Arab J. Math. Sc.
Volume 4, Number 1, June 1998
pp. 29-37

CHARACTERIZATION OF SPHERES IN A EUCLIDEAN SPACE

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ABSTRACT. Let $\psi : M \to R^{n+1}$ be a compact a connected immersed hypersurface without boundary. If all the sectional curvatures of M are bounded below by a constant k_0, then it is shown that the inequality

$$||A||^2 - n\alpha^2 \geq \frac{(n-1)}{k_0}||grad\ \alpha||^2$$

implies that M is a sphere, where A is the shape operator and α is the mean curvature of M. It is also shown that if the mean curvature α and the support function $\rho = <\psi, N>$ of M satisfy $\alpha||\psi||^2 < \frac{n-3}{4n}\rho$ and the scalar curvature S of M satisfies $S||\psi||^2 = n(n-1), n \geq 3$, then ρ is a non-zero constant and M is the sphere $S^n\left(\frac{1}{\rho^2}\right)$, where N is the unit normal vector field to M.

1. INTRODUCTION

It is known that a compact positively curved hypersurface M of constant mean curvature in R^{n+1} is a sphere [5]. On a connected hypersurface in R^{n+1}, vanishing of $grad\ \alpha$ implies that α is a constant. Therefore in view of above result of Nomizu and Smyth, characterizing spheres among compact and connected positively curved hypersurfaces in R^{n+1} is equivalent to obtaining conditions under which $grad\ \alpha$ vanishes identically. In this paper we obtain one such characterization as given in the following theorem.

Theorem 1. *Let M be a compact and connected immersed hypersurface without boundary in R^{n+1}. If all the sectional curvatures of M are bounded below by a constant $k_0 > 0$ and the shape operator A, the mean curvature α of M satisfy*

$$\|A\|^2 - n\alpha^2 \geq \frac{(n-1)}{k_0}\|grad\ \alpha\|^2$$

then M is a sphere

Next we consider the hypersurface M in R^{n+1} with immersion $\psi : M \to R^{n+1}$; and the unit normal vector field N. The support function $\rho : M \to R$ is given by $\rho = <\psi, N>$ where $<,>$ is the Euclidean inner product on R^{n+1}. In [2], it is proved that if the mean curvature of the hypersurface $\psi : M \to R^4$ is nowhere zero and the scalar curvature S satisfies $S\|\psi\|^2 = 6$, then M is a sphere in R^4, and a question was raised whether this result could be generalized for $n > 3$. In this paper we generalize this result and prove the following theorem

Theorem 2. *Let $\psi : M \to R^{n+1}, n \geq 3$, be a compact and connected immersed hypersurface without boundary. If the mean curvature α the support function ρ of M satisfy $\alpha\|\psi\|^2 < \frac{n-3}{2n}\rho$, (or $\alpha\|\psi\|^2 > \frac{n-3}{2n}$) and the scalar curvature S of M satisfies $S\|\psi\|^2 = n(n-1)$, then ρ is a non-zero constant and M is the sphere $S^n(\frac{1}{\rho^2})$.*

2. PRELIMINARIES

Let M be an orientable hypersurface of R^{n+1} with unit normal vector field N. The Euclidean metric $<,>$ on R^{n+1} induces the Riemannian metric g on M. The Riemannian connections $\bar{\nabla}$ on R^{n+1} and ∇ on M are related by the Gauss and Weingarten formulas

$$(2.1)\quad \bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,\ \bar{\nabla}_X N = -A(X),\ \ X, Y \in \mathfrak{X}(M),$$

where A is the shape operator of M and $\mathfrak{X}(M)$ is the Lie-algebra of smooth vector fields on M. The curvature tensor field R and the scalar curvature S of M are given by

$$(2.2)\qquad \begin{aligned} R(X,Y;Z,W) &= g(AY,Z)g(AX,W) \\ &\quad -g(AX,Z)g(AY,W),\ X,Y,Z,W \in \mathfrak{X}(M) \end{aligned}$$

$$(2.3)\qquad S = n^2\alpha^2 - \|A\|^2,$$

where $\alpha = \frac{1}{n}\sum_i g(Ae_i,e_i)$ is the mean curvature of M and $\|A\|^2 = \sum_{ij} g(Ae_i,e_j)^2$ is the square of the length of the shape operator A, $\{e_1,\dots,e_n\}$ being a local orthonormal frame. The shape operator A satisfies the Codazzi equation

$$(\nabla A)(X,Y) = (\nabla A)(Y,X), \qquad X,Y \in \mathfrak{X}(M),$$

where the covariant derivative $(\nabla A)(X,Y)$ is given by

$$(\nabla A)(X,Y) = \nabla_X AY - A(\nabla_X Y).$$

The shape operator A being symmetric the following could be easily verified

$$(2.4)\qquad g((\nabla)(X,Y),Z) = g(Y,(\nabla A)(X,Z)).$$

For the immersion $\psi : M \to R^{n+1}$ of the hypersurface M, the support function $\rho : M \to R$ is defined by $\rho = <\psi, N>$. Taking ψ as position vector field of points of M in R^{n+1}, we can express it as $\psi = t + \rho N$, $t \in \mathfrak{X}(M)$. Taking the covariant derivative with respect to a $X \in \mathfrak{X}(M)$ of this expression of ψ and using (2.1) we obtain

$$(2.5)\qquad \nabla_X t = X + \rho A(X), \quad X(\rho) = -g(A(X),t).$$

If we define $F : M \to R$ by $F = \frac{1}{2}\|\psi\|^2$, then we obtain

$$grad\ F = t,$$

and the second equation in (2.5) gives

$$(2.6)\qquad grad\ \rho = -At.$$

The Laplacian operator Δ acting on smooth functions on a Riemannian manifold (M, g) is defined by $\Delta f = div(grad\ f)$. For two smooth function $f, h : M \to R$, using $\Delta(fh) = f\Delta h + h\Delta f + 2g(grad\ f, grad\ h)$, the following lemma can be easily proved by applying principle of mathematical induction separately for k and m.

Lemma 2.1.

$$\begin{aligned}\Delta(f^k h^m) &= m f^k h^{m-1}\Delta h + k f^{k-1} h^m \Delta f \\ &\quad + m(m-1) f^k h^{m-2} \|grad\ h\|^2 \\ &\quad + k(k-1) f^{k-2} h^m \|grad\ f\|^2 \\ &\quad + 2km f^{k-1} h^{m-1} g(grad\ f, grad\ h).\end{aligned}$$

3. PROOF OF THEOREM 1

First we prove the following Lemmas

Lemma 3.1. *Let M be a complete and connected hypersurface in R^{n+1}. Then*

$$\|\nabla A\|^2 \geq n\|grad\ \alpha\|^2$$

and for a positively curved M the equality holds if and only if M is a sphere.

Proof. Define a symmetric operator $B : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$B(X) = A(X) - \alpha X, \quad X \in \mathfrak{X}(M).$$

Then we get $(\nabla B)(X, Y) = (\nabla A)(X, Y) - g(grad\ \alpha, X)Y,\ X, Y \in \mathfrak{X}(M)$.

Using a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M, we arrive at

$$\begin{aligned}\|\nabla B\|^2 &= \|\nabla A\|^2 + n\|grad\ \alpha\|^2 \\ &\quad -2\sum_{ij} g((\nabla A)(e_i, e_j), e_j) g(grad\ \alpha, e_i)\end{aligned} \tag{3.1}$$

Note that

$$\begin{aligned} nX(\alpha) &= \sum_i Xg(A(e_i), e_i) = \sum_i g((\nabla A)(X, e_i), e_i) \\ &= \sum_i g(X, (\nabla A)(e_i, e_i)), \end{aligned}$$

where we have used the equation (2.5) in the last part of above equation. Thus we get $n\ grad\ \alpha = \sum_i (\nabla A)(e_i, e_i)$ and consequently we have

$$\begin{aligned} \sum_{ij} g((\nabla A)(e_i, e_j), e_j) g(grad\ \alpha, e_i) &= \sum_{ij} g(e_i, (\nabla A)(e_j, e_j)) \\ & \quad g(grad\ \alpha, e_i) \\ &= \sum_i ng(e_i, grad\ \alpha) g(e_i, grad\ \alpha) \\ &= n\|grad\ \alpha\|^2. \end{aligned}$$

Thus (3.1) gives

$$\|\nabla B\|^2 = \|\nabla A\|^2 - n\|grad\ \alpha\|^2 \geq 0.$$

Suppose all the sectional curvatures of M are positive and the equality $\|\nabla A\|^2 = n\|grad\ \alpha\|^2$ holds, which is equivalent to $(\nabla B)(X, Y) = 0$. Since M is irreducible (sectional curvatures of M being positive), $B = \lambda I$ for some constant $\lambda \in R$, consequently we have $A = (\alpha + \lambda)I$ which gives $n\alpha + n\lambda$, that is, $\lambda = 0$. Hence $A = \alpha I$, that is, M is totally umbilical complete and connected hypersurface of R^{n+1}, which is a sphere in R^{n+1} (cf. [4])

Lemma 3.2. *Let λ_i be the eigenvalues of the shape operator A of a compact hypersurface M in R^{n+1}. Then*

$$\int_M \left\{ \sum_{i<j} (\lambda_i - \lambda_j)^2 \right\} dv = n \int_M \|A\|^2 dv - n^2 \int_M \alpha^2 dv$$

Proof. We have for a local orthonormal frame $\{e_1, \ldots, e_n\}$ with $Ae_i = \lambda_i e_i$;

$$\sum_{ij} (\lambda_i - \lambda_j)^2 = \sum_{ij} \lambda_i^2 + \sum_{ij} \lambda_j^2 - 2 \sum_{ij} \lambda_i \lambda_j$$

$$= n\|A\|^2 + n\|A\|^2 - 2\sum_j(\sum_i \lambda_i)\lambda_j$$
$$= 2n\|A\|^2 - 2n\alpha\sum_j \lambda_i = 2n\|A\|^2 - 2n^2\alpha^2,$$

and $\sum_{ij}(\lambda_i - \lambda_j)^2 = 2\sum_{i<j}(\lambda_i - \lambda_j)^2$. This gives

$$\sum_{i<j}(\lambda_i - \lambda_j)^2 = n\|A\|^2 - n^2\alpha^2.$$

Integrating above equation proves the Lemma.

For a compact hypersurface M in R^{n+1} we have the following integral formula (cf. [3])

$$\int_M \left\{ \|\nabla A\|^2 - n^2\|grad\ \alpha\|^2 + \frac{1}{2}\sum_{i\neq j}(\lambda_i - \lambda_j)^2 K_{ij} \right\} dv = 0,$$

where $K_{ij} = R(e_i, e_j, e_j, e_i)$ is the sectional curvature of the plane section spanned by $\{e_i, e_j\}$. Since $\sum_{i\neq j}(\lambda_i - \lambda_j)^2 K_{ij} = 2\sum_{i<j}(\lambda_i - \lambda_j^2)K_{ij}$, above integral becomes

$$\int_M \left\{ \|\nabla A\|^2 - n^2\|grad : \alpha\|^2 + \sum_{i<j}(\lambda_i - \lambda_j)^2 K_{ij} \right\} dv = 0, \tag{3.2}$$

From the hypothesis of the theorem we have $K_{ij} > k_0$ and consequently (3.2) takes the form

$$\int_M \{\|\nabla\alpha\|^2 - n\|grad\ \alpha\|^2\}dv$$
$$+\int_M \left\{ k_0\sum_{i<j}(\lambda_i - \lambda_j)^2 - n(n-1)\|grad\ \alpha\|^2 \right\} dv \leq 0.$$

Using Lemma 3.2 in above integral inequality we get

$$\begin{aligned} &\int_M \{\|\nabla\alpha\|^2 - n\|grad\ \alpha\|^2\}dv \\ &+nk_0\int_M \left\{ \|A\|^2 - n\alpha^2 - \frac{(n-1)}{k_0}\|grad\ \alpha\|^2 \right\} dv \leq 0. \end{aligned} \tag{3.3}$$

Thus if $\|A\|^2 - n\alpha^2 \geq \frac{(n-1)}{k_0}\|grad\ \alpha\|^2$ holds, then the above inequality implies

$$\int_M \left\{\|\nabla A\|^2 - n\|grad\ \alpha\|^2\right\} dv \leq 0,$$

which together with Lemma 3.1 gives $\|\nabla A\|^2 = n\|grad\ \alpha\|^2$. As all the sectional curvatures of M are positive, using Lemma 3.1 again, we get that M is a sphere.

4. PROOF OF THEOREM 2

Using equation (2.5) and (2.6), we get

$$\begin{aligned} \Delta F = div(grad\ F) = div\ t &= \sum_i g(\nabla_{e_i} t, e_i) \\ &= \sum_i g(e_i + \rho A e_i, e_i), \end{aligned}$$

which gives

(4.1) $$\Delta F = n(1 + \rho\alpha).$$

Similarly using equations (2.4), (2.5) and $n\ grad\ \alpha = \sum_i (\nabla A)(e_i, e_i)$, we compute

(4.2) $$\Delta\rho = -nt(\alpha) - n\alpha - \rho\|A\|^2.$$

We use Lemma 2.1 and equations (2.5), (2.7), (4.1) and (4.2) to compute

(4.3) $$\begin{aligned} \Delta(\rho F^{(\frac{n-1}{2})}) = & -nF^{\frac{n-1}{2}} t(\alpha) - nF^{\frac{n-1}{2}}\alpha - \rho F^{\frac{n-1}{2}}\|A\|^2 \\ & + \frac{n(n-1)}{2}\rho F^{\frac{n-3}{2}} + \frac{n(n-1)}{2}\rho^2\alpha F^{\frac{n-3}{2}} \\ & + \rho\frac{(n-1)}{2}\frac{(n-3)}{2}F^{\frac{n-5}{2}}\|t\|^2 \\ & -(n-1)g(At, t)F^{\frac{n-3}{2}} \end{aligned}$$

Also we have

(4.4) $$\begin{aligned} div(F^{\frac{n-1}{2}}\alpha t) = & \ F^{\frac{n-1}{2}} t(\alpha) + \frac{n-1}{2}\alpha F^{\frac{n-3}{2}}\|t\|^2 \\ & + n\alpha F^{\frac{n-1}{2}} + n\rho\alpha^2 F^{\frac{n-1}{2}}, \end{aligned}$$

where we have used $t(F) = \|t\|^2$ and $div\ t = n(1+\rho\alpha)$ which are outcomes of (2.5) and (2.6) respectively.

Next we compute

$$div(F^{\frac{n-3}{2}}\rho t) = -F^{\frac{n-3}{2}}g(At,t) + \frac{n-3}{2}F^{\frac{n-5}{2}}\rho\|t\|^2 + n\rho F^{\frac{n-3}{2}} + n\rho^2\alpha F^{\frac{n-3}{2}}, \tag{4.5}$$

where we have used $t(\rho) = -g(At,t)$ which follows from (2.5).

Use (4.3), (4.4) and (4.5) to arrive at

$$\begin{aligned} &\Delta(\rho F^{\frac{n-1}{2}}) + ndiv(F^{\frac{n-1}{2}}\alpha t) - (n-1)div(^{\frac{n-3}{2}}\rho t) \\ &= \rho F^{\frac{n-1}{2}}S - \frac{n(n-1)}{2}\rho F^{\frac{n-3}{2}} + n(n-1)\alpha \\ &F^{\frac{n-3}{2}}\|t\|^2 - \rho\frac{(n-1)}{2}\frac{(n-3)}{2}F^{\frac{n-5}{2}}\|t\|^2, \end{aligned} \tag{4.6}$$

where we have used (2.3) and the fact $2F = \|t\|^2 + \rho^2$.

Integrating (4.6) and using Stokes theorem we get

$$n(n-1)\int_M F^{\frac{n-5}{2}}\|t\|^2\left(\alpha F - \frac{(n-3)}{4n}\rho\right)dv + \int_M \rho F^{\frac{n-3}{2}}\left(SF - \frac{n(n-1)}{2}\right)dv = 0$$

which is equivalent to

$$\frac{n(n-1)}{2}\int_M F^{\frac{n-5}{2}}\|t\|^2\left(\alpha\|\psi\|^2 - \frac{(n-3)}{2n}\rho\right)dv + \frac{1}{2}\int_M \rho F^{\frac{n-3}{2}}\left(S\|\psi\|^2 - n(n-1)\right)dv = 0$$

Thus in view of the hypothesis above integral gives $\|t\|^2 = 0$ or that $t = 0$, and consequently by the second equation in (2.6) it follows that ρ is a constant. Thus $\|\psi\| = \rho =$ constant, which proves that M is the sphere $S^n(\frac{1}{\rho^2})$.

Remark. For $n = 3$, the Theorem 2 reduces to Theorem 1 in [2].

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Date received August 6, 1997.