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MÖBIUS TRANSFORMATION WITH A 2-CYCLE

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ABSTRACT. This paper shows that any Möbius transformation with at least one 2-cycle has the form $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$, where α , $\beta \in \mathbb{C}$ and $\alpha^2 + \beta \neq 0$. Also, it is shown that $f(z)$ maps any circle passing through its fixed points into itself.

1. PRELIMINARIES

In this section, we mention basic definitions and major properties that is related to the subject of this paper.

1.1. **Möbius Transformation :** The transformation $f : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ defined by $f(z) = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ where a, b, c, and d are complex constants, is called a Möbius transformation. Note that $f(\infty) = \lim_{z \to \infty} f(z) = \dfrac{a}{c}$ and $f\left(\dfrac{-d}{c}\right) = \infty$. Möbius transformation always maps circles and lines into circles and lines. Also it is a conformal mapping.

There is always a Möbius transformation that maps three given distinct points z_1, z_2, and z_3 onto three given distinct points w_1, w_2, and w_3, respectively. The equation (called the cross-ratio equation in a symmetrical form)

$$\frac{(w - w_1)\ (w_2 - w_3)}{(w - w_3)\ (w_2 - w_1)} = \frac{(z - z_1)\ (z_2 - z_3)}{(z - z_3)\ (z_2 - z_1)} \tag{1.1}$$

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gives the desired Möbius transformation when none of these points is infinity. For more details about Möbius transformation, the reader is refered to [2], [3], [6] and [7].

1.2. **Dynamical Systems :** The point $a \in \mathbb{C}$ is called a fixed point of the map $f : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$, if $f(a) = a$.

The point $a \in \mathbb{C}$ is called a $k-cycle$ of f if $f^i(a) \neq a$ for every $i = 1, \ldots, k-1$ and $f^k(a) = a$. The set $\{a, f(a), f^2(a), \ldots, f^{k-1}(a)\}$ is called the orbit of a. For more details about basic definitions and concepts of dynamical systems the reader can refer to to [1] and [4].

In this paper, we assume that the complex constant $c \neq 0$, in other words $f(z) = \dfrac{az+b}{cz+d}$ is not a linear map. In this case, $f(z)$ has at most two fixed points.

2. Introduction

An elementary example of a Möbius transformation is the inversion map $f(z) = \dfrac{1}{z}$, it maps the unit circle into itself, and it maps the interior of the unit circle into its exterior. Also the real-axis and the imaginary-axis are the only two lines mapped into themselves by $f(z) = \dfrac{1}{z}$. If we look at $f(z) = \dfrac{1}{z}$ from a dynamical systems point of view, we notice that its fixed points are 1 and -1 and the only circle centered at $0 = f(\infty)$ and passing through 1 and -1 is the unit circle. We also notice that for every $z \in \mathbb{C}_\infty \setminus \{1, -1\}$, z is a 2-cycle. Also the only line passing through 1 and -1 is the real-axis, and the imaginary-axis is orthogonal to it at the origin.

Is the unit circle the only circle mapped into itself by $f(z) = \dfrac{1}{z}$?.

The answer is negative, for example, the circle $\left|z - \dfrac{3i}{4}\right| = \dfrac{5}{4}$ is mapped into itself by $f(z) = \dfrac{1}{z}$, since the points 1, -1, $2i$ and $\dfrac{-i}{2}$ lie on the circle $\left|z - \dfrac{3i}{4}\right| =$

$\frac{5}{4}$ and $f(2i) = \frac{-i}{2}$, $f\left(\frac{-i}{2}\right) = 2i$. Note that the the circle $\left|z - \frac{3i}{4}\right| = \frac{5}{4}$ passes through the fixed points of $f(z) = \frac{1}{z}$.

Is any circle passing through the two fixed points of $f(z) = \frac{1}{z}$ mapped into itself by $f(z) = \frac{1}{z}$?.

This paper studies the general form of such Möbius transformation, by showing that if $f(z)$ is any Möbius transformation with at least one $2 - cycle$, then it has the form $f(z) = \frac{\alpha z + \beta}{z - \alpha}$, where α , $\beta \in \mathbb{C}$ and $\alpha^2 + \beta \neq 0$, and it maps the circle centered at α and passing through the two fixed points of $f(z)$ into itself. In other words $\frac{1}{z}$ is a special case of $f(z) = \frac{\alpha z + \beta}{z - \alpha}$.

Moreover, $f(z)$ maps any circle passing through its two fixed points into itself.

3. Möbius transformation with a 2-cycle

The first proposition shows that if a Möbius transformation has at least one 2-cycle, then all the points in the extended complex plane are 2-cycles except its fixed points.

Proposition 3.1.

If $f(z)$ is any Möbius transformation which has a 2-cycle, then it has the form $f(z) = \frac{\alpha z + \beta}{z - \alpha}$, where α , $\beta \in \mathbb{C}$, and $f^2(z) = z$ for every $z \in \mathbb{C}_\infty$.

Proof: Case(1): The orbit of α is $\{\alpha, \infty\}$. This means that $f(\alpha) = \infty$ and $f(\infty) = \alpha$. In this case $f(z) = \frac{\alpha z + \beta}{z - \alpha}$ is obviously the desired Möbius transformation.

Case(2): The orbit of α_1 is $\{\alpha_1, \alpha_2\}$, where $\alpha_1 \neq \alpha_2$ and both of them are not ∞. Without any loss of generality, we may assume that the point α will be mapped to ∞. So, $f(z) = \frac{az + b}{z - \alpha}$, where $a\alpha + b \neq 0$.

$f(\alpha_1) = \alpha_2 \Longrightarrow \alpha_1\alpha_2 - \alpha\alpha_2 = a\alpha_1 + b \Longrightarrow \alpha_1\alpha_2 - b = \alpha\alpha_2 + a\alpha_1.$

$f(\alpha_2) = \alpha_1 \Longrightarrow \alpha_2\alpha_1 - \alpha\alpha_1 = a\alpha_2 + b \Longrightarrow \alpha_1\alpha_2 - b = \alpha\alpha_1 + a\alpha_2.$

So, $\alpha\alpha_2 + a\alpha_1 = \alpha\alpha_1 + a\alpha_2 \Longrightarrow \alpha\alpha_2 - \alpha\alpha_1 + a\alpha_1 - a\alpha_2 = 0$

$\Longrightarrow \alpha\ (\alpha_2 - \alpha_1) - a\ (\alpha_2 - \alpha_1) = 0 \Longrightarrow (\alpha_2 - \alpha_1)\ (\alpha - a) = 0 \Longrightarrow a = \alpha$, since $\alpha_1 \neq \alpha_2$.

Therefore, the desired Möbius transformation is $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$.

Now, we prove that $f^2(z) = z$ for every $z \in \mathbb{C}_\infty$.

Obviously, $f^2(\alpha) = \alpha$ and $f^2(\infty) = \infty$.

For every $z \in \mathbb{C} - \{\alpha\}$, we want to prove that $f(z) = f^{-1}(z)$.

Let $w = \dfrac{\alpha z + \beta}{z - \alpha} \Longrightarrow wz - \alpha w = \alpha z + \beta \Longrightarrow wz - \alpha z = \alpha w + \beta \Longrightarrow z = \dfrac{\alpha w + \beta}{w - \alpha}$.

So, $f^{-1}(z) = \dfrac{\alpha z + \beta}{z - \alpha} = f(z)$. Hence, $f^2(z) = z$ for every $z \in \mathbb{C} - \{\alpha\}$.

Therefore, $f^2(z) = z$ for every $z \in \mathbb{C}_\infty$. □

We can also remark that $f^2(\alpha) = \alpha$, $f^2(\infty) = \infty$, $f^2(\alpha_1) = \alpha_1$ and $f^2(\alpha_2) = \alpha_2$, then $f^2(z) = z$ for all $z \in \mathbb{C}_\infty$.

Remark : Proposition 3.1 is closely related to Theorem 1 of N. Eljoseph in [5], but the proof here is completely different, and the assumption that $f(z)$ has at least one $2 - cycle$ is weaker than the heavy assumption $f^n = Id$ used in [5].

The fixed points of $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ are the points satisfying $z = \dfrac{\alpha z + \beta}{z - \alpha}$, by solving the last equation, the fixed points are $\gamma_1 = \alpha + \left(\alpha^2 + \beta\right)^{\frac{1}{2}}$ and $\gamma_2 = \alpha - \left(\alpha^2 + \beta\right)^{\frac{1}{2}}$. If w is not a fixed point of $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$, then w is a $2 - cycle$ of f, by Proposition (3.1).

Note that, $\alpha = \dfrac{\gamma_1 + \gamma_2}{2}$, also α,γ_1 and γ_2 lie on a straight line, call it L_1. Clearly, $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ maps the straight line L_1 into itself.

Now, let C be the circle $|z - \alpha| = R$ where $R = \left|(\alpha^2 + \beta)^{\frac{1}{2}}\right|$, clearly the fixed points γ_1 and γ_2 lie on C.

The next proposition shows that $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ maps the circle C centered at $f(\infty) = \alpha$ and passing through the fixed points γ_1 and γ_2 into itself.

Proposition 3.2.

$f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ *maps the circle C into itself and its interior into its exterior.*

Proof: Let uss first show that $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ maps the circle C into itself. If w lies on C then $|w - \alpha| = R$.

$$|f(w) - \alpha| = \left|\frac{\alpha w + \beta}{w - \alpha} - \alpha\right| = \left|\frac{\alpha^2 + \beta}{w - \alpha}\right| = \frac{|\alpha^2 + \beta|}{|w - \alpha|} = \frac{R^2}{R} = R.$$

Therefore, $f(w)$ lies on the circle C.

Hence, $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ maps the circle C into itself.

Another proof: The only circle orthogonal to L_1 at the fixed points γ_1 and γ_2 is the circle C. Since $\alpha \notin C$, $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ is conformal, and γ_1, γ_2 are fixed points, then $f(z)$ maps C into a circle orthogonal to L_1 at γ_1 and γ_2 which is the circle C itself.

Now, we show that $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ maps the interior of the circle C into its exterior. If w lies inside the circle C then $|w - \alpha| = r < R$.

$|f(w) - \alpha| = \dfrac{|\alpha^2 + \beta|}{|w - \alpha|} = \dfrac{R^2}{r} > \dfrac{R^2}{R} = R.$ Therefore, $f(w)$ lies outside the circle C.

Hence $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ maps the interior of the circle C into the exterior of C. □

We proved that $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ maps the circle C into itself, the next proposition shows that any point lies on C and its image are symmetric with respect to the line L_1.

Proposition 3.3.

If w is not a fixed point of $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$, and w lies on the circle C, then w and $f(w)$ are symmetric with respect to the line L_1.

Proof: If w is not a fixed point of $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$, and w lies on the circle C, then $w = \alpha + \left(\alpha^2 + \beta\right)^{\frac{1}{2}} e^{i\theta}$, where $\theta \in (-\pi, \pi) \setminus \{0\}$ is the oriented angle between the half lines $[\alpha, w)$ and $[\alpha, \gamma_2)$ (see figure 1).

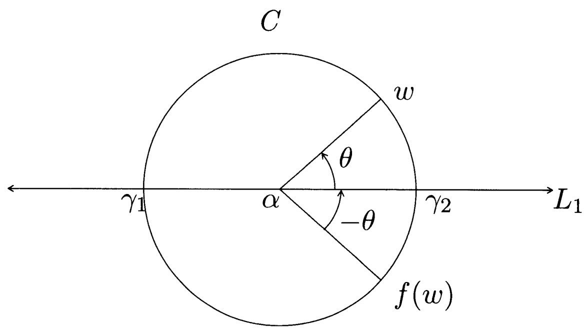

figure 1

$$\begin{aligned} f(w) &= f\left(\alpha + \left(\alpha^2+\beta\right)^{\frac{1}{2}} e^{i\theta}\right) \\ &= \frac{\alpha^2 + \alpha\left(\alpha^2+\beta\right)^{\frac{1}{2}} e^{i\theta} + \beta}{\left(\alpha^2+\beta\right)^{\frac{1}{2}} e^{i\theta}} = \alpha + \frac{\alpha^2+\beta}{\left(\alpha^2+\beta\right)^{\frac{1}{2}} e^{i\theta}} \\ &= \alpha + \left(\alpha^2+\beta\right)^{\frac{1}{2}} e^{-i\theta}. \end{aligned}$$

$f(w)$ lies on the circle C and the angle between L_1 and the half line $[\alpha, f(w))$ equals $-\theta$.

Therefore, w and $f(w)$ are symmetric with respect to the line L_1. □

Another proof:

Let w be any point liying on C and not a fixed point. Let θ be the oriented angle between the half lines $[\alpha, w)$ and $[\alpha, \gamma_2)$. Since $f(z)$ is a conformal mapping and $[\alpha, \gamma_2)$ is invariant by f and $f(w)$ lies on the circle C, then θ is the oriented angle between $[\alpha, f(w))$ and $[\alpha, \gamma_2)$. It follows that w and $f(w)$ are symmetric with respect to the line L_1. □

Let L_2 be the straight line passing through α and perpendicular to L_1, Let ζ_1 and ζ_2 be the points where L_2 and the circle C intersect. Then by Proposition (3.3) $f(\zeta_1) = \zeta_2$ and $f(\zeta_2) = \zeta_1$.

Therefore, $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ maps the line L_2 into itself.

Thus, we have proved the following corollary :

Corollary 3.4.

$f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ *maps the line* L_2 *into itself.*

So far, we proved that $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ maps the straight line L_1 passing through the two fixed points of f into itself, it also maps the straight line L_2 perpendicular to L_1 at α into itself, and it maps the circle C centered at α and passing through the two fixed points of f into itself.

The next proposition shows that $f(z) = \dfrac{\alpha z+\beta}{z-\alpha}$ maps every circle passing through its two fixed points into itself.

Proposition 3.5.

$f(z) = \dfrac{\alpha z+\beta}{z-\alpha}$ *maps every circle passing through* γ_1 *and* γ_2 *into itself.*

Proof:

Let S be any circle passing through the two fixed points γ_1 and γ_2, then S intersects the straight line L_2 in a point say ξ different from α. (see figure 2).

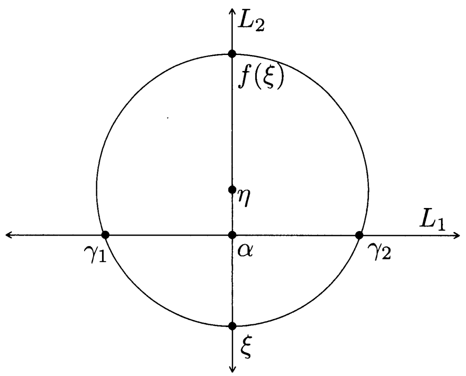

figure 2

Since $\xi \in L_2$, then $\xi = \alpha + \ i\ \lambda\left(\alpha^2+\beta\right)^{\frac{1}{2}}$, where $\lambda \in \mathbb{R}^*$

Now, $f\left(\xi\right) = \dfrac{\alpha\ \left(\alpha + i\lambda\left(\alpha^2+\beta\right)^{\frac{1}{2}}\right)+\beta}{i\lambda\left(\alpha^2+\beta\right)^{\frac{1}{2}}} = \alpha + \dfrac{\alpha^2+\beta}{i\lambda\left(\alpha^2+\beta\right)^{\frac{1}{2}}}$.

So, $f\left(\xi\right) = \alpha + \dfrac{1}{i\lambda}\left(\alpha^2+\beta\right)^{\frac{1}{2}} = \alpha + i\left(\dfrac{-1}{\lambda}\right)\left(\alpha^2+\beta\right)^{\frac{1}{2}}$.

Let $\eta = \dfrac{\xi + f\left(\xi\right)}{2}$, then $\eta = \alpha + \dfrac{i}{2}\left(\lambda - \dfrac{1}{\lambda}\right)\left(\alpha^2+\beta\right)^{\frac{1}{2}}$.

We want to show that the circle S is in fact centered at η and has a radius $|\xi - \eta|$.

In other words, we want to show that $|\xi-\eta| = |f(\xi)-\eta| = |\gamma_1-\eta| = |\gamma_2-\eta|$, and this implies that $f(z) = \dfrac{\alpha z+\beta}{z-\alpha}$ maps the circle S into itself.

We know that $|\xi-\eta| = |f(\xi)-\eta| = \left|\frac{i}{2}\left(\lambda+\frac{1}{\lambda}\right)\left(\alpha^2+\beta\right)^{\frac{1}{2}}\right| = \frac{1}{2}\left|\lambda+\frac{1}{\lambda}\right|\left|\alpha^2+\beta\right|^{\frac{1}{2}}$

Also, we know that $|\gamma_1-\eta| = |\gamma_2-\eta|$ since L_2 is the bisector perpendicular to the line segment $[\gamma_1, \gamma_2]$.

So, it is enough to show that $|\gamma_1-\eta| = \frac{1}{2}\left|\lambda+\frac{1}{\lambda}\right|\left|\alpha^2+\beta\right|^{\frac{1}{2}}$

$$|\gamma_1-\eta| = \left|1-\frac{i}{2}\left(\lambda-\frac{1}{\lambda}\right)\right|\left|\alpha^2+\beta\right|^{\frac{1}{2}} = \left|1+i\left(\frac{1}{2\lambda}-\frac{\lambda}{2}\right)\right|\left|\alpha^2+\beta\right|^{\frac{1}{2}}$$

$$= \sqrt{1+\frac{1}{4}\left(\frac{1}{\lambda^2}-2+\lambda^2\right)}\left|\alpha^2+\beta\right|^{\frac{1}{2}} = \sqrt{\frac{1}{4}\left(\frac{1}{\lambda^2}+2+\lambda^2\right)}\left|\alpha^2+\beta\right|^{\frac{1}{2}}$$

$$= \sqrt{\frac{1}{4}\left(\lambda+\frac{1}{\lambda}\right)^2}\left|\alpha^2+\beta\right|^{\frac{1}{2}} = \frac{1}{2}\left|\lambda+\frac{1}{\lambda}\right|\left|\alpha^2+\beta\right|^{\frac{1}{2}}.$$

Hence, ξ, $f(\xi)$, γ_1 and γ_2 lie on the circle S, and $\alpha \notin S$.

Therefore, $f(z) = \dfrac{\alpha z+\beta}{z-\alpha}$ maps the circle S into itself. □

In the next example, we give a good estimation of a given quantity using Proposition (3.5) rather than using triangle inequality.

Example: If $|z-4i| = 5$, then $\left|\dfrac{9-4iz}{z}\right| = 5$.

To see this, observe that $\dfrac{9-4iz}{z} = \dfrac{9}{z} - 4i$, and consider the Möbius transformation $f(z) = \dfrac{9}{z}$. It has at least one 2-cycle and its fixed points are 3 and -3. The two fixed points obviously lie on the circle $|z-4i| = 5$, by proposition (3.5) $f(z) = \dfrac{9}{z}$ maps the circle $|z-4i| = 5$ into itself. Hence $\left|\dfrac{9}{z} - 4i\right| = 5$, so $\left|\dfrac{9-4iz}{z}\right| = 5$. □

We finish this paper by finding circles not passing through the fixed points of $f(z) = \dfrac{\alpha z + \beta}{z - \alpha}$ but mapped into themselves by $f(z)$.

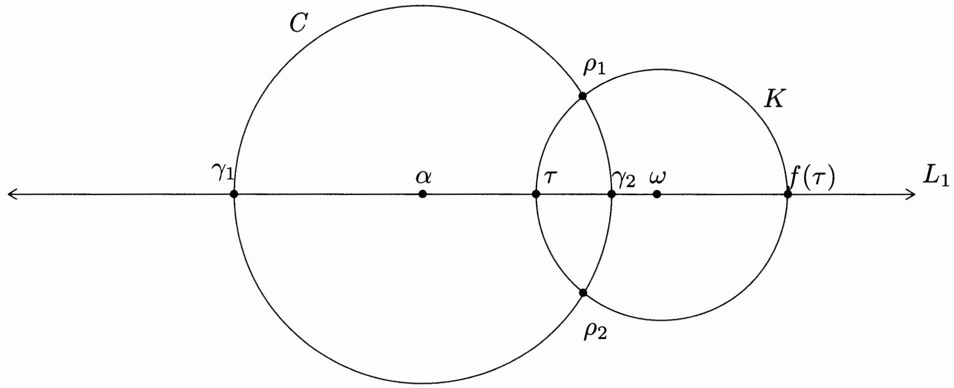

Figure 3

Let τ be any non-fixed point liying on L_1 and $\tau \neq \alpha$, then $f(\tau) \in L_1$. Let K be the circle centered at $\omega = \dfrac{\tau + f(\tau)}{2}$ and passing through τ and $f(\tau)$. Then K is symmetric with respect to the line L_1, and K intersects the circle C at two points ρ_1 and ρ_2 (see figure 3). Since these two points are symmetric with respect to the line L_1, then $f(\rho_1) = \rho_2$ and $f(\rho_2) = \rho_1$. Thus, $f(z)$ maps the circle K into itself.

We can also remark that γ_1, γ_2, τ and $f(\tau)$ are harmonic division , which means that $\dfrac{f(\tau)-\gamma_1}{f(\tau)-\gamma_2} = -1\ \dfrac{\tau-\gamma_1}{\tau-\gamma_2}$, so the circles C and K are orthogonal. Then, the line passing through ρ_1, ρ_2 is orthogonal to L_1.

4. Conclusion

(1) If $f(z)$ is any Möbius transformation which has at least one 2-cycle, then it has the form $f(z) = \dfrac{\alpha z+\beta}{z-\alpha}$, where α , $\beta \in \mathbb{C}$, and $\alpha^2+\beta \neq 0$.

(2) If γ_1 and γ_2 are the two fixed points of $f(z) = \dfrac{\alpha z+\beta}{z-\alpha}$, L_1 is the straight line passing through γ_1 and γ_2 and L_2 is the straight line perpendicular to L_1 at α, then $f(z)$ maps L_1 into itself, and it also maps L_2 into itself.

(3) $f(z)$ maps any circle passing through γ_1 and γ_2 into itself.

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