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REMARKS ON A FOUR-DIMENSIONAL COMPACT ALMOST KÄHLER EINSTEIN MANIFOLD

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ABSTRACT. In this paper, we will prove that the skew-symmetric part of the Ricci $*$-tensor of a 4-dimensional compact almost Kähler Einstein manifold M vanishes somewhere on M. Further, we will give another proof of the theorem in [1] by V. Apostolov and J. Armstrong.

1. INTRODUCTION

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if the Kähler form Ω of M defined by $\Omega(X,Y) = g(X, JY)$ is closed. The condition $d\Omega = 0$ is equivalent to $\mathfrak{S}_{X,Y,Z}\, g((\nabla_X J)Y, Z) = 0$, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to X, Y, Z. An almost Kähler manifolds with integrable almost complex structure is a Kähler manifold. A non-Kähler, almost Kähler manifold is called a strictly almost Kähler manifold.

Concerning the integrability of almost Kähler manifolds, the following conjecture by S.I. Goldberg is well known:

Conjecture. *A compact almost Kähler Einstein manifold is integrable.*

This conjecture is true in the case where the scalar curvature of M is non-negative([8]), and many progresses have been made by many authors under certain additional curvature conditions. However, the conjecture is still open. Recently, the authors have proved the following.

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Theorem A ([6]) *Let $M = (M, J, g)$ be a four-dimensional compact almost Kähler Einstein manifold. If the norm of skew-symmetric part of the Ricci $*$-tensor is constant, then M is a Kähler manifold.*

In this paper, we shall prove the following theorem which was suggested by T. Draghici in a private communication.

Theorem 1 *Let $M = (M, J, g)$ be a four-dimensional compact almost Kähler Einstein manifold. Then, the skew-symmetric part of the Ricci $*$-tensor vanishes somewhere on M.*

Taking account of the result in [7], we can obtain Theorem A as a corollary of Theorem 1. Further, in section , we will give another proof of the following theorem by V. Apostolov and J. Armstrong.

Theorem 2 ([1]) *A four-dimensional compact almost Kähler Einstein manifold M with Hermitian Weyl tensor is a Kähler manifold.*

In this theorem, it seems interesting to study the integrability without compactness.

2. PRELIMINARIES

In this section, we prepare several fundamental formulas which will be used in our argument. Most of them are established in [6].

Let $M = (M, J, g)$ be a four-dimensional almost Hermitian manifold. The Kähler form of M is defined by $\Omega(X,Y) = g(X, JY)$ for X, $Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of all smooth vector fields on M. We assume that M is oriented by the volume form $dM = \Omega^2/2$. We denote by ∇, R, ρ and τ the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of M respectively, where $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. The Ricci $*$-tensor ρ^* of M is defined by

$$\rho^*(x,y) = \frac{1}{2}\,\text{trace of}\,(z \mapsto R(x, Jy)Jz)$$

for x, y, $z \in T_pM$, the tangent space of M at $p \in M$. We see immediately that $\rho^*(x,y) = \rho^*(Jy, Jx)$. Further, it is known that

$$(2.1)\quad \frac{1}{2}\{\rho(x,y) + \rho(Jx,Jy)\} - \frac{1}{2}\{\rho^*(x,y) + \rho^*(y,x)\} = -\frac{\tau^* - \tau}{4} g(x,y)$$

holds for x, $y \in T_pM$, $p \in M$ ([4]). The $*$-scalar curvature τ^* is the trace of the linear endomorphism Q^* of T_pM defined by $g(Q^*x,y) = \rho^*(x,y)$ for x, $y \in T_pM$.

In this paper, for any orthonormal bases (resp. any local orthonormal frame field) $\{e_i\}_{i=1,\ldots,4}$ of T_pM (resp. on a neighborhood of p), we shall adopt the following notational convention:

$$\begin{gathered}\Gamma_{ijk} = g(\nabla_{e_i}e_j, e_k), \quad J_{ij} = g(Je_i, e_j), \quad \nabla_i J_{jk} = g((\nabla_{e_i}J)e_j, e_k),\\ R_{ijkl} = g(R(e_i,e_j)e_k,e_l), \ldots, R_{\bar{i}\bar{j}\bar{k}\bar{l}} = g(R(Je_i,Je_j)Je_k,Je_l),\\ \rho_{ij} = \rho(e_i,e_j), \ldots, \rho_{\bar{i}\bar{j}} = \rho(Je_i,Je_j),\\ \rho^*_{ij} = \rho^*(e_i,e_j), \ldots, \rho^*_{\bar{i}\bar{j}} = \rho^*(Je_i,Je_j),\end{gathered}$$

and so on, where the Latin indices run over the range 1,. . . ,4. Then, we have $\Gamma_{ijk} = -\Gamma_{ikj}$, $J_{ij} = -J_{ji}$, $\nabla_i J_{jk} = -\nabla_i J_{kj}$, $\nabla_i J_{\bar{j}\bar{k}} = -\nabla_i J_{jk}$.

We denote by $\wedge^2 M$ the vector bundle of real 2-forms on M. The bundle $\wedge^2 M$ inherits a natural inner product from the Riemannian metric g (also denoted by g). We have the following decomposition of $\wedge^2 M$:

$$\wedge^2 M = \mathbb{R}\Omega \oplus \wedge_0^{1,1} M \oplus LM, \tag{2.2}$$

where $\wedge_0^{1,1}M$ denotes the vector bundle of real J-invariant 2-forms, LM the vector bundle of real J-skew-invariant 2-forms of M which are perpendicular to Ω, respectively. The subbundle LM is endowed with the natural complex structure J defined by $J\Phi(X,Y) = -\Phi(JX,Y)$ for any local section Φ of LM and X, $Y \in \mathfrak{X}(M)$. The subbundle $\mathbb{R}\Omega \oplus LM$ is identified itself with the bundle $\wedge_+^2 M$ of self-dual 2-forms of M, while LM is the bundle $\wedge_-^2 M$ of anti-self-dual 2-forms of M:

$$\wedge_+^2 M = \mathbb{R}\Omega \oplus LM, \tag{2.3}$$
$$\wedge_-^2 M = \wedge_0^{1,1} M. \tag{2.4}$$

Let $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ be any local unitary frame and we put $\{e^i\} = \{\iota(e_i)\}$, where $\iota : TM \to T^*M$ denotes the duality defined by means of the metric g. Then Ω is given by $\Omega = -e^1 \wedge e^2 - e^3 \wedge e^4$. Further, if we put $\Phi = (e^1\wedge e^3 - e^2\wedge e^4)/\sqrt{2}$, $J\Phi = (e^1\wedge e^4 + e^2\wedge e^3)/\sqrt{2}$, $\Psi_1 = (e^1\wedge e^2 - e^3\wedge e^4)/\sqrt{2}$, $\Psi_2 = (e^1 \wedge e^3 + e^2 \wedge e^4)/\sqrt{2}$, $\Psi_3 = (e^1 \wedge e^4 - e^2 \wedge e^3)/\sqrt{2}$, then $\{\Phi, J\Phi\}$ and $\{\Psi_1, \Psi_2, \Psi_3\}$ are local orthonormal frame of LM and $\wedge_0^{1,1}M$ respectively.

We denote by $\mathcal{W}$ the Weyl conformal curvature operator of M and $\mathcal{W}^+$ (resp. $\mathcal{W}^-$) the so called self-dual (resp. anti-self-dual) Weyl curvature operator. Considering the operators $\mathcal{R}$ and $\mathcal{W}$ as symmetric endomorphisms of $\wedge^2 M$, we have the following $SO(4)$-splitting

$$\mathcal{R} = \frac{\tau}{12} Id + \widetilde{\mathrm{Ric}_0} + \mathcal{W}^+ + \mathcal{W}^-,$$

where $\widetilde{\mathrm{Ric}_0}$ is the Kulkarni-Nomizu extension of the traceless Ricci tensor Ric_0 to an endomorphism of $\wedge^2 M$ anticommuting with the Hodge $*$-operator. It is well-known that M is an Einstein manifold if and only if the bundles $\wedge^2_{\pm} M$ are preserved by the operator $\mathcal{R}$. Further, $\mathcal{W}^+$ decomposes into three pieces corresponding to the decomposition (2.3). With respect to the basis $\{\Omega/\sqrt{2}, \Phi, J\Phi\}$ of $\wedge^2_+ M$, we can write $\mathcal{W}^+$ as the matrix of the form

$$\mathcal{W}^+ = \left(\begin{array}{c|c} \kappa/6 & \mathcal{W}_2^+ \\ \hline (\mathcal{W}_2^+)^* & \mathcal{W}_3^+ - (\kappa/12) I \end{array} \right), \tag{2.5}$$

where the smooth function $\kappa = 3g(\mathcal{W}^+(\Omega), \Omega)$ is the so called conformal scalar curvature, $\mathcal{W}_2^+$ is the part of $\mathcal{W}^+$ which exchanges the two factors in (2.3) and $\mathcal{W}_3^+$ is a traceless self-adjoint endomorphism of LM ([1]). We may easily check that

$$\kappa = \frac{1}{2}(3\tau^* - \tau) \quad \text{and} \quad \frac{\tau^*}{2} = g(\mathcal{R}(\Omega), \Omega).$$

An almost Hermitian manifold is called a manifold with Hermitian Weyl tensor if it satisfies the condition $\mathcal{W}_3^+ = 0$.

Now, we assume that M is a four-dimensional almost Kähler manifold. Then, the equality

$$R_{ijkl} - R_{ij\bar{k}\bar{l}} - R_{\bar{i}\bar{j}kl} + R_{\bar{i}\bar{j}\bar{k}\bar{l}} + R_{\bar{i}j\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{i\bar{j}k\bar{l}} = 2\sum_a (\nabla_a J_{ij})(\nabla_a J_{kl})$$

holds ([3]), and hence we have

$$\|\nabla J\|^2 = 2(\tau^* - \tau).$$

Since almost Kähler manifold is quasi-Kähler, we may observe that there exists a local 1-form α such that

$$\nabla \Omega = \alpha \otimes \Phi - J\alpha \otimes J\Phi.$$

From this equality, we have

$$\|\alpha\|^2 = \frac{1}{2}\|\nabla\Omega\|^2 = \frac{\tau^* - \tau}{2}.$$

We put

$$\begin{aligned} u &= g(\mathcal{R}(\Phi), \Phi) = -(R_{1313} - R_{1324}), \\ v &= g(\mathcal{R}(J\Phi), J\Phi) = -(R_{1414} + R_{1423}), \\ w &= g(\mathcal{R}(\Phi), J\Phi) = -(R_{1314} + R_{1323}), \end{aligned}$$

and define a function K on M by

$$K = (u - v)^2 + 4w^2.$$

If M is in addition an Einstein manifold, then we may observe that

$$u + v = -\frac{\tau^* - \tau}{4}$$

and that the matrix $\mathcal{W}_3^+$ in (2.5) is given by

$$\mathcal{W}_3^+ = \begin{pmatrix} u + \dfrac{\tau^* - \tau}{8} & w \\ w & v + \dfrac{\tau^* - \tau}{8} \end{pmatrix},$$

and hence we have the following.

Lemma 3 *A four-dimensional almost Kähler Einstein manifold M is a manifold with Hermitian Weyl tensor if and only if $K = 0$.*

In the sequel, we assume that M is a four-dimensional strictly almost Kähler Einstein manifold. Then, the set $M_0 = \{\, p \in M \mid (\tau^* - \tau)(p) > 0 \,\}$ is a non-empty open submanifold of M. Further, we put $M_1 = \{\, p \in M_0 \mid G(p) > 0 \,\}$, where G be a function on M defined by

$$G = \sum_{i,j} (\rho^*_{ij} - \rho^*_{ji})^2.$$

Then, we see that $M_0 - M_1$ has no interior point unless it is empty in case $\tau < 0$ (see [6]).

Lemma 4 ([6], Lemma 4) *For each point $p \in M_1$, there exists a special adapted pair $\{\{e_i\}_{i=1,\dots,4}, \{\alpha, J\alpha\}\}$ defined on a neighborhood U of p. Namely, the unitary frame $\{e_i\}$ and 1-forms α, $J\alpha$ on U satisfy*

$$\begin{aligned}\nabla\Omega &= \alpha\otimes\frac{1}{\sqrt{2}}(e^1\wedge e^3 - e^2\wedge e^4) - J\alpha\otimes\frac{1}{\sqrt{2}}(e^1\wedge e^4 + e^2\wedge e^3),\\ \alpha &= \alpha_1 e^1, \qquad \alpha_1^2 = \|\alpha\|^2 = \frac{\tau^*-\tau}{2}\end{aligned}$$

and

$$\rho^*_{13} = \frac{\sqrt{G}}{4}, \qquad \rho^*_{14} = 0.$$

With respect to the special adapted pair, we have the following equalities ([6], (3.9)-(3.12) and (3.19)):

$$\Gamma_{113} = \Gamma_{123} = \Gamma_{124} = \Gamma_{223} = \Gamma_{214} = \Gamma_{224} = 0, \tag{2.6}$$

$$\Gamma_{114} = \Gamma_{213} = -\frac{\alpha_1}{\sqrt{2}}, \tag{2.7}$$

$$\Gamma_{324} - \Gamma_{313} = 0, \qquad \Gamma_{323} + \Gamma_{314} = 0, \tag{2.8}$$

$$\Gamma_{424} - \Gamma_{413} = 0, \qquad \Gamma_{423} + \Gamma_{414} = 0, \tag{2.9}$$

$$\Gamma_{324} - \Gamma_{423} = 0, \qquad \Gamma_{313} + \Gamma_{414} = 0, \tag{2.10}$$

$$\rho^*_{13} = \frac{\alpha_1}{\sqrt{2}}(\Gamma_{314} - \Gamma_{413}) = -\frac{\alpha_1}{\sqrt{2}}(\Gamma_{323} + \Gamma_{424}) \tag{2.11}$$

$$(e_3\tau^*)^2 + (e_4\tau^*)^2 = 4(\tau^*-\tau)K + \frac{(\tau^*-\tau)^3}{4} - 2(\tau^*-\tau)^2(u-v). \tag{2.12}$$

Further, the first Chern from γ is locally represented as follows ([6], (2.44) with $\rho^*_{14} = 0$):

$$8\pi\gamma = \tau(e^1\wedge e^2 + e^3\wedge e^4) + (\tau^*-\tau)e^3\wedge e^4 + 4\rho^*_{13}(e^1\wedge e^4 + e^2\wedge e^3). \tag{2.13}$$

3. PROOF OF THEOREM 1

We assume that $G > 0$ everywhere on M. In this case, M is a strictly almost Kähler manifold. Further, the 2-form $\nu(X,Y) = \rho^*(X,Y) - \rho^*(Y,X)$ is J-skew-invariant and perpendicular to Ω, and hence a global section of the vector bundle LM. Thus, the Euler class $e(LM)$ of LM vanishes, and hence the first Chern class $c_1(LM, J)$ of the complex line bundle LM must vanish.

Now, we calculate the first Chern form $\gamma(LM,J)$ of complex line bundle LM. Let $\{\{e_i\}_{i=1,2,3,4},\{\alpha,J\alpha\}\}$ be a special adapted pair and put

$$\theta^1=\frac{1}{\sqrt{2}}(e^1\wedge e^3-e^2\wedge e^4),\qquad \theta^2=\frac{1}{\sqrt{2}}(e^1\wedge e^4+e^2\wedge e^3).$$

Then, $\{\theta^1,\theta^2\}$ is a local orthonormal frame field of LM. By direct calculation, we have

$$\begin{aligned}\nabla_{e_1}\theta^1 &= \frac{1}{\sqrt{2}}((\nabla_{e_1}e^1)\wedge e^3+e^1\wedge\nabla_{e_1}e^3-(\nabla_{e_1}e^2)\wedge e^4-e^2\wedge\nabla_{e_1}e^4)\\ &= (\Gamma_{112}+\Gamma_{134})\theta^2.\end{aligned}$$

Similarly, we obtain

$$\nabla_{e_i}\theta^1=\Gamma_i\theta^2,\qquad \nabla_{e_i}\theta^2=-\Gamma_i\theta^1, \tag{3.1}$$

for $i=1,\ldots,4$, where $\Gamma_i=\Gamma_{i12}+\Gamma_{i34}$. Thus, taking account of (2.6)–(2.11), we have

$$\begin{aligned}\nabla_{[e_1,e_2]}\theta^1 &= \sum_a(\Gamma_{12a}-\Gamma_{21a})\nabla_{e_a}\theta^1\\ &= \Gamma_{121}\Gamma_1\theta^2-\Gamma_{212}\Gamma_2\theta^2+\frac{\alpha_1}{\sqrt{2}}\Gamma_3\theta^2,\\ \nabla_{[e_1,e_3]}\theta^1 &= -\Gamma_{312}\Gamma_2\theta^2-\Gamma_{313}\Gamma_3\theta^2+(\Gamma_{134}-\Gamma_{314})\Gamma_4\theta^2,\\ \nabla_{[e_1,e_4]}\theta^1 &= \frac{\alpha_1}{\sqrt{2}}\Gamma_1\theta^2-\Gamma_{412}\Gamma_2\theta^2-(\Gamma_{134}+\Gamma_{413})\Gamma_3\theta^2+\Gamma_{313}\Gamma_4\theta^2,\\ \nabla_{[e_2,e_3]}\theta^1 &= \left(\frac{\alpha_1}{\sqrt{2}}+\Gamma_{312}\right)\Gamma_1\theta^2+\Gamma_{314}\Gamma_3\theta^2+(\Gamma_{234}-\Gamma_{313})\Gamma_4\theta^2,\\ \nabla_{[e_2,e_4]}\theta^1 &= \Gamma_{412}\Gamma_1\theta^2-(\Gamma_{234}+\Gamma_{313})\Gamma_3\theta^2-\Gamma_{413}\Gamma_4\theta^2,\\ \nabla_{[e_3,e_4]}\theta^1 &= -\frac{\sqrt{2}}{\alpha_1}\rho^*_{13}\Gamma_1\theta^2+\Gamma_{343}\Gamma_3\theta^2-\Gamma_{434}\Gamma_4\theta^2.\end{aligned} \tag{3.2}$$

We denote by K the curvature tensor of LM. Then, taking account of (3.1), (3.2), (2.6)–(2.11), we have

$$\begin{aligned}K(e_1,e_2)\theta^1 &= \nabla_{e_1}(\nabla_{e_2}\theta^1)-\nabla_{e_2}(\nabla_{e_1}\theta^1)-\nabla_{[e_1,e_2]}\theta^1\\ &= \nabla_{e_1}(\Gamma_2\theta^2)-\nabla_{e_2}(\Gamma_1\theta^2)+\Gamma_{112}\Gamma_1\theta^2+\Gamma_{212}\Gamma_2\theta_2-\frac{\alpha_1}{\sqrt{2}}\Gamma_3\theta^2\\ &= (e_1\Gamma_{212}+e_1\Gamma_{234})\theta^2-(e_2\Gamma_{112}+e_2\Gamma_{134})\theta^2\end{aligned}$$

$$
\begin{aligned}
& +\Gamma_{112}(\Gamma_{112}+\Gamma_{134})\theta^2+\Gamma_{212}(\Gamma_{212}+\Gamma_{234})\theta^2 \\
& -\frac{\alpha_1}{\sqrt{2}}(\Gamma_{312}+\Gamma_{334})\theta^2 \\
= & \left(R_{1212}+R_{1234}+\frac{\tau^*-\tau}{4}\right)\theta^2=\left(-\rho^*_{11}+\frac{\tau^*-\tau}{4}\right)\theta^2 \\
= & -\frac{\tau}{4}\theta^2.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
K(e_1,e_3)\theta^1 &= \rho^*_{14}\theta^2=0, \\
K(e_1,e_4)\theta^1 &= -\rho^*_{13}\theta^2, \\
K(e_2,e_3)\theta^1 &= \rho^*_{24}\theta^2=-\rho^*_{13}\theta^2, \\
K(e_2,e_4)\theta^1 &= -\rho^*_{23}\theta^2=0, \\
K(e_3,e_4)\theta^1 &= -\rho^*_{33}\theta^2=-\frac{\tau^*}{4}\theta^2.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
2\pi\gamma(LM,J) &= -\left(-\frac{\tau}{4}e^1\wedge e^2-\rho^*_{13}e^1\wedge e^4-\rho^*_{13}e^2\wedge e^3-\frac{\tau^*}{4}e^3\wedge e^4\right) \\
&= \frac{1}{4}\{\tau(e^1\wedge e^2+e^3\wedge e^4)+(\tau^*-\tau)e^3\wedge e^4 \\
&\quad +4\rho^*_{13}(e^1\wedge e^4+e^2\wedge e^3)\}.
\end{aligned}
$$

Comparing (2.13) with this equality, we conclude that

$$
c_1(M)=c_1(LM,J)=0. \tag{3.3}
$$

We now recall the Wu's theorem and the equality established in [9, (2.9)]:

$$
p_1(M)+2\chi(M) = c_1(M)^2, \tag{3.4}
$$

$$
p_1(M)+2\chi(M) = \frac{1}{2\pi^2}\int_M \|\mathcal{R}_+\|^2 dM, \tag{3.5}
$$

where $p_1(M)$ and $\chi(M)$ are the Pontrjagin class and the Euler-Poincaré class respectively, and $\mathcal{R}_+$ is the restriction of $\mathcal{R}$ to $\wedge^2_+M$. From (3.3), (3.4) and (3.5), we have $\mathcal{R}_+=0$ and hence $u=v=w=0$, $\tau^*=0$. Therefore, M is a Ricci flat Kähler manifold. But, this is a contradiction and completes the proof.

4. ANOTHER PROOF OF THEOREM 2

We assume that M is not a Kähler manifold. Then, the submanifold M_0 is not empty. Let $p_0 \in M_0$ be a point at which $\tau^* - \tau$ takes its maximum value.

We suppose $p_0 \in M_1$. Then, from Lemma 3 and (2.12), we obtain $(\tau^* - \tau)(p_0) = 0$. But, this is a contradiction.

Next, we suppose $p_0 \notin M_1$, namely $G(p_0) = 0$. If $M_1^c = M - M_1$ has an interior point, then from the result in [7], it follows that M is a Ricci flat Kähler manifold. But, this is also a contradiction. Thus, M_1^c has no interior point. Let $\{p_n\}_{n=1,2,\ldots}$ be a sequence in M_1 with $p_n \to p_0$ as $n \to \infty$. Then, from (2.12), we get

$$\|\text{grad}\,(\tau^* - \tau)\|^2(p_n) \geq (e_3\tau^*)^2(p_n) + (e_4\tau^*)^2(p_n) = \frac{(\tau^* - \tau)^3(p_n)}{4}(> 0)$$

for all n. On one hand, from the continuity the function $\|\text{grad}\,(\tau^* - \tau)\|^2$, we have

$$\lim_{n\to\infty} \|\text{grad}\,(\tau^* - \tau)\|^2(p_n) = \|\text{grad}\,(\tau^* - \tau)\|^2(p_0) = 0.$$

Therefore, we obtain $(\tau^* - \tau)(p_0) = 0$, which is a contradiction. This completes the proof.

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