Arab J. Math. Sc.
Volume 6, Number 1, June 2000
pp. 17-36

PAIRINGS AND TWISTED PRODUCTS

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ABSTRACT. Let H and K be Hopf algebras (co)acting on an algebra A or a coalgebra C. Using so-called pairings, dual pairings and Hopf algebra maps $H \to K$, we construct twistings of the (co)multiplication on A (C). Our construction generalizes the smash product, the smash coproduct, the H-opposite algebra, the Drinfel'd double, and the Doi-Takeuchi product.

0. INTRODUCTION

Let H and K be bialgebras. Doi and Takeuchi [6] introduced the notion of skew pairing $\sigma : H \otimes K \to k$. For a convolution invertible skew pairing σ, they introduced a new bialgebra $H \bowtie_\sigma K$, which is equal to $H \otimes K$ as a coalgebra, but with a newly defined complicated multiplication. One of the interesting features of their construction is the fact that the Drinfel'd double can be viewed as a special case.

In this note, we will explain how the Doi-Takeuchi product appears in a natural way. To this end, we introduce the notion of pairing $\sigma : H \otimes K \to k$ (which is nothing else than a skew pairing $H^{\mathrm{cop}} \otimes K \to k$). Using such a pairing, we can twist the multiplication on an (H, K)-bicomodule algebra A in a natural way (see Proposition 2.1). Our main result is Theorem 4.3, stating that the Doi-Takeuchi product is a special case of sucha twisting.

[1]This work was supported by the NSF of China and the NSF of Henan province.

Mathematics Subject Classification. 16W30.

Our construction has five variations: one can also twist the multiplication on an (H,K)-dimodule algebra and an (H,K)-bimodule algebra, and three analogous constructions exist for the comultiplication on coalgebras with compatible (co)action by H and K. There exist several duality results between the six constructions, under the assumption that H, K or A is finitely generated and projective. Other special cases of our construction include the smash product of an H-module algebra and an H-comodule algebra ([9] is the oldest reference for this construction), Molnar's smash coproduct of an H-module coalgebra and an H-comodule coalgebra (see [13]), Long's H-opposite algebra (see [11]), and the twisted coalgebra introduced in [3].

Our paper is organized as follows. In Section 1, we give the definition of pairing, and its dual version, which we call a dual pairing element. Actually, there is a "semi-dual", consisting of just bialgebra maps $H \to K$. In Section 2, we discuss how we can twist the multiplication on an algebra (with compatible (co)action by two bialgebras H and K), and in Section 3, we obtain the dual results for coalgebras. In Section 4, we prove that our constructions include previously introduced constructions as special cases.

Doi and Takeuchi mention that their construction contains the Drinfel'd double as a special case; in [4], it is shown that the Drinfel'd double may also be viewed as a smash product. Since both the smash product and the Doi-Takeuchi construction are special cases of our new construction, the Drinfel'd double appears in two ways as a special case. This is made explicit in Section 4, using one of our duality Theorems.

We will use the Sweedler-Heyneman notation for comultiplications and coactions. If C is a coalgebra, then we write for all $c \in C$:

$$\Delta_C(c) = \sum c_{(1)} \otimes c_{(2)}$$

If M is a right C-comodule, then we write for all $m \in M$:

$$\rho^r(m) = \sum m_{[0]} \otimes m_{[1]}$$

Similarly, for a left C-comodule M:

$$\rho^l(m) = \sum m_{[-1]} \otimes m_{[0]}$$

If M is a left-right (C, D)-bicomodule, then

$$\begin{aligned}(I_C \otimes \rho^r)(\rho^l(m)) &= \sum m_{[-1]} \otimes m_{[0][0]} \otimes m_{[0][1]} = \\ (\rho^l \otimes I_D)(\rho^r(m)) &= \sum m_{[0][-1]} \otimes m_{[0][0]} \otimes m_{[1]}\end{aligned}$$

and we write

$$(I_C \otimes \rho^r)(\rho^l(m)) = (\rho^l \otimes I_D)(\rho^r(m)) = \sum m_{[-1]} \otimes m_{[0]} \otimes m_{[1]}$$

For more details about Hopf algebras, actions and coactions, we refer to the literature, see for example [14], [15].

1. PAIRINGS AND DUAL PAIRINGS

In the sequel, H and K will be bialgebras over a commutative ring k. A map $\sigma : H \otimes K \to k$ will be called a *pairing* if the following conditions hold, for all $h, h' \in H$ and $k, k' \in K$:

$$\sigma(hh' \otimes k) = \sum \sigma(h \otimes k_{(1)})\sigma(h' \otimes k_{(2)}) \tag{1}$$

$$\sigma(h \otimes kk') = \sum \sigma(h_{(1)} \otimes k)\sigma(h_{(2)} \otimes k') \tag{2}$$

$$\sigma(1 \otimes k) = \varepsilon(k) \tag{3}$$

$$\sigma(h \otimes 1) = \varepsilon(h) \tag{4}$$

$\mathrm{P}(H, K)$, the set of all pairings, is a multiplicative subset of $H \otimes K^*$.

An element $R = \sum R^1 \otimes R^2 = \sum r^1 \otimes r^2 \in H \otimes K$ is called a *dual pairing element* if

$$\sum R^1_{(1)} \otimes R^1_{(2)} \otimes R^2 = \sum R^1 \otimes r^1 \otimes R^2 r^2 \tag{5}$$

$$\sum R^1 \otimes R^2_{(1)} \otimes R^2_{(2)} = \sum R^1 r^1 \otimes R^2 \otimes r^2 \tag{6}$$

$$\sum R^1 \varepsilon(R^2) = 1_H \tag{7}$$

$$\sum \varepsilon(R^1) R^2 = 1_K \tag{8}$$

$\mathrm{DP}(H, K)$, the set of all dual pairings elements, is a multiplicative subset of $H \otimes K$.

Remarks

1) Let $H = kG$ and $K = kG'$ be group algebras. Then $P(H,K)$ consists of bimultiplicative maps $G \times G' \to k$.

2) Pairings of Hopf algebras have been considered before by several authors; see for example [2] (in the case where $H = K$ is commutative and cocommutative), and [7]. In [6], skew pairings are introduced. The following statements are then equivalent:

1) $\sigma : H \otimes K \to k$ is a pairing;
2) $\sigma : H \otimes K^{op} \to k$ is a skew pairing in the sense of [6];
3) $\sigma : H^{cop} \otimes K \to k$ is a skew pairing in the sense of [6].

If (H, σ) is a coquasitriangular algebra (also called dual quasitriangular or braided) bialgebra, then $\sigma : H \otimes H \to k$ is a skew pairing.

3) In a similar way, the following are equivalent:

1) $R \in H \otimes K$ is a dual pairing;
2) $R \in H^{op} \otimes K$ is a bialgebra copairing in the sense of [3];
3) $R \in H \otimes K^{cop}$ is a bialgebra copairing in the sense of [3].

If (H, R) is a quasitriangular bialgebra, then $R \in H \otimes H$ is a bialgebra copairing.

Proposition 1.2. *If $\sigma \in \mathrm{P}(H,K)$ has a convolution inverse σ^{-1}, then $\sigma^{-1} \in \mathrm{P}(H^{\mathrm{cop}}, K^{\mathrm{cop}})$.*

Proof. We have to check that σ^{-1} satisfies conditions (1-4). (1) takes the form

$$\sigma^{-1}(hh' \otimes k) = \sum \sigma^{-1}(h \otimes k_{(2)})\sigma^{-1}(h' \otimes k_{(1)}) \tag{9}$$

Define $f_1, f_2, g_1, g_2 : H \otimes H \otimes K \to k$ by the formulas

$$\begin{aligned} f_1(h \otimes h' \otimes k) &= \sigma(hh' \otimes k) \\ f_2(h \otimes h' \otimes k) &= \sum \sigma(h \otimes k_{(1)})\sigma(h' \otimes k_{(2)}) \\ g_1(h \otimes h' \otimes k) &= \sigma^{-1}(hh' \otimes k) \\ g_1(h \otimes h' \otimes k) &= \sum \sigma^{-1}(h \otimes k_{(2)})\sigma^{-1}(h' \otimes k_{(1)}) \end{aligned}$$

We know that $f_1 = f_2$. It is easily verified that g_i is the convolution inverse of f_i, implying that $g_1 = g_2$. Verification of (2-4) is left to the reader.

We have a similar result for dual pairing elements:

Proposition 1.3. *If $R \in \mathrm{DP}(H,K)$ is invertible with inverse $S = R^{-1}$, then $R^{-1} \in \mathrm{DP}(H^{\mathrm{op}}, K^{\mathrm{op}})$.*

Proof. We have to prove that S satisfies (5-8). (5) takes the form

$$\sum \Delta(S^1) \otimes S^2 = \sum S^1 \otimes s^1 \otimes s^2 S^2 \tag{10}$$

Now we compute easily that

$$\sum (\Delta(S^1) \otimes S^2)(\Delta(R^1) \otimes R^2) = (\Delta \otimes I)(SR) = 1 \otimes 1 \otimes 1$$

and

$$\begin{aligned}(\sum S^1 \otimes s^1 \otimes s^2 S^2)(\sum R^1 \otimes r^1 \otimes R^2 r^2) &= \sum S^1 R^1 \otimes s^1 r^1 \otimes s^2 S^2 R^2 r^2 \\ &= S_{23} S_{13} R_{13} R_{23} = 1 \otimes 1 \otimes 1\end{aligned}$$

We conclude that the two sides of (10) are the inverses of the two sides of (5). We know that (5) holds, so (10) also holds. The other conditions can be proved in a similar way

Proposition 1.4. *Assume that K is finitely generated and projective as a k-module. Then $\mathrm{P}(H,K) \cong \mathrm{Hopf}(H, K^*)$.*

Proof. It is well-known that $(H \otimes K)^* \cong \mathrm{Hom}\,(\mathrm{H}, \mathrm{K}^*)$ as k-algebras. The map $f : H \to K^*$ corresponding to $\sigma \in (H \otimes K)^*$ is given by the formula

$$\langle f(h), k \rangle = \sigma(h \otimes k)$$

It is straightforward to show that this correspondence restricts to an isomorphism $\mathrm{P}(H,K) \to \mathrm{Hopf}(H,K^*)$.

Proposition 1.5. *Assume that H is finitely generated and projective as a k-module. Then $\mathrm{DP}(H,K) \cong \mathrm{Hopf}(H^*, K)$ as k-algebras.*

Proof. If H is finitely generated and projective, then we have an isomorphism

$$F : H \otimes K \to \mathrm{Hom}\,(\mathrm{H}^*, \mathrm{K})$$

defined as follows: $F(R) = f$, with

$$f(h^*) = \sum \langle h^*, R^1 \rangle R^2$$

If $f \in \mathrm{Hom}\,(\mathrm{H}^*, \mathrm{K})$, then

$$F^{-1}(f) = \sum_i h_i \otimes f(h_i^*)$$

where $\{h_i, h_i^* \mid i = 1, \cdots, n\}$ is a dual basis for H. A straightforward computation shows that F is a k-algebra isomorphism, the multiplication on $\mathrm{Hom}\,(\mathrm{H}^*, \mathrm{K})$ being the convolution. Standard calculations show that F restricts to an isomorphism $\mathrm{DP}(H, K) \to \mathrm{Hopf}(H^*, K)$.

2. TWISTING THE MULTIPLICATION ON AN ALGEBRA

Let A be left-right (H, K)-bicomodule algebra. This means that A is at the same time a left H-comodule algebra, a right K-comodule algebra, and a left-right (H, K)-bicomodule. For a pairing $\sigma \in \mathrm{P}(H, K)$, we put ${}^\sigma A$ equal to A as a k-module, but with new multiplication

$$a \cdot_\sigma b = \sum \sigma(a_{[-1]}, b_{[1]}) a_{[0]} b_{[0]} \tag{11}$$

The subscript σ will be omitted if no confusion is possible.

Proposition 2.1. *If $\sigma : H \otimes K \to k$ is a pairing, then ${}^\sigma A$ is an associative algebra with unit* 1.

Proof. This result can be found in [7, Theorem 2.1], using a different notation. We give the proof for completeness sake. The proof of the associativity goes as follows:

$$\begin{aligned} (a \cdot b) \cdot c &= \sum \sigma(a_{[-1]}, b_{[1]}) (a_{[0]} b_{[0]}) \cdot c \\ &= \sum \sigma(a_{[-2]}, b_{[1]}) \sigma(a_{[-1]} b_{[-1]}, c_{[1]}) a_{[0]} b_{[0]} c_{[0]} \\ (1) \quad &= \sum \sigma(a_{[-2]}, b_{[1]}) \sigma(a_{[-1]}, c_{[1]}) \sigma(b_{[-1]}, c_{[2]}) a_{[0]} b_{[0]} c_{[0]} \end{aligned}$$

where equation (1) is used to obtain the last equality

$$\begin{aligned} a \cdot (b \cdot c) &= \sum \sigma(b_{[-1]}, c_{[1]}) a \cdot (b_{[0]} c_{[0]}) \\ &= \sum \sigma(b_{[-1]}, c_{[2]}) \sigma(a_{[-1]}, b_{[1]} c_{[1]}) a_{[0]} b_{[0]} c_{[0]} \\ (2) \quad &= \sum \sigma(b_{[-1]}, c_{[2]}) \sigma(a_{[-2]}, b_{[1]}) \sigma(a_{[-1]}, c_{[1]}) a_{[0]} b_{[0]} c_{[0]} \end{aligned}$$

The fact that 1 is a neutral element follows easily from equations (3-4).

Now let A be a left-left (H,K)-dimodule algebra (cf. [11]). By this we mean that A is at the same time a left H-comodule algebra and a left K-module algebra, such that the following compatibility condition holds:

$$\rho^l(k \rightharpoonup a) = \sum a_{[-1]} \otimes k \rightharpoonup a_{[0]} \tag{12}$$

This condition means that the left K-action is left H-colinear, i.e. A is a dimodule in the sense of [11]. For a Hopf algebra map $f: H \to K$, we define a new multiplication on A as follows:

$$a \cdot_f b = \sum a_{[0]}(f(a_{[-1]}) \rightharpoonup b) \tag{13}$$

${}^fA$ will be our notation for A furnished by this new multiplication. We will often drop the subscript f: $a \cdot_f b = a \cdot b$.

Proposition 2.2. *If $f: H \to K$ is a bialgebra map, then ${}^fA$ is an associative algebra with unit* 1.

Proof. Let us verify the associativity; the fact that 1 is unit can be seen easily.

$$\begin{aligned}
(a \cdot b) \cdot c &= \sum \Big(a_{[0]}(f(a_{[-1]}) \rightharpoonup b)\Big) \cdot c \\
&= a_{[0]}(f(a_{[-2]}) \rightharpoonup b_{[0]})\Big((f(a_{[-1]})f(b_{[-1]})) \rightharpoonup c\Big) \\
&= \sum a \cdot \Big(b_{[0]}(f(b_{[-1]}) \rightharpoonup c)\Big) \\
&= a \cdot (b \cdot c)
\end{aligned}$$

Now let A be a left-right (K,H)-bimodule algebra, and $R \in \mathrm{DP}(H,K)$ a dual pairing element. ${}^RA$ will be equal to A as a k-module, but with new multiplication defined by

$$a \cdot_R b = \sum (a \leftharpoonup R^1)(R^2 \rightharpoonup b) \tag{14}$$

Proposition 2.3. *If $R \in \mathrm{DP}(H,K)$ is a dual pairing element, then ${}^RA$ is an associative algebra with unit* 1.

Proof. We easily compute that

$$\begin{aligned}(a\cdot b)\cdot c &= \sum((a\leftharpoonup R^1)(R^2\rightharpoonup b))\cdot c\\ &= \sum(a\leftharpoonup R^1r^1_{[1]})(R^2\rightharpoonup b\leftharpoonup r^1_{[2]})(r^2\rightharpoonup c)\\ a\cdot(b\cdot c) &= \sum a\cdot((b\leftharpoonup R^1)(R^2\rightharpoonup c))\\ &= \sum(a\leftharpoonup r^1)(r^2_{[1]}\rightharpoonup b\leftharpoonup R^1)(r^2_{[2]}R^2\leftharpoonup c)\end{aligned}$$

To see that $(a\cdot b)\cdot c = a\cdot(b\cdot c)$, it therefore suffices to show that

$$\sum R^1r^1_{[1]}\otimes R^2\otimes r^1_{[2]}\otimes r^2=\sum r^1\otimes r^2_{[1]}\otimes R^1\otimes r^2_{[2]}R^2$$

Using (5), we find that the left hand side is equal to

$$\sum R^1r^1\otimes R^2\otimes r'^1\otimes r^2r'^2$$

(6) implies that the right hand side equals

$$\sum r^1r'^1\otimes r^2\otimes R^1\otimes r'^2R^2$$

and the associativity follows.

Now assume that K is a finitely generated projective bialgebra, and that the Hopf algebra map $f:\ H\to K^*$ corresponds to the pairing $\sigma:\ H\otimes K\to k$. Let A be a left-right (H,K)-comodule algebra. We define a left K^*-coaction on A by the formula

$$k^*\rightharpoonup a=\sum\langle k^*,a_{[1]}\rangle a_{[0]}$$

Then A is a left-left (H,K^*)-dimodule algebra. Moreover

Proposition 2.4. *With notation as above, ${}^{\sigma}A={}^{f}A$ as k-algebras.*

Proof. For all $a,b\in A$, we have

$$\begin{aligned}a\cdot_\sigma b &= \sum\sigma(a_{[-1]},b_{[1]})a_{[0]}b_{[0]}\\ &= \sum f(a_{[-1]}),b_{[1]}\rangle a_{[0]}b_{[0]}\\ &= \sum a_{[0]}(f(a_{[-1]})\rightharpoonup b)\\ &= a\cdot_f b\end{aligned}$$

Now assume that H is finitely generated and projective as a k-module. If A is a left-left (H^*, K)-dimodule algebra, then it is also a left-right (K, H)-bimodule algebra; the right H-action is given by the formula $a \leftharpoonup h = \sum \langle h, a_{[-1]} \rangle a_{[0]}$

Proposition 2.5. *With notation as above, ${}^R A = {}^f A$ as k-algebras.*

Proof. For all $a, b \in A$, we have

$$\begin{aligned} a \cdot_R b &= \sum (a \leftharpoonup R^1)(R^2 \rightharpoonup b) \\ &= \sum \langle R^1, a_{[-1]} \rangle a_{[0]} (R^2 \leftharpoonup b) \\ &= \sum a_{[0]} (f.(a_{[-1]}) \rightharpoonup b) \\ &= a \cdot_f b \end{aligned}$$

3. CONSTRUCTING NEW COALGEBRAS

As in the previous Section, H and K will be bialgebras. Let C be a left-right (K, H)-comodule coalgebra. This means that C is at the same time a left-right (K, H)-bicomodule and a k-coalgebra such that

$$\sum c_{[0](1)} \otimes c_{[0](2)} \otimes c_{[1]} = \sum c_{(1)[0]} \otimes c_{(2)[0]} \otimes c_{(1)[1]} c_{(2)[1]} \tag{15}$$

$$\sum c_{[-1]} \otimes c_{[0](1)} \otimes c_{[0](2)} = \sum c_{(1)[-1]} c_{(2)[-1]} \otimes c_{(1)[0]} \otimes c_{(2)[0]} \tag{16}$$

for all $c \in C$. For any pairing $\sigma : H \otimes K \to k$, we introduce a new coalgebra $C^\sigma = (C, \Delta_\sigma)$, with comultiplication Δ_σ given by the formula

$$\Delta_\sigma(c) = \sum \sigma(c_{(1)[1]} \otimes c_{(2)[-1]}) c_{(1)[0]} \otimes c_{(2)[0]} \tag{17}$$

Proposition 3.1. *Let C and σ be as above. Then C^σ is a coassociative coalgebra with counit ε_C.*

Proof. We only have to verify the coassociativity. For all $c \in C$, we have

$$\begin{aligned} &(I_C \otimes \Delta_\sigma)\Delta_\sigma(c) \\ &\quad = \sum \sigma(c_{(1)[1]} \otimes c_{(2)[-1]}) \sigma(c_{(2)[0](1)[1]} \otimes c_{(2)[0](2)[-1]}) c_{(1)[0]} \end{aligned}$$

$$\begin{aligned}
& \otimes c_{(2)[0](1)[0]} \otimes c_{(2)[0](2)[0]} \\
\text{(16)} \quad &= \sum \sigma(c_{(1)[1]} \otimes c_{(2)[-1]}c_{(3)[-2]})\sigma(c_{(2)[1]} \otimes c_{(3)[-1]})c_{(1)[0]} \\
& \otimes c_{(2)[0]} \otimes c_{(3)[0]} \\
\text{(1)} \quad &= \sum \sigma(c_{(1)[1]} \otimes c_{(2)[-1]})\sigma(c_{(1)[2]} \otimes c_{(3)[-2]})\sigma(c_{(2)[1]} \otimes c_{(3)[-1]})c_{(1)[0]} \\
& \otimes c_{(2)[0]} \otimes c_{(3)[0]}
\end{aligned}$$

and

$$\begin{aligned}
& (\Delta_\sigma \otimes I_C)\Delta_\sigma(c) \\
&= \sum \sigma(c_{(1)[1]} \otimes c_{(2)[-1]}) \otimes \sigma(c_{(1)[0](1)[1]} \otimes c_{(1)[0](2)[-1]})c_{(1)[0](1)[0]} \\
& \otimes c_{(1)[0](2)[0]} \otimes c_{(2)[0]} \\
\text{(15)} \quad &= \sum \sigma(c_{(1)[2]}c_{(2)[1]} \otimes c_{(3)[-1]})\sigma(c_{(1)[1]} \otimes c_{(2)[-1]})c_{(1)[0]} \\
& \otimes c_{(2)[0]} \otimes c_{(3)[0]} \\
\text{(2)} \quad &= \sum \sigma(c_{(1)[2]} \otimes c_{(3)[-2]})\sigma(c_{(2)[1]} \otimes c_{(3)[-1]})\sigma(c_{(1)[1]} \otimes c_{(2)[-1]})c_{(1)[0]} \\
& \otimes c_{(2)[0]} \otimes c_{(3)[0]}
\end{aligned}$$

and the coassociativity follows.

Now let C be a right-right (K, H)-dimodule coalgebra. This means that C is a right K-module coalgebra, a right H-comodule coalgebra, and a (K, H)-dimodule. This means that (15) holds, and

$$\Delta(c \leftharpoonup k) = \sum (c_{(1)} \otimes k_{(1)}) \otimes (c_{(2)} \otimes k_{(2)}) \tag{18}$$

$$\rho^r(c \leftharpoonup k) = \sum (c_{[0]} \leftharpoonup k) \otimes c_{[1]} \tag{19}$$

for all $c \in C$ and $k \in K$. For any bialgebra map $f : H \to K$, we let $C^f = C$ as a k-module, with new comultiplication

$$\Delta_f(c) = \sum c_{(1)[0]} \otimes (c_{(2)} \leftharpoonup f(c_{(1)[1]})) \tag{20}$$

Proposition 3.2. *Let C and f be as above. Then C^f is a coassociative coalgebra with counit ε_C.*

Proof. For all $c \in C$, we have

$$(\Delta_f \otimes I)\Delta_f(c) = \sum c_{(1)[0](1)[0]} \otimes (c_{(1)[0](2)} \leftharpoonup f(c_{(1)[0](1)[1]}))$$

$$\otimes(c_{(2)}\leftharpoonup f(c_{(1)[1]}))$$

$$\text{(apply (15) to } c_{(1)}) \quad = \sum c_{(1)[0]} \otimes (c_{(2)[0]}\leftharpoonup f(c_{(1)[1]})) \otimes (c_{(3)}\leftharpoonup f(c_{(1)[2]}c_{(2)[1]}))$$

and

$$\begin{aligned}(I\otimes\Delta_f)\Delta_f(c) &= \sum c_{(1)[0]} \otimes (c_{(2)[0]}\leftharpoonup f(c_{(1)[1]})) \otimes (c_{(3)}\leftharpoonup f(c_{(1)[2]}))\leftharpoonup f(c_{(2)[1]}) \\ &= \sum c_{(1)[0]} \otimes (c_{(2)[0]}\leftharpoonup f(c_{(1)[1]})) \otimes (c_{(3)}\leftharpoonup f(c_{(1)[2]}c_{(2)[1]}))\end{aligned}$$

proving that Δ_f is coassociative.

Now let C be a left-right (H,K)-bimodule coalgebra, and R a dual pairing element in $H\otimes K$. We define $C^R = C$ as a k-module, with new comultiplication

$$\Delta_R(c) = \sum (R^1\rightharpoonup c_{(1)}) \otimes (c_{(2)}\leftharpoonup R^2) \tag{21}$$

Proposition 3.3. *With notation as above, C^R is a coassociative coalgebra with counit ε_C.*

Proof. For all $c\in C$, we have

$$\begin{aligned}(\Delta_R\otimes I)\Delta_R(c) &= \sum ((r^1\rightharpoonup(R^1_{(1)}\rightharpoonup c_{(1)})) \otimes ((R^1_{(2)}\rightharpoonup c_{(2)}\leftharpoonup r^2)) \otimes (c_{(3)}\otimes R^2) \\ (12) \quad &= \sum ((r^1R^1)\rightharpoonup c_{(1)}) \otimes (r'^1\rightharpoonup c_{(2)}\leftharpoonup r^2) \otimes (c_{(3)}\leftharpoonup(R^2r'^2))\end{aligned}$$

and

$$\begin{aligned}(I\otimes\Delta_R)\Delta_R(c) &= \sum (R^1\rightharpoonup c_{(1)}) \otimes (r^1\rightharpoonup c_{(2)}\leftharpoonup R^2_{(1)}) \otimes (c_{(3)}\leftharpoonup R^2_{(2)}r^2) \\ (13) \quad &= \sum ((R^1r'^1)\rightharpoonup c_{(1)}) \otimes (r^1\rightharpoonup c_{(2)}\leftharpoonup R^2) \otimes (c_{(3)}\leftharpoonup r'^2r^2)\end{aligned}$$

and the coassociativity of Δ_R follows.

Assume that K is finitely generated and projective as a k-module, and take $\sigma\in \mathrm{P}(H,K)$ corresponding to the bialgebra map $f: H\to K^*$, as in Proposition 1.4. If C is a left-right (K,H)-comodule coalgebra, then it isalso a right-right (K^*,H)-dimodule coalgebra, if we put

$$c\leftharpoonup k^* = \sum \langle k^*, c_{[-1]}\rangle c_{[0]}$$

Proposition 3.4. *With notation as above, we have that* $C^\sigma = C^f$.

Proof. It suffices to prove that the comultiplications Δ_f and Δ_σ coincide. Indeed, we have for all $c \in C$ that

$$\begin{aligned}
\Delta_\sigma(c) &= \sum \sigma(c_{(1)[1]} \otimes c_{(2)[-1]}) c_{(1)[0]} \otimes c_{(2)[0]} \\
&= \sum \langle f(c_{(1)[1]}), c_{(2)[-1]} \rangle c_{(1)[0]} \otimes c_{(2)[0]} \\
&= \sum c_{(1)[0]} \otimes (c_{(2)} \leftharpoonup f(c_{(1)[1]})) \\
&= \Delta_f(c)
\end{aligned}$$

Now let H be finitely generated and projective, and let $R \in \mathrm{DP}(H,K)$ correspond to $f : H^* \to K$, as in Proposition 1.5. If C is a right-right (K,H^*)-dimodule coalgebra, then it is also a left-right (H,K)-bimodule algebra, if we let

$$h^* \rightharpoonup c = \sum \langle h^*, c_{[1]} \rangle c_{[0]}$$

Proposition 3.5. *With notation as above, we have that* $C^R = C^f$.

Proof. For all $c \in C$, we have

$$\begin{aligned}
\Delta_f(c) &= \sum c_{(1)[0]} \otimes (c_{(2)} \leftharpoonup f(c_{(1)[1]})) \\
&= \sum \langle c_{(1)[1]}, R^1 \rangle c_{(1)[0]} \otimes c_{(2)} \leftharpoonup R^2 \\
&= \sum (R^1 \rightharpoonup c_{(1)}) \otimes (c_{(2)} \leftharpoonup R^2)
\end{aligned}$$

If C is a left-right (H,K)-bimodule coalgebra, then C^* is a left-right (K,H)-bimodule algebra. The left and right action are given by the formula

$$\langle k \rightharpoonup c^* \leftharpoonup h, c \rangle = \langle c^*, hck \rangle$$

for all $h \in H$, $k \in K$ and $c^* \in C^*$.

Proposition 3.6. *With notation as above, we have that* ${}^R(C^*) \cong (C^R)^*$ *as* k*-algebras.*

Proof. The multiplication $\bullet$ in $(C^R)^*$ is given by

$$\langle c^* \bullet d^*, c \rangle = \langle c^* \otimes d^*, \Delta_R(c) \rangle$$

$$
\begin{aligned}
&= \sum \langle c^*, R^1 \rightharpoonup c_{(1)} \rangle \langle d^*, c_{(2)} \leftharpoonup R^2 \rangle \\
&= \sum \langle c^* \leftharpoonup R^1, c_{(1)} \rangle \langle R^2 \rightharpoonup d^*, c_{(2)} \rangle
\end{aligned}
$$

hence

$$
c^* \bullet d^* = \sum (c^* \leftharpoonup R^1) \otimes (R^2 \rightharpoonup d^*) = c^* \cdot_R d^*
$$

4. APPLICATIONS

The smash product

Let A be a left H-module algebra, and B a left H-comodule algebra. Then $A \otimes B$ is a left-left (H, H)-dimodule algebra: the action and coaction are given by the formulas

$$
\begin{aligned}
h \rightharpoonup (a \otimes b) &= (h \rightharpoonup a) \otimes b \\
\rho^l(a \otimes b) &= \sum b_{[-1]} \otimes a \otimes b_{[0]}
\end{aligned}
$$

The identity map $I : H \to H$ is obviously a bialgebra map, so we can consider the algebra ${}^I(A \otimes B)$ constructed in Proposition 2.2. .The multiplication is given by the formula

$$
\begin{aligned}
(a \otimes b) \cdot_I (c \otimes d) &= \sum (a \otimes b_{[0]})((b_{[-1]} \rightharpoonup c) \otimes d) \\
&= \sum a(b_{[-1]} \rightharpoonup c) \otimes b_{[0]} d
\end{aligned}
$$

This is exactly the multiplication rule on the smash product $A\#B$ (see e.g. [16], [4, Remark 3.1]). We can conclude:

Proposition 4.1. *Let A be a left H-module algebra, B a left H-comodule algebra, and I the identity map of H. Then*

$$
{}^I(A \otimes B) = A\#B
$$

The Drinfel'd double I

Let H be a finitely generated projective Hopf algebra with twisted antipode $\overline{S}$. Then we can consider the Drinfel'd double $\mathrm{D}(H)$. In [4, Prop. 3.2], it was shown that, as a k-algebra, the Drinfel'd double is isomorphic to a smash product $H\#H^*$ (the underlying bialgebra is $H^{\rm op}\otimes H$). Therefore the Drinfel'd double can be viewed as a special case of the construction in Proposition 2.2. We will make this explicit. Recall that $\mathrm{D}(H) = H \bowtie H^*$, with multiplication

$$(h \bowtie h^*)(k \bowtie k^*) = \sum h_{(2)}k \bowtie h^* * (h_{(1)} \rightharpoonup k^* \leftharpoonup \overline{S}(h_{(3)})) \tag{22}$$

The H-bimodule structure of H^* is given by

$$\langle h \rightharpoonup k^* \leftharpoonup k, l\rangle = \langle k^*, klh\rangle$$

Now it is easy to verify that H is a left $H \otimes H^{\rm op}$-comodule algebra, and that H^* is a left $H \otimes H^{\rm op}$-module algebra. The structure maps are given by (see [4]):

$$\rho^l(h) = \sum (h_{(1)} \otimes \overline{S}(h_{(3)})) \otimes h_{(2)} \tag{23}$$

$$(h \otimes k) \triangleright h^* = h \rightharpoonup h^* \leftharpoonup k \tag{24}$$

The multiplication on ${}^I(H^* \otimes H)$ is given by

$$\begin{aligned}(h^* \otimes h) \cdot_I (k^* \otimes k) &= \sum h^* * (h_{[-1]} \triangleright k^*) \otimes h_{[0]}k \\ &= \sum h^* * (h_{(1)} \rightharpoonup k^* \leftharpoonup \overline{S}(h_{(3)})) \otimes h_{(2)}k\end{aligned}$$

and we see that the switch map ${}^I(H^*\otimes H) \to \mathrm{D}(H)$ is an algebra isomorphism.

The H-opposite of an algebra

Let A be a left-left (H, H)-dimodule algebra, and consider the identity map $I : H \to H$. Then ${}^I(A^{\rm op}) = \overline{A}$, the H-opposite algebra of A, which plays an important role in the theory of H-Azumaya algebras and the Brauer-Long group (see [11], [1]). The multiplication is given by the formula

$$\overline{a} \cdot \overline{b} = \sum (a_{(-1)} \rightharpoonup b) a_{(0)}$$

We remark that the algebra ${}^IA$ has been considered also in [8].

The smash coproduct

Let C be a right H-module coalgebra, and D a right H-comodule coalgebra. We can make $C \otimes D$ into a right-right H-dimodule coalgebra as follows:

$$\begin{aligned}(c \otimes d) \leftharpoonup h &= (c \leftharpoonup h) \otimes d \\ \rho^r(c \otimes d) &= \sum c \otimes d_{[0]} \otimes d_{[1]}\end{aligned}$$

Again, we take I the identity map. The comultiplication on ${}^I(C \otimes D)$ is given by the formula

$$\Delta_I(c \otimes d) = \sum (c_{(1)} \otimes d_{(1)[0]}) \otimes (c_{(2)} \leftharpoonup d_{(1)[1]}) \otimes d_{(2)})$$

This is nothing other than the comultiplication on the smash coproduct $C \rtimes D$ (cf. [13]).

Proposition 4.2. *Let C be a right H-module coalgebra, D a right H-comodule coalgebra, and I the identity map of H. Then*

$${}^I(C \otimes D) = C \rtimes D$$

The bialgebra of Doi and Takeuchi

Let $\sigma : H \otimes K \to k$ be a convolution invertible skew pairing. In [6], Doi and Takeuchi introduced a new bialgebra $H \bowtie_\sigma K$. The comultiplication is the one from $H \otimes K$, and the multiplication is given by the formula

$$(h \bowtie k)(g \bowtie l) = \sum \sigma(g_{(1)} \otimes k_{(1)})\sigma^{-1}(g_{(3)} \otimes k_{(3)})(hg_{(2)} \bowtie k_{(2)}l) \tag{25}$$

Our next aim is to show that $H \bowtie_\sigma K$ can be viewed as a special case of the construction in Proposition 2.1.

Let $\sigma : H \otimes K \to k$ be an invertible pairing. We have seen in Proposition 1.2 that $\sigma^{-1} \in \mathrm{P}(H^{\mathrm{cop}}, K^{\mathrm{cop}})$. Thus we obtain a new pairing

$$\kappa = (\sigma \otimes \sigma^{-1}) \circ (I \otimes \tau \otimes I) : (H \otimes H^{\mathrm{cop}}) \otimes (K \otimes K^{\mathrm{cop}}) \to k$$

Here τ denotes the switch map, as usual. On H, we define a left $H \otimes H^{\mathrm{cop}}$-coaction as follows:

$$\rho^l(h) = \sum (h_{(1)} \otimes h_{(3)}) \otimes h_{(2)}$$

It is easy to verify that this coaction makes H into a left $H\otimes H^{\rm cop}$-comodule algebra. In a similar way, K is a right $K\otimes K^{\rm cop}$-comodule algebra, under the coaction

$$\rho^r(k)=\sum k_{(2)}\otimes(k_{(3)}\otimes k_{(1)})$$

$H\otimes K$ is then a left-right $(H\otimes H^{\rm cop},K\otimes K^{\rm cop}$-bicomodule algebra, and the comultiplication on ${}^\kappa(H\otimes K)$ is given by the formula

$$(g\otimes l)\cdot_\kappa(h\otimes k)=\sum\sigma(g_{(1)}\otimes k_{(3)})\sigma^{-1}(g_{(3)}\otimes k_{(1)})g_{(2)}h\otimes lk_{(2)} \tag{26}$$

Now assume that $\sigma:\ H\otimes K\to k$ is a convolution invertible *skew* pairing. Then $\sigma:\ H^{\rm op}\otimes K^{\rm opcop}\to k$ is an invertible pairing, and we can consider the algebra ${}^\kappa(H^{\rm op}\otimes K^{\rm opcop})$. The multiplication is given by

$$(g\otimes l)\cdot_\kappa(h\otimes k)=\sum\sigma(g_{(1)}\otimes k_{(1)})\sigma^{-1}(g_{(3)}\otimes k_{(3)})(hg_{(2)}\bowtie k_{(2)}l) \tag{27}$$

Comparing (25) and (27), we find

Theorem 4.3. *Let $\sigma:\ H\otimes K\to k$ be a convolution invertible skew pairing. Then*

$$\kappa=(\sigma\otimes\sigma^{-1})\circ(I\otimes\tau\otimes I):\ (H^{\rm op}\otimes H^{\rm opcop})\otimes(K^{\rm opcop}\otimes K^{\rm op})\to k$$

is a pairing, $H^{\rm op}\otimes K^{\rm opcop}$ is a $(H^{\rm op}\otimes H^{\rm opcop},K^{\rm opcop}\otimes K^{\rm op})$-bicomodule algebra, and

$$H\bowtie_\sigma K=({}^\kappa(H^{\rm op}\otimes K^{\rm opcop}))^{\rm op}$$

The Drinfel'd double II

Let H be a finitely generated projective bialgebra. In [6, Remark 2.3], it is observed that the Drinfel'd double

$$\mathrm{D}(H)=(H^*)^{\rm cop}\bowtie_\sigma H$$

where $\sigma(h^*\otimes h)=\langle h^*,h\rangle$. Thus the Drinfel'd double is a special case of the construction in Proposition 2.1. This has also been observed by Ferrer Santos in [7]. Let us make this clear. Let H be a finitely generated projective bialgebra with twisted antipode $\overline{S}$. We have already seen that H is a left

$H \otimes H^{\mathrm{op}}$-comodule algebra (see (23)). We can also make H^* into a right $H \otimes H^{\mathrm{op}}$-comodule algebra, by putting

$$\rho^r(h^*) = \sum_{i,j} (h_i \rightharpoonup h^* \leftharpoonup h_j) \otimes h_i^* \otimes h_j^* \tag{28}$$

Here $\{h_i, h_i^* \mid i = 1, \cdots, n\}$ is a dual basis for H. In fact this right action corresponds to the left $H \otimes H^{\mathrm{op}}$-action on H^* given in (24). The evaluation map

$$\mathrm{ev}: \ (H \otimes H^{\mathrm{op}}) \otimes (H \otimes H^{\mathrm{op}})^* \to k$$

is a pairing, corresponding to the identity $I: \ H \otimes H^{\mathrm{op}} \to H \otimes H^{\mathrm{op}}$, using the dictionary of Proposition 1.4. We can therefore consider $H \otimes H^*$ as a left-right $(H \otimes H^{\mathrm{op}}, H \otimes H^{\mathrm{op}})$-bicomodule algebra, and the multiplication on ${}^{\mathrm{ev}}(H \otimes H^*)$ is given by

$$\begin{aligned}
&(h \otimes h^*) \cdot_{\mathrm{ev}} (k \otimes k^*) \\
&\quad = \sum \langle h_{(1)} \otimes \overline{S}(h_{(3)}), h_i^* \otimes h_j^* \rangle h_{(2)} k \otimes h^* * (h_i \rightharpoonup k^* \leftharpoonup h_j) \\
&\quad = \sum h_{(2)} k \otimes h^* * \Big((\langle h_{(1)}, h_i^* \rangle h_i) \rightharpoonup k^* \leftharpoonup (\langle \overline{S}(h_{(3)}), h_j^* \rangle h_j) \Big) \\
&\quad = \sum h_{(2)} k \otimes h^* * (h_{(1)} \rightharpoonup k^* \leftharpoonup \overline{S}(h_{(3)}))
\end{aligned}$$

This is exactly the multiplication on the Drinfel'd double, see (22). Our two interpretations of the Drinfel'd double - using respectively a Hopf algebra map (the identity) and a pairing (the evaluation map) are put together using Proposition 2.4.

The bialgebra constructed in [3]

Let $R \in H \otimes K$ be a bialgebra copairing in the sense of [3], with inverse S. In [3, Sec. 3], the comultiplication on $H \otimes K$ is twisted as follows: $H \bowtie^S K = H \otimes K$ as a k-module, with comultiplication

$$\Delta(h \bowtie k) = \sum (h_{(1)} \bowtie S^2 k_{(1)} R^2) \otimes (S^1 h_{(2)} R^1 \bowtie k_{(2)}) \tag{29}$$

This will turn out to be a special case of the construction in Proposition 3.3. The arguments are dual to the ones for the Doi-Takeuchi bialgebra. We start with $R \in \mathrm{DP}(H, K)$. Using Proposition 1.3, we find that

$$T = (I \otimes \tau \otimes I)(R \otimes S) = \sum (R^1 \otimes S^1) \otimes (R^2 \otimes S^2) \in \mathrm{DP}(H \otimes H^{\mathrm{op}}, K \otimes K^{\mathrm{op}})$$

H is a left $H \otimes H^{\mathrm{op}}$-coalgebra, and K is a right $K \otimes K^{\mathrm{op}}$-coalgebra. The actions are defined by the formulas

$$(h' \otimes h'') \cdot h = h'hh'' \quad \text{and} \quad k \cdot (k' \otimes k'') = k''kk'$$

It follows that $H \otimes K$ is a left-right $(H \otimes H^{\mathrm{op}}, K \otimes K^{\mathrm{op}})$-bimodule coalgebra, and, according to Proposition 3.3, the comultiplication on ${}^{T}(H \otimes K)$ is given by the formula

$$\Delta_T(h \otimes k) = \sum (R^1 h_{(1)} S^1 \otimes k_{(1)}) \otimes (h_{(2)} \otimes S^2 k_{(2)} R^2)$$

Now let $R \in H \otimes K$ be a bialgebra copairing. Then $R \in \mathrm{DP}(H^{\mathrm{opcop}}, K^{\mathrm{cop}})$, and the comultiplication on ${}^{T}(H^{\mathrm{opcop}} \otimes K^{\mathrm{cop}})$ is given by the formula

$$\Delta_T(h \otimes k) = \sum (S^1 h_{(2)} R^1 \otimes k_{(2)}) \otimes (h_{(1)} \otimes S^2 k_{(1)} R^2) \tag{30}$$

Comparing (29) and (30), we conclude:

Proposition 4.4. *Let $R \in H \otimes K$ be a bialgebra copairing with inverse S. Then $T = (I \otimes \tau \otimes I)(R \otimes S) \in (H^{\mathrm{opcop}} \otimes H^{\mathrm{cop}}) \otimes (K^{\mathrm{cop}} \otimes K^{\mathrm{opcop}}$ is a copairing element, and*

$$({}^{T}(H^{\mathrm{opcop}} \otimes K^{\mathrm{cop}}))^{\mathrm{cop}} \cong H \bowtie^{S} K$$

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Date received May 1, 1999.