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REFLEXIVITY OF FUZZY BANACH SPACE

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ABSTRACT. The concept of a fuzzy dual space of a fuzzy Banach space is introduced to give some results of fuzzy dual spaces analogous to those of normed spaces in ordinary analysis. Introducing fuzzy dual spaces allows us to prove the reflexivity of a fuzzy Banach space.

1. Introduction

The concept of fuzzy normed spaces was introduced by Felbin who used a fuzzy numbers to develop their theory of fuzzy normed spaces [2]. Itoh and Cho [6]defined and applied the concept of fuzzy bounded linear operator on fuzzy normed space and its fuzzy operator norm to get similar results to those of normed (or Banach) spaces. This paper introduce a definition of fuzzy dual spaces of fuzzy normed spaces and show that their properties are related. An alternative formula for the definition of fuzzy operator norm is also introduced and the meaning of the linear isometry of fuzzy canonical mapping has been described.We proved that a finite dimensional fuzzy normed space is reflexive and the reflexivity of infinite dimensional fuzzy Banach spaces is achieved for those which are uniformly convex.

2. Preliminaries

Let $\mathbf{R}$ and $\mathbf{I}$ be the set of real numbers and closed interval [0,1],respectively. Denote $\mathbf{R}(\mathbf{I}) = \{\eta : \mathbf{R} \to \mathbf{I}/\eta$ is normal,fuzzy convex and upper

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semi-continuous} the set of all fuzzy real numbers.If $\eta \in \mathbf{R}(\mathbf{I})$ and $\eta(t) = 0$ for $t < 0$ then η is called non-negative fuzzy real number.[2,6] The set of all non-negative fuzzy real numbers of $\mathbf{R}(\mathbf{I})$ is denoted by $\mathbf{R}^*(\mathbf{I})$.

The α-level set of a fuzzy real number $\eta \in \mathbf{R}^*(\mathbf{I})$, $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\} = [\eta_1^\alpha, \eta_2^\alpha]$, where η_1^α and η_2^α are two real numbers[2,5].

Let $\eta, \delta \in \mathbf{R}(\mathbf{I}), [\boldsymbol{\eta}]_{\boldsymbol{\alpha}} = [\boldsymbol{\eta_1^\alpha}, \boldsymbol{\eta_2^\alpha}]$ and $[\delta]_\alpha = [\delta_1^\alpha, \delta_2^\alpha]$ then for all $\alpha \in (0,1]$,

$$[\eta \oplus \delta]_\alpha = [\eta_1^\alpha + \delta_1^\alpha, \eta_2^\alpha + \delta_2^\alpha]$$

$$[\eta \ominus \delta]_\alpha = [\eta_1^\alpha - \delta_1^\alpha, \eta_2^\alpha - \delta_2^\alpha]$$

$$[\eta \otimes \delta]_\alpha = [\eta_1^\alpha \cdot \delta_1^\alpha, \eta_2^\alpha \cdot \delta_2^\alpha]$$

$$[\tilde{1} \oslash \eta]_\alpha = [\frac{1}{\eta_2^\alpha}, \frac{1}{\eta_1^\alpha}] if \eta_1^\alpha > 0$$

A partial order $\preccurlyeq$ in $\mathbf{R}(\mathbf{I})$ is defined by $\eta \preccurlyeq \delta$ if and only if $\eta_1^\alpha \leq \delta_1^\alpha$ and $\eta_2^\alpha \leq \delta_2^\alpha$. The additive and multiplicative identities in $\mathbf{R}(\mathbf{I})$ are $\tilde{0}$ and $\tilde{1}$ respectively (cf.[2,6]) .

Definition 2.1. [2]

Let X be vector space over $\mathbf{R}(\mathbf{I})$, let $\|\cdot\| : X \longrightarrow \mathbf{R}^*(\mathbf{I})$ be a mapping and let L, R : $[01] \times [1,0]$ be symmetric and non-decreasing mappings satisfying $L(0,0) = 0, R(1,1) = 1$. Write

$$[\|x\|]_\alpha = [\|x\|_1^\alpha, \|x\|_2^\alpha]$$

for $x \in X$, and $\alpha \in (0,1]$
The mapping $\|\cdot\|$ is called a fuzzy norm if:
i. $\|x\| = \tilde{0}$ if and only if $x = 0$.
ii. $\|rx\| = |r|\|x\|, x \in X$ and $r \in \mathbf{R}$.
iii. for all $x, y \in X$,

$$(a) \quad whenever \quad s \leq \|x\|_1^1, t \leq \|y\|_1^1 \quad and \quad s + t \leq \|x+y\|_1^1,$$
$$\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t));$$

$$(b) \quad whenever \quad s \geq \|x\|_1^1, t \geq \|y\|_1^1 \quad and \quad s+t \geq \|x+y\|_1^1,$$
$$\|x+y\|(s+t) \leq L(\|x\|(s), \|y\|(t)).$$

And then the quadruple $(X, \|\cdot\|, L, R)$ is called a fuzzy normed linear space and we write $(X, \|\cdot\|)$ or simply X when L and R are as indicated just above.

Proposition 2.1. [2]

In a fuzzy normed linear space $(X, \|\cdot\|)$ the triangle inequality of Defintion 2.1 (iii) is equivalent to $\|x+y\| \preccurlyeq \|x\| \oplus \|y\|$,holds if $L = min.$ and $R = max.$

Definition 2.2. [6] Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be fuzzy normed spaces and T be a linear operator from X into Y.
An operator T is called a fuzzy bounded operator if there exists a fuzzy number $K \succcurlyeq \tilde{0}(K \in \mathbf{R}^*(\mathbf{I}))$ with $\sup\{K_2^\alpha : \alpha \in (0,1]\} < \infty$, such that:

$$(a) \quad wherever \quad s \leq \|x\|_1^1, t \leq K_1^1 \quad and \quad st \leq \|Tx\|_1^1,$$
$$\|Tx\|(st) \geq \min(\|x\|(s), K(t));$$

and

$$(b) \quad wherever \quad s \geq \|x\|_1^1, t \geq K_1^1 \quad and \quad st \geq \|Tx\|_1^1,$$
$$\|Tx\|(st) \leq \max(\|x\|(s), K(t));$$

where

$$[K]_\alpha = [K_1^\alpha, K_2^\alpha]$$

and

$$[\|Tx\|]_\alpha = [\|Tx\|_1^\alpha, \|Tx\|_2^\alpha].$$

Proposition 2.2. [6] *The inequalities (a) and (b) of Definition 2.2 are equivalent to the following inequality : $\|Tx\| \preccurlyeq K \otimes \|x\|$ for all $x \in X$.*

Definition 2.3. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be fuzzy normed spaces. Let $T : X \longrightarrow Y$ be a bounded linear operator. The fuzzy operator norm of T, denoted $\|T\|$, is defined by: $\|T\| = \inf\{K \succcurlyeq \tilde{0}(K \in \mathbf{R}^*(\mathbf{I}))/\|Tx\| \preccurlyeq K \otimes \|x\|, \quad forevery \quad x \in X\}$ We write $[\|T\|]_\alpha = [\|T\|_1^\alpha, \|T\|_2^\alpha]$ as the α-level set of the fuzzy real number $\|T\|$.

Proposition 2.3. [6] *If T is a fuzzy bounded operator on a fuzzy normed space X, then $\|Tx\| \preccurlyeq \|T\| \otimes \|x\|$ for all $x \in X$.*

Theorem 2.1. *The norm of T in Definition 2.3 is equal to*

$$\|T\| = \sup_{\|x\|=\tilde{1}} \|Tx\|_1^\alpha,$$

for every $x \in X$ where

$$[\|T\|]_\alpha = [\|T\|_1^\alpha, \|T\|_2^\alpha]$$

and

$$[\sup_{\|x\|=\tilde{1}} \|Tx\|]_\alpha = [\sup_{\|x\|_1^\alpha=1} \|Tx\|_1^\alpha, \sup_{\|x\|_2^\alpha=1} \|Tx\|_2^\alpha]$$

for any $\alpha \in (0,1]$.

Proof: Let $\|T\| = \sup\{\|Tx\| : x \in X, \|x\| = \tilde{1}\}$. If $K \succcurlyeq \tilde{0}(K \in \mathbf{R}^*(\mathbf{I}))$ with $\|Tx\| \preccurlyeq K \otimes \|x\|$ then for every $\alpha \in (0,1]$,

$$\|Tx\|_1^\alpha \le K_1^\alpha \cdot \|x\|_1^\alpha$$

and

$$\|Tx\|_2^\alpha \le K_2^\alpha \cdot \|x\|_2^\alpha$$

where

$$[K]_\alpha = [K_1^\alpha, K_2^\alpha] \quad and \quad [\|x\|]_\alpha = [\|x\|_1^\alpha, \|x\|_2^\alpha].$$

For

$$x \in X \quad with \quad \|x\| = \tilde{1},$$

then

$$\|Tx\|_1^\alpha \le K_1^\alpha \quad and \quad \|Tx\|_2^\alpha \le K_2^\alpha,$$

so that

$$\|T\| = \inf\{K \succcurlyeq \tilde{0} / \|Tx\| \preccurlyeq K \otimes \|x\|, \quad forevery \quad x \in X\}$$

Conversely, let T be fuzzy bounded linear operator from X into Y. If $x = 0$, then $Tx = 0$ since T is linear.

If $x(\neq 0) \in X$, then by Proposition 2.3, $\|Tx\| \preccurlyeq \|T\| \otimes \|x\|$.

For every $\alpha \in (0,1]$,

$$\|Tx\|_1^\alpha \le \|T\|_1^\alpha \cdot \|x\|_1^\alpha$$

And

$$\|Tx\|_2^\alpha \le \|T\|_2^\alpha \cdot \|x\|_2^\alpha.$$

If $\|x\| = \tilde{1}$, then $\|Tx\|_1^\alpha \leq \|T\|_1^\alpha$ and $\|Tx\|_2^\alpha \leq \|T\|_2^\alpha$

This show that

$$\|T\|_1^\alpha = \sup_{\|x\|_1^\alpha} \|Tx\|_1^\alpha$$

and

$$\|T\|_2^\alpha = \sup_{\|x\|_2^\alpha} \|Tx\|_2^\alpha,$$

with

$$[\sup_{\|x\|=\tilde{1}} \|Tx\|]_\alpha = [\sup_{\|x\|_1^\alpha=1} \|Tx\|_1^\alpha, \sup_{\|x\|_2^\alpha=1} \|Tx\|_2^\alpha]$$

Hence

$$\|T\| = \sup_{\|x\|=1} \|Tx\|,$$

for every $x \in X$

Lemma 2.1. [2] *Let $\{x_1, x_2, ..., x_n\}$ be linearly independent set of vectors in a fuzzy normed linear space $(X, \|\cdot\|)$ (of any dimension). Then there exists an $\eta \succ \tilde{0}(\eta \in \mathbf{R}^*(\mathbf{I}))$ with*

$$\sup_{\alpha\in(0,1]} \eta_2^\alpha < \infty,$$

where $[\eta]_\alpha = [\eta_1^\alpha, \eta_2^\alpha]$ and such that for every choice of scalars $a_1, a_2, ..., a_n$ we have

$$\|\sum_{i=1}^n a_i x_i\| \succcurlyeq (\sum_{i=1}^n |a_i|)\eta.$$

Theorem 2.2. *If $(X, \|\cdot\|)$ is an n-dimensional fuzzy normed space, then every fuzzy linear operator on X is a fuzzy bounded.*

Proof:

Let $\{e_1, e_2, ..., e_n\}$ be any basis for X. Then every $x \in X$ has a unique representation

$$x = \sum_{i=1}^n a_i e_i,$$

for every scalars $a_1, a_2, ..., a_n$

Consider any fuzzy linear operator T on X.

Since T is a fuzzy linear then we have

$$\|Tx\| = \|\sum_{i=1}^{n} a_i Te_i\| \preccurlyeq \sum_{i=1}^{n} |a_i|\|Te_i\| \preccurlyeq |a_1|\|Te_i\| \oplus |a_2|\|Te_2\| \oplus ... \oplus |a_n|\|Te_n\|$$

For any $\alpha \in (0, 1]$ then

$$\|Tx\|_1^\alpha \leq |a_1|\|Te_1\|_1^\alpha + |a_2|\|Te_2\|_1^\alpha + ... + |a_n|\|Te_n\|_1^\alpha \leq (\max_{1\leq i\leq n} \|Te_i\|_1^\alpha)(\sum_{i=1}^{n} |a_i|)$$

and

$$\|Tx\|_2^\alpha \leq |a_1|\|Te_1\|_2^\alpha + |a_2|\|Te_2\|_2^\alpha + ... + |a_n|\|Te_n\|_2^\alpha \leq (\max_{1\leq i\leq n} \|Te_i\|_2^\alpha)(\sum_{i=1}^{n} |a_i|)$$

So

$$\|Tx\| \preccurlyeq (\max_{1\leq i\leq n} \|Te_i\|(\sum_{i=1}^{n} |a_i|)$$

Take

$$K = \max_{1\leq i\leq n} \|Te_i\| \quad with \quad \sup K_2^\alpha < \infty,$$

where

$$[K]_\alpha = [K_1^\alpha, K_2^\alpha] = [\max_{1\leq i\leq n} \|Te_i\|_1^\alpha, \max_{1\leq i\leq n} \|Te_i\|_2^\alpha].$$

Now letting $\inf\{\eta_1^\alpha/\alpha \in (0, 1]\} > 0$ and $\sup\{\eta_2^\alpha/\alpha \in (0, 1]\} < \infty$, then by Lemma 3.1, we have

$$\|Tx\|_1^\alpha \leq K_1^\alpha \cdot \sum_{i=1}^{n} |a_i| \leq \frac{K_1^\alpha \cdot \|x\|_1^\alpha}{\eta_1^\alpha} \leq K_1^\alpha \cdot \|x\|_1^\alpha$$

and

$$\|Tx\|_2^\alpha \leq K_2^\alpha \cdot \sum_{i=1}^{n} |a_i| \leq \frac{K_2^\alpha \cdot \|x\|_2^\alpha}{\eta_2^\alpha} \leq K_2^\alpha \cdot \|x\|_2^\alpha$$

for any $x \in X$.

Therefore $\|Tx\| \preccurlyeq K \otimes \|x\|$ for every $x \in X$ i.e. T is a fuzzy bounded operator.

3. Fuzzy dual space and reflexivity

Definition 3.1. Let $(X, \|\cdot\|)$ be fuzzy normed space. A sequence $\{x_\eta\}_1^\alpha \subseteq X$ is said to be converge to $x \in X$, denoted by

$$\lim_{n\to\infty} x_n = x$$

if

$$\lim_{n\to\infty} \|x_n - x\|_2^\alpha = 0 \quad forevery \quad \alpha \in (0,1],$$

and is called a Cauchy sequence if

$$\lim_{n,m\to\infty} \|x_m - x_n\|_2^\alpha = 0 \quad forevery \quad \alpha \in (0,1].$$

A fuzzy set $A \subseteq X$ is said to be complete if any Cauchy sequence in A converges in A. A complete fuzzy normed space is called a fuzzy Banach space.

Definition 3.2. Let $(X, \|\cdot\|)$ be fuzzy normed space. An operator T on X is said to be fuzzy continuous at $x \in X$ if

$$\lim_{n\to\infty} x_n = x$$

implies that

$$\lim_{n\to\infty} Tx_n = Tx$$

for arbitrary $x_n \subseteq X$.The T is said to be fuzzy continuous if it is continuous at every point of X.

Definition 3.3. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be fuzzy normed spaces. We denoted by FBL(X, Y) for the set of all fuzzy bounded linear operators from X into Y. If $Y = \mathbf{R}(\mathbf{I})$, then FBL$(X, \mathbf{R}(\mathbf{I}))$, the space of all fuzzy bounded linear functionals from X into $\mathbf{R}(\mathbf{I})$ is denoted by X^*, is called the fuzzy dual space of the fuzzy normed space X.

Proposition 3.1. *If $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are fuzzy normed spaces. then FBL(X, Y) is a fuzzy normed space. Moreover, FBL(X, Y) is a fuzzy Banach space if $(Y, \|\cdot\|)$ is a fuzzy Banach space.*

Proof:

Let $\{T_n\}$ be a Cauchy sequence in FBL(X, Y)

(3.1) $$\lim_{n,m\to\infty} \|T_n - T_m\|_2^\alpha = 0 \quad forevery \quad \alpha \in (0,1]. \quad \ldots(1)$$

For $x \in X$ then the sequence $\{T_n(x)\}$ is a Cauchy sequence in Y and hence converge to some $y \in Y$, say $T(x) = y$. So that

(3.2) $$\|T_n(x) - T_m(x)\| \preccurlyeq \|T_n - T_m\| \otimes \|x\|. \quad \ldots(2)$$

Let $x_1, x_2 \in X$ and $\epsilon > 0$ then there exist $N_1(\alpha), N_2(\alpha)$ and $N_3(\alpha)$ such that for every $\alpha \in (0,1]$,

$$\|T_n(x_1+x_2) - T(x_1+x_2)\|_2^\alpha < \frac{1}{3} \quad \epsilon \quad forevery \quad n \geq N_1(\alpha),$$

$$\|T_n(x_1) - T(x_1)\|_2^\alpha < \frac{1}{3} \quad \epsilon \quad forevery \quad n \geq N_2(\alpha),$$

$$\|T_n(x_2) - T(x_2)\|_2^\alpha < \frac{1}{3} \quad \epsilon \quad forevery \quad n \geq N_3(\alpha).$$

Take $N_0(\alpha) = \max(N_1(\alpha), N_2(\alpha), N_3(\alpha))$
Then for every $n \geq N_0(\alpha)$ we have

$$\|T(x_1+x_2) - T(x_1) - T(x_2)\|_2^\alpha \leq \|T(x_1+x_2) - T_n(x_1+x_2)\|_2^\alpha$$

$$+\|T_n(x_1) - T(x_1)\|_2^\alpha + \|T_n(x_2) - T(x_2) - T(x_2)\|_2^\alpha < \epsilon$$

This proves that

$$T(x_1+x_2) = T(x_1) + T(x_2) \tag{3.3}$$

Now since

$$\lim_{n\to\infty} T_n(kx) = T(kx) \quad then \quad \lim_{n\to\infty} \|T_n(kx) - T(kx)\| = \tilde{0}$$

i.e.

$$\lim_{n\to\infty} \|T_n(kx) - T(kx)\|_2^\alpha = 0,$$

or

$$\|\lim_{n\to\infty} T_n(kx) - T(kx)\|_2^\infty = \|k \lim_{n\to\infty} T_n(x) - T(kx)\|_2^\alpha\| = 0 \quad i.e. \quad k \lim_{n\to\infty} T_n(x) = T(kx)$$

Hence

$$T(kx) = kT(x). \tag{3.4}$$

Therefore from (3.3) and (3.4) we get that T is a linear operator.

Also since

$$\|(T - T_m)(x)\| = \|Tx - T_m x\| = \lim_{n\to\infty} T_n x - T_m x\| = \lim_{n\to\infty} \|T_n x - T_m x\|.$$

For $\epsilon > 0$. Choose a sufficiently large N such that

$$\|T_n - T_m\|_1^\alpha \leq \epsilon \quad and$$
$$\|T_n - T_m\|_2^\alpha \leq \epsilon \quad \forall n, m \geq N$$

Fix $m \geq N$. Then for each $x \in X$ and every $\alpha \in (0, 1]$

We have,

$$\|Tx - T_m x\|_1^\alpha \leq \lim_{n\to\infty} \|T_n - T_m\|_1^\alpha \cdot \|x\|_1^\alpha \leq \epsilon \cdot \|x\|_1^\infty \quad and$$
$$\|Tx - T_m x\|_2^\alpha \leq \lim_{n\to\infty} \|T_n - T_m\|_2^\alpha \cdot \|x\|_2^\chi \leq \epsilon \cdot \|x\|_2^\infty$$

$\therefore \|(T - T_m)x\| \preccurlyeq \tilde{\epsilon} \otimes \|x\|$ where $\tilde{\epsilon} \in \mathbf{R}^*(\mathbf{I})$

This proves that $T - T_m$ is a fuzzy bounded operator.

Now as $T = (T - T_m) + T_m$ so that T is a fuzzy bounded operator and since

$$\lim_{m\to\infty} \|T - T_m\| = \tilde{0} \quad then \quad \|T - \lim_{m\to\infty} T_m\| = \tilde{0}.$$

Therefore

$$\lim_{m\to\infty} T_m = T.$$

Corollary 3.1. *The dual space X^* of any fuzzy normed linear space X is a fuzzy Banach space.*

Proof:

Since $(\mathbf{R}(\mathbf{I}), \|.\|)$ is finite dimensional fuzzy normed spaced, then it is complete (cf[2]) and so $(\mathbf{R}(\mathbf{I}), \|.\|)$ is fuzzy Banach space. Therefore X^*=FBL$(X, \mathbf{R}(\mathbf{I}))$ is a fuzzy Banach space.

Theorem 3.1. *Let X be a real linear space, let $\|.\|$ be a mapping:$X \to \mathbf{R}^*(\mathbf{I})$, then $(X, \|.\|)$ is a fuzzy normed space.*

Proof: Let r be any real number. We will show that the mapping $\|.\|$ has the following properties:-

i. $\|x\| = \tilde{0}$ if and only if $x = 0$.

ii. $\|rx\| = |r|\|x\|$ for all $x \in \mathbf{R}$.

iii. $\|x + y\| \preccurlyeq \|x\| \oplus \|y\|$ for every $x, y \in \mathbf{R}$.

i. Let$\|x\| = \tilde{0}$. then for every $\alpha \in (0,1]$,we have $\|x\|_1^\alpha = 0$ and $\|x\|_2^\alpha = 0$ and so $x = 0$.
If $x = 0$ then $\|x\|_1^\alpha = 0$ and $\|x\|_2^\alpha = 0$ so that $\|x\| = \tilde{0}$.

ii. If $r = 0$ then for every $x \in \mathbf{R}$.
$[\|rx\|]_\alpha = [\|0\|]_\alpha = \{0\} = [|r|\|x\|]_\alpha$
i.e.$\|rx\| = |r|\|x\|$.
Let $r \neq 0$,then
$[\|rx\|]_\alpha = [\|rx\|_1^\alpha, \|rx\|_2^\alpha]$
$= [|r|\|x\|_1^\alpha, |r|\|x\|_1^\alpha]$
$[|r|\|x\|]_\alpha$
Hence $\|rx\| = |r|\|x\|$.

iii.since $\|.\|$ is a fuzzy norm in $\mathbf{R}^*(\mathbf{I})$, then for every $\alpha \in (0,1]$ and for all $x, y \in \mathbf{R}$, $[\|x+y\|]_\alpha = [\|x+y\|_1^\alpha, \|x+y\|_2^\alpha]$

As $\|x+y\|_1^\alpha \leq \|x\|_1^\alpha + \|y\|_1^\alpha$ and
$\|x+y\|_2^\alpha \leq \|x\|_2^\alpha + \|y\|_2^\alpha$,it follows that
$\|x+y\| \preccurlyeq \|x\| \oplus \|y\|$.
That is $((\mathbf{R}), \|.\|_{(\mathbf{R})})$ is a fuzzy normed space.

Definition 3.4. Let $(X, \|.\|)$ be a fuzzy normed space, the fuzzy second dual of X, denoted by X^{**}, is the space of all fuzzy bounded linear functionals $F_x$defined on X^*,that is $X^{**} = (X^*)^*, X^{**}$ is a fuzzy Banach space if it equipped with the norm $\|.\|$ defined dy

$$\|F_x\| = \sup\{\|F_x(T)\| : T \in X^*, \|T\| = \tilde{1}\}.$$

Proposition 3.2. *If $(X, \|.\|)$ be a fuzzy normed space, then there exists a fuzzy mapping $S : X \to X^{**}$ which is linear and preserves fuzzy norms.*

Proof:
For any $x \in X$, defined a fuzzy bounded linear functional $f \in X^*$ with the properties:$\|f\| = \tilde{1}$ and $f(x) = \|x\|$. defined a fuzzy functional F_x on X^* by $F_x(f) = f(x)$.
For each $x \in X$, then F_x is linear. Indeed,
$F_x(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2)(x)$
$= (\alpha f_1)(x) + (\beta f_2)(x)$
$= \alpha(f_1(x)) + \beta(f_2(x))$

$= \alpha F_x(f_1) + \beta F_x(f_2)$ Also F_x is a fuzzy bounded linear functional on X^* because T is a fuzzy bounded linear functional on X so $F_x \in X^{**} \forall x \in X$. Now defined

$$S : X \to X^{**}$$
$$x \mapsto F_x$$

It is clear that S is linear. Moreover,

$$\begin{aligned} \|F_x\| &= \sup\{\|F_x(T)\| : T \in X^*, \|T\| = \tilde{1}\} \\ &= \sup\{\|T(x)\| : T \in X^*, \|T\| = \tilde{1}\} \\ &\preccurlyeq \sup\{\|T\| \otimes \|x\| : T \in X^*, \|T\| = \tilde{1}\} \end{aligned}$$

it follows that for each $\alpha \in (0, 1]$

$$\|F_x\|_1^\alpha \leq \sup\{\|T\|_1^\alpha . \|x\|_1^\alpha\} \leq \|x\|_1^\alpha$$

and

$$\|F_x\|_2^\alpha \leq \sup\{\|T\|_2^\alpha . \|x\|_2^\alpha\} \leq \|x\|_2^\alpha,$$

where $\|T\|\tilde{1}$ and $\|T\|_\alpha = [\|T\|_1^\alpha . \|T\|_2^\alpha]$
So that

$$\|F_x \preccurlyeq \|x\| \ldots (1) \tag{3.5}$$

Now $\|S(x)\| = \|F_x\| \preccurlyeq \|x\| = \tilde{1} \otimes \|x\|$,then S is a fuzzy bounded linear operator form X into X^{**}. In fact $\|F_x\| = \|x\|$. If $x(\neq 0) \in X$, using the properties of T we have,

$$\|x\| = \|T(x)\| = \|F_x(T)\| \preccurlyeq \|F_x\| \otimes \|T\|.$$

For any $\alpha \in (0, 1]$,

$$\|x\|_1^\alpha \leq \|F_x\|_1^\alpha . \|T\|_1^\alpha,$$

and

$$\|x\|_2^\alpha \leq \|F_x\|_2^\alpha . \|T\|_2^\alpha$$

Thus

$$\|x\| \preccurlyeq \|F_x\| \ldots (2) \tag{3.6}$$

Hence $F_x \in X^{**}$ and therefore from (3.5) and ((3.6)) we have $\|S(x)\| = \|x\|$. That is, the mapping

$$S : X \to X^{**}$$
$$x \mapsto F_x$$

preserves fuzzy norms

Definition 3.5. Let$(X, \|.\|)$ be a fuzzy normed space, then $S : X \to X^{**}$,defined in the previous proposition, is called the canonical mapping from X into X^{**}.

Definition 3.6. Let$(X, \|.\|)$ be a fuzzy normed linear space,and let S be the fuzzy canonical mapping from X into X^{**}, then X is said to be reflexive fuzzy normed linear space if $S(X) = X^{**}$.

Theorem 3.2. *The fuzzy dual space of the finite dimensional fuzzy normed space* $(X, \|.\|)$ *is* $(X, \|.\|)$.

Proof:
Let X^* be the fuzzy dual space of the n-dimensional fuzzy normed space X, $T \in X^*$, and $\{e_1, e_2, ..., e_n\}$ be any basis for X. Then every $x \in X$ has a unique representation

$$x = \sum_{k=1}^{n} a_k e_k, foranyscalarsa_1, a_2, ..., a_n$$

Define $\alpha_k = Te_k$ such that $\alpha_k \in R(I)$. Since T is linear and bounded, then

$$\begin{aligned} T(x) &= T(\sum_{k=1}^{n} a_k e_k) \\ &= \sum_{k=1}^{n} a_k T e_k \\ &= \sum_{k=1}^{n} a_k \alpha_k \end{aligned}$$

Foe a given positive integer k choose x be such that $\|e_k\| \preccurlyeq \tilde{1}$, so that

$$\begin{aligned} \|\alpha_k\| &= \|Te_k\| \\ &\preccurlyeq \|T\| \otimes \|e_k\|. \end{aligned}$$

$$\begin{aligned} \|\alpha_k\|_1^\alpha &\leq \|T\|_1^\alpha . \|e_k\|_1^\alpha \\ &\leq \|T\|_1^\alpha \end{aligned}$$

and

$$\begin{aligned} \|\alpha_k\|_2^\alpha &\leq \|T\|_2^\alpha . \|e_k\|_2^\alpha \\ &\leq \|T\|_2^\alpha \end{aligned}$$

where

$$[\|T\|]_\alpha = [\|T\|_1^\alpha, \|T\|_2^\alpha] \quad and \quad [\|e_k\|]_\alpha = [\|e_k\|_1^\alpha, \|e_k\|_2^\alpha].$$

So that $\|\alpha_k\| \preccurlyeq \|T\|$ where $\alpha = \{\alpha_k\} \subseteq X$ and $\|\alpha\| \preccurlyeq \|T\|$.
Conversely, suppose $\alpha = \{\alpha_k\} \subseteq X$ is given,define T on X by $T(x) = \sum_{k=1}^{n} a_k\alpha_k$ for $x = \sum_{k=1}^{n} a_k e_k \in X$. then by Lemma 3.1, there is an $\eta \succ \tilde{0}$ such that $\|x\| \succcurlyeq (\sum_{k=1}^{n} |a_k|)\eta$.
If $[\eta]_\alpha = [a^\alpha, b^\alpha]$ for every $\alpha \in (0,1]$ we have

$$\|x\|_2^\alpha \geq (\sum_{k=1}^{n} |a_k|).b^\alpha \quad and \quad \|x\|_1^\alpha \geq (\sum_{k=1}^{n} |a_k|).a^\alpha,$$

or

$$\sum_{k=1}^{n} |a_k| \leq \frac{\|x\|_1^\alpha}{a^\alpha} \leq \|x\|_1^\alpha$$

and

$$\sum_{k=1}^{n} |a_k| \leq \frac{\|x\|_2^\alpha}{b^\alpha} \leq \|x\|_2^\alpha.$$

Now, since

$$\begin{aligned} \|Tx\| &= \|\sum_{k=1}^{n} a_k\alpha_k\| \\ &\preccurlyeq \sup_n(\sum_{k=1}^{n} \|a_k\alpha_k\|) \end{aligned}$$

Then for each $\alpha \in (0,1]$ we have

$$\begin{aligned} \|Tx\|_1^\alpha &\leq \|\sum_{k=1}^{n} a_k\alpha_k\|_1^\alpha \\ &\leq (\sup_n \|\alpha_n\|_1^\alpha.\|x\|_1^\alpha).\sum_{k=1}^{n} |a_k| \\ &\leq (\sup_n \|\alpha_n\|_1^\alpha.\|x\|_1^\alpha), \end{aligned}$$

and similarly

$$\|Tx\|_2^\alpha \leq (\sup_n \|\alpha_n\|_2^\alpha).\|x\|_2^\alpha.$$

Thus

$$\|Tx\| \preccurlyeq (\sup_n \|\alpha_n\|) \otimes \|x\|.$$

Taking the supremum over all x of norm $\tilde{1}$, we get

$$\begin{aligned} \|T\| &= \sup \|Tx\| \\ &\preccurlyeq \sup_n \|\alpha_n\| \end{aligned}$$

Hence $T \in X^*$ and $\|T\| \preccurlyeq \|\alpha\|$.
Therefore we observe thatX^* and X are isometrically isomorphic.
Remark:

If we write equality for isometric isomorphism we have, $X^{**} = (X^*)^* = X^* = X$,where X^* and X^{**} are the fuzzy dual space and the second fuzzy dual space of the finite dimensional fuzzy normed space X respectively. Therefore every finite dimensional fuzzy normed space X is reflexive.

Theorem 3.3. *Every reflexive fuzzy norm space is a fuzzy Banach space.*

Proof:

If X is a reflexive fuzzy normed space then it is linearly isometric to X^{**},thus the proof follows by using proposition 3.1 .

Definition 3.7. Let $(X, \|.\|)$ be a fuzzy normed space. A set $A \subset X$ is called a fuzzy bounded set if for each $\alpha \in (0, 1]$ there exists $M = M(\alpha) > 0$ such that $\|x\|_2^\alpha \leq M \quad \forall x \in A$.

Example 3.1. Let $T : \ell_1 \to \ell_\infty$ be the linear operator defined by: for $x = \{\zeta_1, \zeta_2, ...\} \in \ell_1, Tx = y = \{\gamma_1, \gamma_2, ...\} \in \ell_\infty$, where $\|\gamma \leq M$ and M is a real number, with norm

$$\|y\|_\infty = \sup_j \|\gamma_j\|$$

Let $\|.\|_1$ be an ordinary ℓ_1 norm define by

$$\|x\|_1 = \sum_{j=1}^{\infty} \|\zeta_j\|$$

where $x = \{\zeta_j\}$ and every $x \in \ell_1$ has a unique representation

$$x = \sum_{k=1}^{\infty} \zeta_k e_k$$

where$\{e_k\}$ be schauder basis, and let $\|.\|_{(\ell_1)}$ and $\|.\|_{(\ell_\infty)}$ be triangle fuzzy norms such that $\|x\|_{(\ell_1)} = (m\|x\|_1, \|x\|_1, n\|x\|_1)$ where $0 < m < 1, 1 < n < \infty$ and $\|y\|_{(\ell_\infty)} = r\|y\|_\infty, \|y\|_\infty, s\|y\|_\infty)$ where $0 < r < 1, 1 < s < \infty$.

Now for $\gamma_k = f(e_k)$ then

$$\begin{aligned} \|Tx\|_\infty &= \|y\|_\infty = \|\sum_{k=1}^{\infty} \zeta_k \gamma_k\| \\ &\leq \sum_{k=1}^{\infty} \|\zeta_k \gamma_k\| \\ &= \sup_j \|\gamma_j\| \sum_{k=1}^{\infty} \|\zeta_k\| \\ &= \sup_j \|\gamma_j\| \|x\|_1 \end{aligned}$$

Hence $\|Tx\|_{(\ell_1)} \preccurlyeq K \otimes \|x\|_{(\ell_1)}$,
where K a fuzzy number $\succcurlyeq \tilde{0}(K \in \mathbf{R}^*(\mathbf{I}))$
such that

$$[K]_\alpha = [K_1^\alpha, K_2^\alpha] = [\sup_j \|\gamma\|_1^\alpha, \sup_j \|\gamma\|_2^\alpha]$$

with $\{K_2^\alpha / \alpha \in (0,1]\} < \infty$.
Then T is a fuzzy bounded operator from a fuzzy norm space $((\ell_1), \|.\|_{(\ell_1)})$ into the fuzzy linear normed space $((\ell_\infty), \|.\|_{(\ell_\infty)})$.

Example 3.2. $(\ell_1)^*$ is (ℓ_∞) i.e.the fuzzy dual space of $((\ell_1), \|.\|_{(\ell_1)})$ is $((\ell_\infty), \|.\|_{(\ell_\infty)})$
The fuzzy dual space of $(\ell_1) = ((\ell_1), \|.\|_{(\ell_1)})$ is $(\ell_\infty)((\ell_\infty), \|.\|_{(\ell_\infty)})$ where $\|.\|_{(\ell_1)}$ and $\|.\|_{(\ell_\infty)}$ are the norms defined in example 3.1 .

Let $\{e_1, e_2, ...\}$ be a schauder basis for $(\ell_1, \|.\|_{(\ell_1)})$ then every $x = \{x_n\} \in \ell_1$ has a unique representation

$$x = \sum_{k=1}^{\infty} x_k e_k = x_1 e_1 + x_2 e_2 + ... \tag{3.7}$$

Let $T \in (\ell_1)^*$ the fuzzy dual space of ℓ_1.
Since T is linear and fuzzy bounded then

$$Tx = x_1 Te_1 + x_2 Te_2 + ... = x_1 \gamma_2 + x_2 \gamma_2 + ... \tag{3.8}$$

Where $\gamma_k = T(e_k)$ for every $k = 1, 2, ..$
For every $\alpha \in (0,1]$ and $\|e_k\|_{(\ell_1)} = \tilde{1}$. then $\|\gamma\|_2^\alpha = \|T(e_k)\|_2^\alpha \leq \ |T\|_2^\alpha . \|e_k\|_2^\alpha = \|T\|_2^\alpha$

Take $M = M(\alpha) = \|T\|_2^\alpha$ then $\|\gamma\|_2^\alpha \leq M$
and

(3.9) $$\sup_k \|\gamma_k\| \preccurlyeq \|T\|$$

Hence $\{\gamma_k\} \in (\ell_\infty)$
On the other hand for $\{\beta_k\} \in (\ell_\infty)$ we may define g on (ℓ_1) by $g(x) = x_1\beta_1 + x_2\beta_2 + ...$, where $x = \{x_k\} \in (\ell_1)$
We have

$$\|g(x)\| \preccurlyeq \sum_1^\infty \|x_k\beta_k\| \preccurlyeq \sup_j \|\beta_j\| \otimes \sum_{k=1}^\infty \|x_k\|$$
$$= K \otimes \|x\|, \quad taking \quad K = \sup_j \|\beta_j\|$$

$\therefore \|g(x)\| \preccurlyeq K \otimes \|x\|$ thus g is linear and fuzzy bounded.
Hence $g \in (\ell_1)^*$
Now, from (3.8) we have

$$\|Tx\| \preccurlyeq \sup_j \|\gamma_j\| \otimes \|x\|$$

Taking sup over all x of norm $\tilde{1}$ we get

(3.10) $$\|T\| \preccurlyeq \sup_j \|\gamma_j\|$$

From ((3.9)) and ((3.10)) we get

$$\|T\| = \sup_j \|\gamma_j\| \quad whichisthenormon(\ell_\infty).$$

It follows that the objective fuzzy linear mapping of $(\ell_1)^*$ onto (ℓ_∞) defined by $T \mapsto \{\gamma_k\}$ is an isomorphism. That is,$(\ell_1)^*$ and (ℓ_∞) are isometrically isomorphic.

Definition 3.8. A sequence $\{x_n\}_2^\infty$ in X is said to be weakly convergent if there is an $x \in X$ such that for every $T \in X^*$,

$$\lim_{n\to\infty} \|T(x_n) - T(x)\| = \tilde{0},$$

i.e.

$$\lim_{n\to\infty} \|T(x_n) - T(x)\|_1^\alpha = \lim_{n\to\infty} \|T(x_n) - T(x)\|_2^\alpha = 0 \quad forevery \quad \alpha \in (0, 1]$$

written $x_n \longrightarrow^w x$ in X. A fuzzy normed space X is called uniformly convex if

$$\lim_{n\to\infty} \|x_n - y_n\| = \tilde{0}$$

where $\{x_n\}$ and $\{y_n\}$ are infinite sequence in X with $\|x_n\| \preccurlyeq \tilde{1}$ and $\|y_n\|\tilde{1}$ and

$$\lim_{n\to\infty} \|\frac{x_n - y_n}{2}\| = \tilde{1}.$$

Definition 3.9. Let $(X, \|.\|)$ be a fuzzy normed space and X^* be its fuzzy dual,a sequence $\{f_n\}$ in X^*

$$\lim_{n\to\infty} \|f_n(x) - f(x)\| = \tilde{0} \quad \forall x \in X,$$

i.e.

$$\lim_{n\to\infty} \|f_n(x) - f(x)\|_\alpha^1 = \lim_{n\to\infty} \|f_n(x) - f(x)\|_\alpha^1 = 0$$

for every $\alpha \in (0, 1]$ and every $x \in X$. In this case we write $f_n \longrightarrow^{w^*} f$ in X^*.

Lemma 3.1. *Let $(X, \|.\|)$ be a fuzzy normed space, if a sequence $\{f_n\}$ of fuzzy bounded linear functionals on X is weakly* convergent to f in X^* then*

$$\|f\| \preccurlyeq \liminf_{n\to\infty} \|f_n\|.$$

Proof:-

By definition, a sequence $\{f_n\}$ in X^* is weakly* convergent to f in X^*,if

$$\lim_{n\to\infty} \|f_n(x) - f(x)\| = \tilde{0},$$

for all $x \in X$.
Since for all $\alpha \in (0, 1]$,

$$|\|f_n(x)\|_1^\alpha - \|f(x)\|_1^\alpha| \leq \|f_n(x) - f(x)\|_1^\alpha \longrightarrow 0 asn \to \infty,$$

and

$$|\|f_n(x)\|_2^\alpha - \|f(x)\|_2^\alpha| \leq \|f_n(x) - f(x)\|_2^\alpha \longrightarrow 0 asn \to \infty,$$

It follows that for any $\alpha \in (0, 1]$,

$$\lim_{n\to\infty} \|f_n(x)\|_1^\alpha = \|f(x)\|_1^\alpha \quad and \quad \lim_{n\to\infty} \|f_n(x)\|_2^\alpha = \|f(x)\|_2^\alpha$$

Thus

$$\lim_{n\to\infty} \|f_n(x)\| = \|f(x)\|$$

If $\{f_n\}$ in X^* then there exists an operator norm $\|f_n\|$ such that $\|f_n(x)\| \preccurlyeq \|f_n\| \otimes \|x\|$ (see Itoh and Cho [6] theorem 2.6).

Now,

$$\begin{aligned} \|f(x)\| &= \lim_{n\to\infty} \|f_n(x)\| \\ &\preccurlyeq \lim_{n\to\infty} \|f_n\| \otimes \|x\| \end{aligned}$$

Then for any $\alpha \in (0,1]$ we have

$$\|f(x)\|_1^\alpha \leq \liminf_{n\to\infty} \|f_n\|_1^\alpha . \|x\|_1^\alpha$$

And

$$\|f(x)\|_2^\alpha \leq \liminf_{n\to\infty} \|f_n\|_2^\alpha . \|x\|_2^\alpha$$

for any $x \in X$.
Therefore

$$\|f\| \preccurlyeq \liminf_{n\to\infty} \|f_n\|$$

Theorem 3.4. *Every uniformly convex fuzzy Banach space is reflexive.*

Proof:- Assume that X uniformly convex fuzzy Banach space. Let $x^{**} \in X^{**}$ with $\|x^{**}\| = \tilde{1}$. It suffices to show that $x^{**} \in X$ holds for every $x^{**} \in X^{**}$ such that $\|x^{**}\| = \tilde{1}$.
Let $\{x_n\}$ be a sequence in the canonical image $S(X)$ of X in X^{**} such that $x_n \longrightarrow^{w^*} x^{**}$ and $\|x_n\| \preccurlyeq \tilde{1}$.
Since for all $\alpha \in (0,1]$

$$[\|x^{**}\|]_\alpha = [\|x^{**}\|_1^\alpha, \|x^{**}\|_2^\alpha], [\|x_n\|]_\alpha = [\|x_n\|_1^\alpha, \|x_n\|_2^\alpha],$$

by Lemma 3.1 we have,

$$\|x^{**}\|_1^\alpha \leq \liminf_{n\to\infty} \|x_n\|_1^\alpha \quad and \quad \|x^{**}\|_2^\alpha \leq \liminf_{n\to\infty} \|x_n\|_2^\alpha$$

Then

$$\|2x^{**}\|_1^\alpha \leq \liminf_{n,m\to\infty} \|x_n + x_m\|_1^\alpha \quad and \quad \|2x^{**}\|_2^\alpha \leq \liminf_{n,m\to\infty} |x_n + x_m\|_2^\alpha$$

So we have

$$2 = \|2x^{**}\|_1^\alpha \leq \liminf_{n,m\to\infty} \|x_n + x_m\|_1^\alpha \leq \limsup_{n,m\to\infty} \|x_n + x_m\|_1^\alpha \leq 2 \tag{3.11}$$

and

$$2 = \|2x^{**}\|_2^\alpha \leq \liminf_{n,m\to\infty} \|x_n + x_m\|_2^\alpha \leq \limsup_{n,m\to\infty} \|x_n + x_m\|_2^\alpha \leq 2. \tag{3.12}$$

From (3.11) and (3.12) we get

$$\lim_{n,m\to\infty} \|\frac{x_n + x_m}{2}\|_1^\alpha = 1 \quad and \quad \lim_{n,m\to\infty} \|\frac{x_n + x_m}{2}\|_2^\alpha = 1$$

Hence

$$\lim_{n,m\to\infty} \|\frac{x_n + x_m}{2}\| = \tilde{1}$$

By uniform convexity this implies that

$$\lim_{n,m\to\infty} \|x_n - x_m\| = \tilde{0}.$$

i.e. $\{x_n\}$ is cauchy sequence. As X is complete then $\{x_n\}$ converges in X and its limit equals x^{**}, whence $x^{**} \in X$ this proves that X is reflexive.

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