Arab J. Math. Sc.
Volume 12, Number 1, June 2006
pp.7-15

INTERSECTION, FIXED POINTS AND MINIMAX INEQUALITIES WITH A GENERALIZED COERCIVITY IN H-SPACES

SOUHAIL CHEBBI[1]

ABSTRACT. We introduce a generalized coercivity type condition for correspondences defined on topological vector spaces endowed with a generalized convex structure. An extension of the Fan's matching theorem is obtained and used to prove results on fixed points and minimax inequalities with a weakned compactness condition.

1. INTRODUCTION

This paper is a study of a coercivity type condition for correspondences defined on topological spaces endowed with a generalized convex structure. We introduce the concept of coercing family in H-spaces and we propose the systematic development of the method based on the Fan's matching type theorem.

We firstly recall the structure of H-convexity defined by Horvath in [10] and H-KKM correspondence defined by Bardaro and Cepitelli in [4]. We then introduce the notion of H-coercing family for correspondences defined in H-spaces and give some examples from the literature. In section 3, we prove a Fan's type theorem on the intersection of correspondences defined in H-spaces and satisfying a weakened compactness condition. Theorem 1 and Theorem 2 of this section generalize recent results of Lassonde [[12], Theorem I], Horvath [[10], Theorem 1] and Bardaro and Cepitelli [[4], Theorem 1 and Theorem 2] as

[1]Supported by the Science Research Center of College at King Saud University (project No. Math/2005/09)

Mathematics Subject Classification. 54H25.

Keywords: H-spaces, H-compacts, H-coercing, fixed points, minimax inequalities.

well as corresponding results obtained in Fan [8], Ben El-Mechaiekh, Deguire and Granas [3], Ben El-Mechaiekh, Chebbi and Florenzano [2] when the H-convexity is replaced by the usual convexity of a topological vector space.

In Section 4, we generalize the results on coincidence and fixed point obtained by Horvath in [11], Bardaro and Cepitelli in [5] and Lassonde in [12] to coercive correspondences defined on non-compact H-spaces and we prove minimax inequalities for functions defined on H-spaces and satisfying a generalized coercivity type condition. Our results generalize minimax inequalities obtained by Allen in [1], Granas in [9], Ben El-Mechaiekh, Deguire and Granas in [3], Lassonde in [12], Horvath in [11] and Ding and Tan in [6].

2. PRELIMINARIES

Let $\langle X\rangle$ denote the family of all non-empty finite subsets of X. In order to define the setting of this paper, we firstly recall some basic concepts:

Definition 2.1. (a) (X,Γ) is said to be an *H-space* if X is a topological space and $\Gamma : \langle X\rangle \to 2^X$ a map such that $\Gamma(A) \subset \Gamma(B)$ if $A \subset B$ and assumed to have non-empty C^∞ values

(b) A subset $C \subset X$ is said to be *H-convex* if for every $A \in \langle C\rangle$, $\Gamma(A) \subset C$.

(c) A subset $K \subset X$ is said to be *H-compact* if for every $A \in \langle X\rangle$, there is a compact H-convex set D such that $A \cup K \subset D$.

Note that the class of H-spaces, which was firstly defined by Horvath in [11], contains topological vector spaces as well as a number of spaces with abstract topological convexity (the pseudo-convexity of Horvath in [10] and the concept of convex space due to Lassonde in [12] for example). For More details about generalized convexity, refer to [12], [10], [11], [4], [5] and [13]. The notion of H-compactness generalizes the c-compactness in [12].

A subset X of a topological space is said to be C^∞ (or *∞-connected*) if for each integer n, any continuous function $f : \partial\Delta_n \to X$ can be continuously extended to a continuous function $g : \Delta_n \to X$.

Definition 2.2. Let (X, Γ) be an H-space. A set-valued map (simply called correspondence) $F : X \to X$ is called *H-KKM* if and only if:

$$\forall A \in \langle X \rangle, \quad \Gamma(A) \subset \bigcup_{x \in A} F(x)$$

Definition 2.3. As was defined in [12], we say that a subet A of a topological space X is *compactly closed* (*open*, respectively) in X if for every compact set $C \subset X$, the set $A \cap C$ is closed (open respectively) in X.

We now introduce the concept of a generalized coercivity condition for correspondences as follow:

Definition 2.4. Let (X, Γ) be an H-space and Y a topological space. A family $\{(C_i, K_i)\}_{i \in I}$ is said to be *H-coercing* for a correspondence $F : X \to Y$ if and only if:

(i) For each $i \in I$, C_i is an H-compact subset of X and K_i is a compact subset of Y;
(ii) For each $i, j \in I$, there exists $k \in I$ such that $C_i \bigcup C_j \subseteq C_k$;
(iii) For each $i \in I$, there exists $k \in I$ such that:

$$\bigcap_{x \in C_k} F(x) \subseteq K_i.$$

Example 2.1. If $F : X \to X$ is a correspondence satisfying the following condition given in [11]: For some $x_0 \in X$, $F(x_0)$ is compact. Then F admits a coercing family.

Proof. Take, for all $i \in I$, $C_i = \{x_0\}$ and $K_i = F(x_0)$. □

Example 2.2. If $F : X \to X$ is a correspondence satisfying the following condition given in [6]: There exists an H-compact subset X_0 of X such that

$$\cap_{x \in X_0} F(x)$$

is compact. Then F admits a coercing family.

Proof. Take, for all $i \in I$, $C_i = \{X_0\}$ and $K_i = \cap_{x \in X_0} F(x)$. □

Note that when X is a subset of a topological vector space, the notion of coercing family in this generality was used by Ben El-Mechaiekh, Chebbi and Florenzano in [2] and generalized the concept of coercivity (with two sets K and C) used in [1],[3] and [8]. For more details about coercing family in topological vector space, see [2].

3. INTERSECTION THEOREMS

The main result of this paper is the following extension of Theorem 4 in [8]:

Theorem 3.1. *Let (X,Γ) be a an H-space, Y any topological space and $F : X \to Y$ a correspondence such that:*

(1) *For every $x \in X$, $F(x)$ is compactly closed in X.*
(2) *For some continuous map $s : X \to Y$, the correspondence $G : X \to X$ given by :*

$$G(x) = s^{-1}(F(x))$$

is H-KKM.

(3) *There exists an H-coercing family $\{(C_i, K_i)\}_{i\in I}$ for F. Then $\bigcap\limits_{x\in X} F(x) \neq \emptyset$.*

Proof. For $j = (i,a) \in J = I \times \langle X \rangle$, let $\hat{C}_j = C_i \cup a$ and $\hat{K}_j = K_i \cup a$. Since C_i is an H-compact subset, the family $\{(\hat{C}_j, \hat{K}_j)\}_{j\in J}$ is also H-coercing for F and, furthermore, $X = \bigcup\limits_{j\in J} \hat{C}_j$.

For every $j \in J$, let Z_j be the compact and H-convex set containing $\hat{C}_j$ and Let $Y_j = s(Z_j)$. We consider the correspondence $G_j : \hat{C}_j \to Z_j$ defined by :

$$G_j(x) = s_j^{-1}(F(x) \cap Y_j)$$

where s_j is the restriction of s to Z_j.

By (1), for each $x \in \hat{C}_j$, $G_j(x)$ is compact and it is easy to check that G_j is H-KKM since $G_j(x) = G(x) \cap Z_j$ and G is H-KKM. It follows from Corollary 1 in [10] that $\bigcap\limits_{x\in\hat{C}_j} G_j(x)$ is not empty, so $\bigcap\limits_{x\in\hat{C}_j} F(x)$ is also not empty . Using condition (ii) of Definition 2.4, we can see that the family $\{\bigcap_{x\in\hat{C}_j} F(x)\}_{j\in J}$ has the finite intersection property. Since for some $j \in J$, $\bigcap_{x\in\hat{C}_j} F(x)$ is contained in a compact set, we conclude that $\bigcap_{j\in J}\bigcap_{x\in\hat{C}_j} F(x)$ is not empty. Since $X = \bigcup_{j\in J}\hat{C}_j$, we just have to notice that $\bigcap_{j\in J}\bigcap_{x\in\hat{C}_j} F(x) = \bigcap_{x\in X} F(x)$, in order to complete the proof. □

Theorem 3.1 extends Theorem 1 in [4] which in turn generalizes Corollary 1 of Horvath in [10]. When I is a singleton and the H-convexity is replaced by the convexity of Lassonde, then Theorem 3.1 is reduced to Theorem I in [12].

For any correspondence $F : X \to Y$, let $F^* : Y \to X$ be the correspondence defined by:

$$F^*(y) = X \setminus F^{-1}(y)$$

The following result is more specially adapted to the study of minimax inequalities:

Theorem 3.2. *Let (X, Γ) be an H-space and $F, G : X \to X$ two correspondences such that:*

(a) *For every $x \in X$, $G(x)$ is compactly closed and $F(x) \subset G(x)$.*
(b) *For every $x \in X$, $x \in F(x)$*
(c) *$F^*(x)$ is H-convex .*
(d) *There exists an H-coercing family $\{(C_i, K_i)\}_{i\in I}$ for G. Then $\bigcap_{x\in X} G(x) \neq \emptyset$.*

Proof. By virtue of Theorem 3.1, it suffices to show that G is H-KKM. Suppose that for some finite subset $A \subset X$, there exists $y \in \Gamma(A)$ and $y \notin G(x)$ for every $x \in A$ and so $A \subset G^*(y)$. Since $G^*(y) \subset F^*(y)$ and by (c) $\Gamma(A) \subset F^*(y)$, hence $y \in F^*(y)$ which is equivalent to $y \notin F(y)$ and this contradicts (b). □

Note that Condition (d) of Theorem 3.2 extends the non-compactness condition of Theorem 2 in [10] and Theorem 1 in [11].

4. FIXED POINTS AND MINIMAX INEQUALITIES

As application of section 3, we prove a generalization of Ky Fan's fixed point theorem as presented by Ben-El Mechaiekh, Deguire and Granas in [3]:

Proposition 4.1. *Let (X, Γ) be an H-space, Y a topological space and $S : X \to Y$ a correspondence such that:*

(i) *For each $x \in X$, $S(x)$ is compactly open in Y.*
(ii) *For each $y \in X$, $S^{-1}(y)$ is non-empty and H-convex.*
(iii) *There exists an H-coercing family $\{(C_i, K_i)\}_{i \in I}$ for the correspondence $F : X \to Y$ defined by $F(x) = Y \backslash S(x)$, $\forall x \in X$.*

Then, for each continuous function s from X to Y, there exists an $x_0 \in X$ such that $s(x_0) \in S(x_0)$. In particular, S has a fixed point.

Proof. By (i), $F(x)$ is compactly closed for each $x \in X$. Let $s : X \to Y$ be any continuous map and $G : X \to X$ a correspondence defined, for all $x \in X$, by $G(x) = s^{-1}(F(x))$. G is not H-KKM, otherwise condition (ii) is not satisfied. Thus, there is a finite set $A \subset X$ and $x_0 \in \Gamma(A)$ such that $s(x_0) \in \cap_{x \in A} S(x)$ and so $s(x_0) \in S(x_0)$. □

Theorem 3.2 can be also used to prove results on minimax inequalities:

Proposition 4.2. *Let (X, Γ) be an H-space and let (E, C) be an order complete topological Riesz space, where C is the closed positive cone with a non-empty interior $int(C)$. Let $f,\ g : X \times X \to (E, C)$ be two functions satisfying the following conditions :*

(a) *For every $(x, y) \in X \times X$, $g(x, y) \leq f(x, y)$.*
(b) *For every $y \in X$ and any $\lambda \in E$, the set $\{x \in X : f(x, y) \in \lambda + int(C)\}$ is H-convex.*
(c) *For every $x \in X$ and any $\lambda \in E$, the set $\{y \in X : g(x, y) \in \lambda + int(C)\}$ is compactly open.*
(d) *There exists a family $\{(C_i, K_i)\}_{i \in I}$ satisfying condition (i) and (ii) of Definition 2.4 and the following one:*

$\forall i \in I,\ \exists k \in I$ such that $\{\mathrm{y} \in \mathrm{Y} : \mathrm{g(x, y)} \notin \lambda + \mathrm{int(C)} \quad \forall \mathrm{x} \in \mathrm{C_k}\} \subset \mathrm{K_i}$

Then, for every $\lambda \in E$, the following alternative holds:

(1) *There exists $y_0 \in X$ such that for every $x \in X$, $g(x, y_0) \notin \lambda + int(C)$.*

(2) *There exists $x_0 \in X$, such that $f(x_0, x_0) \in \lambda + int(C)$.*

Proof. For fixed $\lambda \in E$, we define $F(x) = \{y \in X : f(x, y) \notin \lambda + int(C)\}$ and $G(x) = \{y \in X : g(x, y) \notin \lambda + int(C)\}$. Condition (a) implies $F(x) \subset G(x)$ for every $x \in X$; indeed if $y \notin G(x)$, then $g(x, y) \in \lambda + int(C)$ and there is a neighborhood V of $0 \in E$ such that $g(x, y) + V \subset \lambda + int(C)$. But $g(x, y) \leq f(x, y)$ implies $\lambda < g(x, y) + v \leq f(x, y) + v$, for every $v \in V$ thus $f(x, y) + V \subset \lambda + int(C)$, that is $y \notin F(x)$.

If there exists $x_0 \in X$ with $x_0 \notin F(x_0)$, then $f(x_0, x_0) \in int(C)$ so we have (2). Otherwise $x \in F(x)$ for each $x \in X$. Hence all assumptions of Theorem 3.2 are satisfied, then $\bigcap_{x \in X} G(x) \neq \emptyset$ which implies property (1) of the alternative □

Corollary 4.1. *Let (X, Γ) be an H-space, (E, C) a completely ordered topological Riesz space. Suppose that $f : X \times X \to (E, C)$ is a function satisfying the following properties:*

(a) *f is upper bounded on the set $\triangle = \{(x, x) : x \in X\}$.*

(b) *For every $y \in X$ and any $\lambda \in E$, the set $\{x \in X : f(x, y) > \lambda\}$ is H-convex.*

(c) *For every $x \in X$ and any $\lambda \in E$, the set $\{y \in X : f(x, y) \leq \lambda\}$ is compactly closed.*

(d) *There exists a family $\{(C_i, K_i)\}_{i \in I}$ satisfying condition (i) and (ii) of Definition 4.2 and the following one:*

$$\forall i \in I,\ \exists k \in I \ \text{ such that } \{\mathrm{y} \in \mathrm{Y} : \mathrm{f(x, y)} \leq \lambda \ \ \forall \mathrm{x} \in \mathrm{C_k}\} \subset \mathrm{K_i} \quad \forall \lambda \in \mathrm{E}$$

Then: $\inf_{y \in X} \sup_{x \in X} f(x, y) \leq \sup f(x, x)$
(whenever the "inf" in the left-hand side exists).

Proof. Take $\lambda = \sup_{x \in X} f(x, x)$ which is well defined. By Proposition 4.2, there exists $y_0 \in X$ such that:

$$f(x, y_0) \leq \sup_{x \in X} f(x, x) \qquad \forall x \in X$$

Since (E, C) is completely ordered, it follows that $\sup_{x \in X} f(x, y_0)$ exists and the result follows. □

Note that Proposition 4.2 generalizes Theorem 3 in [3] and Proposition 5.1 in [10] by relaxing the compactness condition. In case $E = \mathbb{R}$, Corollary 4.1 is reduced to minimax inequalities obtained in [1] and [7].

REFERENCES

1. G. Allen, Variational inequalities, complementarity problems, and duality theorems, *J. Math. Anal. Appl.*, **58** (1977), 1-10.
2. H. Ben-El-Mechaiekh, S. Chebbi and M. Florenzano, A generalized KKMF principle,*J. Math. Anal.Appl.*, **309** (2005), 583-590.
3. H. Ben-El-Mechaiekh, P. Deguire and A. Granas, Points fixes et coincidences pour les applications multivoques II (applications de type ϕ et $\phi*$), *C. R. Acad. Sc. Paris* Série I Math., **295** (1982), 337-340.
4. C. Bordaro and R. Ceppitelli, Some further generalisations of Knaster-Kuratowski-Mazurkiwicz Theorem and Minimax Inequalities, *J. Math. Anal.Appl.*, **132** (1989), 484-490.
5. C. Bordaro and R. Ceppitelli, Fixed point theorems and vector valued minimax theorems, *J. Math. Anal.Appl.*, **146** (1990), 363-373.
6. X.P. Ding and K.K. Tan, Matching Theorems, Fixed point theorems and minimax inequalities without convexity, *J. Austral. Math. Soc.*, **49** (1990), 111-128.
7. K. Fan, A minimax inequality and applications, in: O. Shisha ed., *Inequalities Vol. III,* pp 103-113, Academic Press, New York-London, 1972.
8. K. Fan, Somme properties of convex sets related to fixed point theorems, *Math. Ann.*, **266** (1984), 519-537.
9. A. Granas, KKM maps and their applications to nonlinear problem, *Birkhauser Boston*, (1982).
10. C.D. Horvath, Points fixes et coincidence pour les applications multivoques sans convexité, *C.R. Acad. Sci. Paris*, **296** (1983), 403-406.

11. C.D. Horvath, contractibility and generalised convexity, *J. Math. Anal. Appl.*, **156** (1991), 341-357.
12. M. Lassonde, On the use of KKM multifunctions in fixed point theory and related topics, *J. Math. Anal. Appl.*, **97** (1983), 151-201.
13. S. Park and H. Kim, Admissible classes of multifunctions on generalized convex spaces, *Proc. Coll. Sci, SNU*, **18** (1993), 1-21.

SOUHAIL CHEBBI

Depatment of Mathematics, College of Science, King Saud University.

P.O. Box 2455, Riyadh 11451, Saudia Arabia.

E-mail: schebbi@ksu.edu.sa

Date received September 17, 2005