\left(ra_1\right)x = \eta\left(b\right)x$. So the result holds for $n = 1$. Now let the result holds for $n-1$ and let $B = \sum_{i=1}^{n-1} Ra_i$, $C = B \cap Ra_n$. By the induction hypothesise, σ_B is an $R-$homomorphism of BM into JM. Let $a \in C$, $x \in M$. As $C = \theta\left(M\right)C$, $a = \sum b_i c_i$ for some $b_i \in \theta\left(M\right), c_i \in C$

and $ax = \sum c_i b_i x$. By induction hypothesise, $\sigma_B(c_i b_i x) = \eta(c_i) b_i x$. Also, as $c_i \in Ra_n$, $\sigma_{Ra_n}(c_i b_i x) = \eta(c_i) b_i x$. This gives $\sigma_B(ax) = \sigma_{Ra_n}(ax)$. Since $A = B + Ra_n$, it follows that σ_A is a well defined $R-$homomorphism of AM into JM. And if $a \in A$, $x \in M$ then, $a = b + ra_n$ for some $b \in B$, $r \in R$. Hence, $\sigma_A(ax) = \sigma_A(bx + ra_n x) = \eta(b) x + \eta(a_n) rx = (\eta(b) + \eta(ra_n)) x = \eta(b + ra_n) x = \eta(a) x$. Hence, $\sigma_A(ax) = \eta(a) x = \sigma_{Ra}(ax)$. This proves that given any $a \in A$, σ_A extends σ_{Ra}. Thus for any two finitely generated ideals A, B contained in I, σ_B extends σ_A, whenever $A \subseteq B$. As I is a direct limit of ideals of type A, we get an $R-$homomorphism σ of IM into JM such that $\sigma(ax) = \eta(a) x$ for any $a \in I$, $x \in M$. The uniqueness of σ is clear. (ii) follows from [5, Lemma 2.6] as $\Gamma(IM) := (IM : M) \cap \theta(M) = I$ by Lemma 2.2 (i). □

The following Theorem is a dual of Theorem 2.2.

Theorem 2.3. *Let M be a faithful multiplication $R-$module and $I, J \subseteq \theta(M)$ be ideals of R. There exists an isomorphism λ of $Hom_R(I, J)$ onto $Hom_R(IM, JM)$ such that for any $\eta \in Hom_R(I, J)$, $a \in I$, $x \in M$, $\lambda(\eta)(ax) = \eta(a) x$.*

Proof. Follows from Lemma 2.5. □

Corollary 2.4. *Let M be a faithful multiplication $R-$module. If I is a finitely generated ideal of R with $I \subseteq \theta(M)$,then $Hom_R(M, IM) \cong I$.*

Proof. By Lemma 2.1 (ii), $I = \theta(M) I$. As I is finitely generated, by a standard determinant argument, $R = \theta(M) + (0 : I)$. So, $1 = a + b$ for some $a \in \theta(M)$, $b \in (0 : I)$. Let $\sigma \in Hom_R(\theta(M), I)$. For any $x \in \theta(M)$, $x = ax + bx$, $\sigma(x) = \sigma(ax) + \sigma(bx)$. As $\sigma(x) \in I$, $\sigma(bx) = b\sigma(x) = 0$. So $\sigma(x) = \sigma(a) x$ for every $x \in \theta(M)$. It follows that $Hom_R(\theta(M), I) \cong I$. Hence, by Theorem 2.3, $Hom_R(M, IM) = Hom_R(\theta(M) M, IM) \cong Hom_R(\theta(M), I) \cong I$. □

3. APPLICATIONS

As an application of Theorem 2.2, we have the following Theorem.

Theorem 3.1. *Let M be a faithful multiplication $R-$module, $I \subseteq \theta(M)$ be an ideal of R. Then $\theta(M)/I$ is quasi-projective if and only if M/IM is.*

Proof. Suppose that $\theta(M)/I$ is quasi-projective. Any homomorphic image of M/IM is isomorphic to M/JM for some ideal J of R such that $I \subseteq J \subseteq \theta(M)$. Now consider an $R-$homomorphism $f : M/IM \to M/JM$. By Theorem 2.2, there exists $\lambda^{-1}(f) : \theta(M)/I \to \theta(M)/J$ such that $f(\overline{a}\,\overline{x}) = \lambda^{-1}(f)(\overline{a})\overline{x}$, in M/JM, for any $x \in M$, $a \in \theta(M)$. As $\theta(M)/I$ is quasi-projective, there exists an $R-$homomorphism $\eta : \theta(M)/I \to \theta(M)/I$ such that $\lambda^{-1}(f) = \pi \circ \eta$, where $\pi : \theta(M)/I \to \theta(M)/J$ is the natural homomorphism. By Theorem 2.2, there exists an $R-$homomorphism $\mu : M/IM \to M/IM$ such that $\mu(\overline{a}\,\overline{x}) = \eta(\overline{a})\overline{x}$ for any $a \in \theta(M)$, $x \in M$. Hence, $f(\overline{a}\,\overline{x}) = \lambda^{-1}(f)(\overline{a})\overline{x} = (\pi \circ \eta)(\overline{a})\overline{x} = \pi'(\eta(\overline{a})\overline{x})$, by Corollary 2.2, where $\pi' : M/IM \to M/JM$ is the natural homomorphism. Thus, $f(\overline{a}\,\overline{x}) = \pi'(\mu(\overline{a}\,\overline{x}))$ and that gives $f = \pi' \circ \mu$. This proves that M/IM is quasi-projective. Similarly one can prove the converse. □

Remark 3.1. There exists a faithful multiplication ideal which is not projective. To construct the example, let R be any non noetherian self-injective commutative von Neumann regular ring. Suppose that every faithful ideal of R is projective. Let A be any ideal of R, C a complement of A in R and $B = A + C = A \oplus C$. Since R is an essential extention of B and R is von Neuman regular ring, B is a faithful ideal. Hence it is projective and hence A is projective. Therefore R is hereditary. So, by [4, Theorem 4.23], every homomorphic image of ${}_RR$ is injective. Hence, by [4, Corollary 6.47], R is semisimple, which is a contradiction. Hence there exists a faithful ideal A of R which is not projective. This A is a multiplication ideal.

An endomorphism σ of an $R-$module M is said to be trivial if there exists an element $r \in R$ such that $\sigma(x) = rx$ for all $x \in M$.

It is well known that if M is a finitely generated multiplication $R-$module, then any endomorphism of M is trivial. In the following result M need not be finitely generated.

Theorem 3.2. *Let M be a faithful multiplication $R-$module, $I \subseteq \theta(M)$ be an ideal of R.*

(i) *If R/I is self-injective, then any $R-$endomorphism of M/IM is trivial.*
(ii) *If R is self-injective, then any endomorphism of IM is trivial.*

Proof. To prove (i), let σ be an endomorphism of M/IM. By Theorem 2.2 (i), $\sigma(\overline{a}\,\overline{x}) = \lambda^{-1}(\sigma)(\overline{a})\overline{x}$ for all $a \in \theta(M), x \in M$. Since R/I is self-injective, there exists $\overline{r} \in R/I$ such that $\lambda^{-1}(\sigma)(\overline{a}) = \overline{r}\,\overline{a}$ for all $a \in \theta(M)$. Now for any $x \in M$, there exists $b_x \in \theta(M)$ such that $x = b_x x$. Hence, $\sigma(\overline{x}) = \sigma\left(\overline{b_x}\,\overline{x}\right) = \lambda^{-1}(\sigma)\left(\overline{b_x}\right)\overline{x} = \overline{r}\,\overline{b_x}\,\overline{x} = \overline{r}\,\overline{x}$ and σ is trivial. (ii) follows by a similar argument using Theorem 2.3. □

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