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ON A SUBCLASS OF PRESTARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

NAILAH A. ALDIHAN

ABSTRACT. The object of this paper is to derive several intersting properties of a new class $T_\lambda^n(\mu, \beta, \xi)$, consisting of analytic univalent functions with negative coefficients, for which coefficient inequalities, distortion theorems, closure theorems, radii of close to convexity, starlikeness and convexity are determined. Moreover, the integral operator and the modified Hadamard product of two functions belonging to $T_\lambda^n(\mu, \beta, \xi)$ are considered. Finally, using the operator of fractional calculus, we have obtained some distortion theorems for $T_\lambda^n(\mu, \beta, \xi)$.

1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{2}^{\infty} a_n z^n \tag{1.1}$$

analytic and univalent in $E : E = \{z : |z| < 1\}$. T denotes the subclass of A consisting of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \geq 0 \tag{1.2}$$

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we define

$$D_\lambda^0 f(z) = f(z)$$

$$D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda\ f(z), \lambda \geq 0$$

$$D_\lambda^n f(z) = D_\lambda\left(D_\lambda^{n-1} f(z)\right) \tag{1.3}$$

This operator was introduced by AL-Oboudi in [1] and when $\lambda = 1$ we get the Sălăgean differential operator [7].

It can be easily seen that

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]^n a_k z^k,$$

and

$$D_\lambda^n f(z) = (\underbrace{\psi_\lambda * \psi_\lambda * \psi_\lambda * \cdots * \psi_\lambda}_{n \text{ time}} * f)(z),$$

where

$$\psi_\lambda(z) = z\left[\frac{1-(1-\lambda) z}{(1-z)^2}\right]$$

and $*$ denotes the well-known convolution symbol and

$$D_\lambda^n f(z) = \varphi_n(z) * f(z) \tag{1.4}$$

$$\varphi_{\lambda,n}(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]^n z^k \tag{1.5}$$

Now we propose

Definition 1.1. Let f be given by (1.1). Then $f \in R_\lambda^n(\mu, \beta, \xi)$ if and only if

$$\left| \frac{\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} - 1}{2\xi\left(\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} - \mu\right) - \left(\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} - 1\right)} \right| < \beta \tag{1.6}$$

where $D_\lambda^n f$ is defined as in (1.4) and $0 < \beta \leq 1$, $0 \leq \mu < 1$, $\frac{1}{2} < \xi \leq 1$.

when $\lambda = 1$ we have the salagean operator.

when $\lambda = 1$ and $n = 0$ we have a subclass of prestarlike functions.

when $\lambda = 1$, and $\varphi_n = z(1-z)^{-2(1-\alpha)}$ we have the class of α-prestarlike functions introduced by Ruscheweyh [6] and studied by Silverman and Silvia [8], Owa and Ahuja [4] and Uralegaddi and Sarangi [10].

Let

$$T_\lambda^n(\mu,\beta,\xi) = R_\lambda^n(\mu,\beta,\xi) \cap T \tag{1.7}$$

Our main tool in the present paper is the following theorem.

2. COEFFICIENT ESTIMATES AND INCLUSION

Theorem 2.1. *Let f be defined by (1.2). Then f is in the class $T_\lambda^n(\mu,\beta,\xi)$ if and only if*

$$\sum_{k=2}^{\infty} \left[(k-1)\lambda(1-\beta+2\beta\xi) - 2\beta\xi(\mu-1)\right] c_k(n,\lambda)\, a_k \leq 2\beta\xi(1-\mu) \tag{2.1}$$

and

$$c_k(n,\lambda) = [1+(k-1)\lambda]^n \tag{2.2}$$

Proof. Assume $f \in T_\lambda^n(\mu,\beta,\xi)$. Then f satisfies

$$\left| \frac{\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} - 1}{2\xi\left(\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} - \mu\right) - \left(\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} - 1\right)} \right| < \beta$$

$$\left| \frac{\sum\limits_{k=2}^{\infty} c_k(n,\lambda)(k-1) a_k z^k}{2\xi(1-\mu)\beta z + \sum\limits_{k=2}^{\infty} c_k(n,\mu)\left[(k-1)\lambda\beta(1-2\xi) + 2\beta\xi(\mu-1) a_k z^k\right]} \right| < 1$$

letting $z \to^{-} 1$. Then we have

$$\sum_{k=2}^{\infty} \left[(k-1)\lambda(1-\beta+2\beta\xi) - 2\beta\xi(\mu-1)\right] c_k(n,\lambda)\, a_k \leq 2\beta\xi(1-\mu), \quad k \geq 2.$$

□

This result is sharp as can be seen by

(2.3) $f_k(z) =$

$$z - \frac{2\beta\xi(1-\mu)}{[(k-1)\lambda(1-\beta+2\beta\xi) - 2\beta\xi(\mu-1)]\, c_k(n,\lambda)} z^k, \quad k \geq 2$$

Corollary 2.1. *Let $f \in T_\lambda^n(\mu,\beta,\xi)$ and be defined by (1.2). Then*

$$a_k \leq \frac{2\beta\xi(1-\mu)}{c_k(n,\lambda)[(k-1)\lambda(1-\beta+2\beta\xi) - 2\beta\xi(\mu-1)]} \tag{2.4}$$

The equality in (2.4) is a attained for the function f_k given by (2.3).

Theorem 2.2. *For $0 < \beta \leq 1$, $n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$*

$$T_\lambda^{n+1}(\mu,\beta,\xi) \subseteq T_\lambda^n(\mu,\beta,\xi)$$

Proof. The proof follows from Theorem 2.1. □

3. GROWTH AND DISTORTION THEOREMS

Theorem 3.1. *Let the function f given by (1.2) be in the class $T_\lambda^n(\mu,\beta,\xi)$. Then for $|z| = r < 1$*

$$\left|D_\lambda^i f(z)\right| \geq r - \frac{2\beta\xi(1-\mu)}{(1+\lambda)^{n-i}[\lambda(1-\beta+2\beta\xi) + 2\beta\xi(1-\mu)]} r^2 \tag{3.1}$$

and

$$\left|D_\lambda^i f(z)\right| \leq r + \frac{2\beta\xi(1-\mu)}{(1+\lambda)^{n-i}[\lambda(1-\beta+2\beta\xi) + 2\beta\xi(1-\mu)]} r^2, \tag{3.2}$$

where $0 \leq i \leq n$. The equalities in (3.1) and (3,2) are attained for the function $f(z)$ given by

$$f_2(z) = z - \frac{2\beta\xi(1-\mu)}{(1+\lambda)^{n-i}[\lambda(1-\beta+2\beta\xi) + 2\beta\xi(1-\mu)]} z^2, \quad z = \pm r \tag{3.3}$$

Proof. Note that $f \in T_\lambda^n(\mu,\beta,\xi)$ if and only if

$$D_\lambda^i f \in T_\lambda^{n-i}(\mu,\beta,\xi)$$

and that

$$D_\lambda^i f(z) = z - \sum_{k=2}^{\infty} c_k(i,\lambda)\, a_k z^k \tag{3.4}$$

where c_k is given by (2.2).

By Theorem 2.1, we know that

$$(1+\lambda)^{n-i}\left[\lambda\left(1-\beta+2\beta\xi\right)+2\beta\xi\left(1-\mu\right)\right]\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{i}a_k$$

$$\leq\sum_{k=2}^{\infty}\left[(k-1)\lambda\left(1-\beta+2\beta\xi\right)+2\beta\xi\left(1-\mu\right)\right]c_k\left(n,\lambda\right)a_k$$

$$< 2\beta\xi\left(1-\mu\right), \tag{3.5}$$

that is,

$$\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{i}a_k\leq\frac{2\beta\xi\left(1-\mu\right)}{(1+\lambda)^{n-i}\left[\lambda\left(1-\beta+2\beta\xi\right)+2\xi\left(1-\mu\right)\beta\right]}. \tag{3.6}$$

The assertions of (3.1) and (3.2) of Theorem 3.1 would now follow immediately. Finally, we note the equalities (3.1) and (3.2) are attained for the function $f_2(z)$ defined by (3.3). □

Corollary 3.1. *Let f be of the form (1.2) and be in the class $T_{\lambda}^{n}(\mu,\beta,\xi)$. Then for $|z|=r<1$*

$$|f(z)|\geq r-\frac{2\beta\xi\left(1-\mu\right)}{(1+\lambda)^{n}\left[\lambda\left(1-\beta+2\beta\xi\right)+2\beta\xi\left(1-\mu\right)\right]}r^2 \tag{3.7}$$

and

$$|f(z)|\leq r+\frac{2\beta\xi\left(1-\mu\right)}{(1+\lambda)^{n}\left[\lambda\left(1-\beta+2\beta\xi\right)+2\beta\xi\left(1-\mu\right)\right]}r^2 \tag{3.8}$$

The equalities in (3.7) and (3.8) are attained for the function f_2 given in (3.3).

4. CONVEX LINEAR COMBINATIONS

Theorem 4.1. *$T_{\lambda}^{n}(\mu,\beta,\xi)$ is a convex set.*

Proof. Let $f_i(z)=z-\sum_{k=2}^{\infty}a_{i,k}z^k$ $(a_{i,k}\geq 0,\ i=1,2)$ be in $T_{\lambda}^{n}(\mu,\beta,\xi)$. It is sufficient to show that the function h defined by

$$h(z)=\eta f_1(z)+(1-\eta)f_2(z)\qquad 0\leq\eta\leq 1 \tag{4.1}$$

is also in $T_\lambda^n(\mu,\beta,\xi)$.

$$h(z) = z - \sum_{k=2}^{\infty} (\eta a_{1,k} + (1-\eta) a_{2,k}) z^k \tag{4.2}$$

Using Theorem 2.1, we obtain

$$\sum_{2}^{\infty} [(k-1)\lambda(1-\beta+2\beta\xi) + 2\beta\xi(1-\mu)] c_k(n,\lambda) [\eta a_{1,k} + (1-\eta) a_{2,k}] \leq 2\beta\xi(1-\mu),$$

which implies that $h(z) \in T_\lambda^n(\mu,\beta,\xi)$ and hence $T_\lambda^n(\mu,\beta,\xi)$ is a convex set. □

5. RADII OF CLOSE-TO-CONVEXITY, STARLIKENESS, AND CONVEXITY

Theorem 5.1. *Let $f \in T_\lambda^n(\mu,\beta,\xi)$, and be defined by (1.2). Then f is close-to-convex of order $0 \leq \rho < 1$ in $|z| < r_1$ where*

$$r_1 = \inf_k \left[\frac{(1-\rho)[1+(k-1)\lambda]^n [(k-1)\lambda(1-\beta+2\beta\xi) + 2\xi(1-\mu)]}{2\beta\xi(1-\mu)k}\right]^{\frac{1}{k-1}}. \tag{5.1}$$

The result is sharp, the extremal function $f(z)$ being given in (2.3).

Proof. We must show that

$$|f'(z) - 1| \leq 1-\rho \text{ for } |z| < r_1,$$

where r_1 is given by (5.1). From (1.2) we have

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1} \leq 1-\rho$$

$$\sum_{k=2}^{\infty}\left(\frac{k}{1-\rho}\right)a_k\left|z\right|^{k-1}\leq 1 \tag{5.2}$$

By Theorem 2.1, (5.2) will be true if

$$\left(\frac{k}{1-\rho}\right)\left|z\right|^{k-1}\leq\frac{c_k\left(n,\lambda\right)\left[\left(k-1\right)\lambda\left(1-\beta+2\beta\xi\right)+2\xi\left(1-\mu\right)\right]}{2\beta\xi\left(1-\mu\right)},$$

where $c_k\left(n,\lambda\right)$ is given by (2.2).

That is,

$$\left|z\right|\leq\left[\frac{\left(1-\rho\right)c_k\left(n,\lambda\right)\left[\left(k-1\right)\lambda\left(1-\beta+2\beta\xi\right)+2\beta\xi\left(1-\mu\right)\right]}{2\beta\xi\left(1-\mu\right)k}\right]^{\frac{1}{k-1}} \tag{5.3}$$

□

Now, Theorem 5.1 follows from (5.3)

Theorem 5.2. *Let f be defined by (1.2) and be in $T_{\lambda}^{n}\left(\mu,\beta,\xi\right)$. Then f is starlike of order ρ $(0\leq\rho<1)$ in $\left|z\right|<r_2$ where*

$$r_2=\inf_k\left[\frac{c_k\left(n,\lambda\right)\left(1-\rho\right)\left[\left(k-1\right)\lambda\left(1-\beta+2\beta\xi\right)+2\beta\xi\left(1-\mu\right)\right]}{2\left(k-\rho\right)\beta\xi\left(1-\mu\right)}\right]^{\frac{1}{k-1}}. \tag{5.4}$$

This result is sharp as can be seen by considering f given in (2.3).

Proof. It is sufficient to show that

$$\left|\frac{zf'\left(z\right)}{f\left(z\right)}-1\right|\leq\left(1-\rho\right)\qquad\text{for } \left|z\right|<r_2, \tag{5.5}$$

where r_2 is given by (5.4).

If f is given by (1.2), then

$$\left|\frac{zf'\left(z\right)}{f\left(z\right)}-1\right|\leq\frac{\sum\limits_{k=2}^{\infty}\left(k-1\right)a_k\left|z\right|^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_k\left|z\right|^{k-1}}.$$

Thus,

$$\sum_{k=2}^{\infty}(k-1)\,a_k\,|z|^{k-1} \leq (1-\rho)-(1-\rho)\sum_{k=2}^{\infty}a_k\,|z|^{k-1}$$
$$\Rightarrow \sum_{k=2}^{\infty}\left(\frac{k-\rho}{1-\rho}\right)a_k\,|z|^{k-1}\leq 1.$$

But by Theortem 2.1 we have

$$\left(\frac{k-\rho}{1-\rho}\right)|z|^{k-1}\leq\frac{c_k\,(n,\lambda)\,[(k-1)\,\lambda\,(1-\beta+2\beta\xi)+2\beta\xi\,(1-\mu)]}{2\xi\,(1-\mu)\,\beta},\qquad k\geq 2$$

where $c_k\,(n,\lambda)$ is given by (2.2),

$$|z|\leq\left[\frac{(1-\rho)\,c_k\,(n,\lambda)\,[(k-1)\,\lambda\,(1-\beta+2\beta\xi)+2\beta\xi\,(1-\mu)]}{2\,(k-\rho)\,\beta\xi\,(1-\mu)}\right]^{\frac{1}{k-1}}.$$

□

Corollary 5.1. *Let the function f defined by (1.2) be in $T_\lambda^n\,(\mu,\beta,\xi)$. Then $f\,(z)$ is convex of order ρ $(0\leq\rho<1)$ in $|z|<r_3$, where*

$$r_3=\inf_k\left[\frac{(1-\rho)\,c_k\,(n,\lambda)\,[(k-1)\,\lambda\,(1-\beta+2\beta\xi)+2\beta\xi\,(1-\mu)]}{2\xi k\,(k-\rho)\,(1-\mu)}\right]^{\frac{1}{k-1}}$$

and $c_k\,(n,\lambda)$ is given by (2.2), $k\geq 2$.

6. A FAMILY OF INTEGRAL OPERATORS

Theorem 6.1. *Let $f\in T_\lambda^n\,(\mu,\beta,\xi)$ and be defined by (1.2) and let c be any real number such that $c>-1$. Then the function*

$$F\,(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f\,(z)\,dt\qquad c>-1 \tag{6.1}$$

belongs to the same class $T_\lambda^n\,(\mu,\beta,\xi)$.

Proof. From the representation (6.1)

$$F\,(z)=z-\sum_{k=2}^{\infty}b_k z^k$$

where

$$b_k=\left(\frac{c+1}{c+k}\right)a_k.$$

Therefor, we have

$$\begin{aligned}
& \sum_{k=2}^{\infty} c_k(n,\lambda)\left[(k-1)\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)\right]b_k \\
= & \sum_{k=2}^{\infty} c_k(n,\lambda)\left[(k-1)\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)\right]\left[\frac{c+1}{c+k}\right]a_k \\
\leq & \sum_{k=2}^{\infty} c_k(n,\lambda)\left[(k-1)\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)\right]a_k \\
\leq & \; 2\beta\xi(1-\mu).
\end{aligned}$$

Hence by Theorem 2.1, $f \in T_{\lambda}^{n}(\mu,\beta,\xi)$. □

Theorem 6.2. *Let*

$$F(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \geq 0$$

be in the class $T_{\lambda}^{n}(\mu,\beta,\xi)$*, and let* c *be a real number such that* $c > -1$*. Then the function* $f(z)$ *given by (6.1) is univalent in* $|z| < R^*$*, where*

$$R^* = \inf_k \left[\frac{(c+1)\,c_k(n,\lambda)\left[(k-1)\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)\right]}{2\beta\xi k(1-\mu)(c+k)}\right]^{\frac{1}{k-1}} \tag{6.2}$$

and $c_n(n,\lambda)$ *is given by (2.2).*

Proof. From (6.1) we have,

$$f(z) = \frac{z^{1-c}\left[zF(z)\right]'}{c+1} = z - \sum_{k=2}^{\infty}\left(\frac{c+k}{c+1}\right)a_k z^k$$

To obtain the required result, it is sufficient to show that

$$\left|f'(z)-1\right| < 1 \quad \text{whenever} \quad |z| < R^*.$$

Now,

$$\left|f'(z)-1\right| \leq \sum_{k=2}^{\infty}\frac{k(c+k)}{c+1}a_k|z|^{k-1} \tag{6.3}$$

Thus, $|f'(z)-1| < 1$ if

$$\sum_{k=2}^{\infty}\frac{k(c+k)}{c+1}a_k|z|^{k-1} < 1.$$

But from Theorm 2.1 we have

$$\sum_{k=2}^{\infty} \frac{c_k(n,\lambda)\left[(k-1)\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)\right]}{2\beta\xi(1-\mu)} a_k \leq 1$$

Hence (6.3) will be satisfied if

$$\frac{k(c+k)}{c+1}|z|^{k-1} < \frac{c_k(n,\lambda)\left[(k-1)\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)\right]}{2\beta\xi(1-\mu)}$$

That is

$$|z| < \left[\frac{(c+1)c_k(n,\lambda)\left[(k-1)\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)\right]}{2\beta\xi k(1-\mu)(c+k)}\right]^{\frac{1}{k-1}} \tag{6.5}$$

Therefore, the function f is univalent in $|z| < R^*$.

The sharpness can be seen from the function

$$f_0(z) = z - \frac{2\beta\xi k(1-\mu)(c+k)}{(c+1)c_k(n,\lambda)\left[(k-1)\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)\right]} z^k$$

$k \geq 2$ and $c_k(n,\lambda)$ is given by (2.2). □

7. DISTORTION THEOREMS INVOLING FRACTIONAL CALCULUS

We shall now prove some distortion theorems for functions belonging to the class $T_\lambda^n(\mu,\beta,\xi)$. Each of these would involve operators of fractional calculus which are defined as follows (see for example [2,3,5,9,10]).

Definition 7.1. The fractional integral of order γ is defined by

$$D_z^{-\gamma} f(z) = \frac{1}{\Gamma(\gamma)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\gamma}} d\xi \tag{7.1}$$

$\gamma > 0$, f is analytic in a simply connected region containing the origin, and multiplicity of $(z-\xi)^{\gamma-1}$ is removed by requiring $Log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 7.2. The fractional derivative of order γ is defined by

$$D_z^{\gamma}(f(z)) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\gamma}} d\xi \tag{7.2}$$

where $0 \leq \gamma < 1$. f is analytic in a simply connected region containing the origin and the multiplicty of $(z-\xi)^{-\gamma}$ is removed as in Definition 7.1

$$D_z^{n+\gamma} f(z) = \frac{d^n}{dz^n} D_z^{\gamma}(f(z)) \tag{7.3}$$

where $0 \leq \gamma < 1$, $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Theorem 7.1. *Let f be given by 1.2 and in $T_\lambda^n(\mu, \beta, \xi)$. Then*

$$\left|D_z^{-\gamma} f(z)\right| \geq \frac{|z|^{1+\gamma}}{\Gamma(2+\gamma)}\left[1 - \frac{2\beta\xi(1-\mu)}{(1+\lambda)^n[\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)]}|z|\right] \tag{7.4}$$

and

$$\left|D_z^{-\gamma} f(z)\right| \leq \frac{|z|^{1+\gamma}}{\Gamma(2+\gamma)}\left[1 + \frac{\beta\xi(1-\mu)}{(1+\lambda)^n[\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)]}|z|\right] \tag{7.5}$$

for $\gamma > 0$, $z \in E$. The bounds are sharp.

Proof. Let

$$\begin{aligned} F(z) &= \Gamma(2+\gamma) z^{-\gamma} D_z^{-\gamma} f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\gamma)}{\Gamma(n+1+\gamma)} a_n z^n. \end{aligned} \tag{7.6}$$

For $\gamma > 0$, we note that

$$0 < \frac{\Gamma(n+1)\Gamma(2+\gamma)}{\Gamma(n+1+\gamma)} < n. \tag{7.7}$$

For $\gamma > 0$, we have $n \geq 2$ and $c_k(\lambda, n+1) \geq c_k(\lambda, n)$ for all $\lambda \geq 0$ and $k \geq 2$. Consequently by using Theorem 2.1, we have

$$\begin{aligned} |F(z)| &\geq |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\gamma)}{\Gamma(n+1+\gamma)} a_n \\ |F(z)| &\geq |z| - \frac{2\beta\xi(1-\mu)}{(1+\lambda)^n[\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)]}|z|^2 \end{aligned}$$

which implies (7.4);
and

$$\begin{aligned}|F(z)| &\le |z| + |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\gamma)}{\Gamma(n+1+\gamma)} a_n \\ &\le |z| + \frac{2\beta\xi(1-\mu)}{(1+\lambda)^n[\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)]} |z|^2\end{aligned}$$

which gives (7.5).

The inequalities are sharp for the function given by

$$D_z^{-\gamma} f(z) = \frac{|z|^{1+\gamma}}{\Gamma(2+\gamma)} \left[1 - \frac{2\beta\xi(1-\mu)}{(1+\lambda)^n[\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)]} |z|\right].$$

□

Corollary 7.1. *Let $f \in T_\lambda^n(\mu,\beta,\xi)$ and be given by (1.2), $\lambda \ge 0$, $0 \le \mu < 1$, $\frac{1}{2} < \xi \le 1$, $0 < \beta \le 1$, $n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$. Then $D_z^{-\gamma}(f(z))$ is included in a disk with center at the origin and radius r_4 given by*

$$r_4 = \frac{1}{\Gamma(2+\gamma)} \left(1 + \frac{2\beta\xi(1-\mu)}{(1+\lambda)^n[\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)]}\right)$$

where $\gamma > 0$.

Theorem 7.2. *Let f be given by (1.2) and be in $T_\lambda^n(\mu,\beta,\xi)$. Then*

$$(7.8)\quad |D_z^\gamma(f(z))| \ge \frac{|z|^{1-\gamma}}{\Gamma(2-\gamma)} \left[1 - \frac{4\beta\xi(1-\mu)}{(1+\lambda)^n[\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)]} |z|\right]$$

and

$$(7.9)\quad |D_z^\gamma(f(z))| \le \frac{|z|^{1-\gamma}}{\Gamma(2-\gamma)} \left[1 + \frac{4\beta\xi(1-\mu)}{(1+\lambda)^n[\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)]} |z|\right]$$

for $0 \le \gamma < 1$, $z \in E$. The bounds are sharp.

Proof. Let

$$\begin{aligned}G(z) &= \Gamma(2-\gamma) z^\gamma D_z^\gamma f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} a_k z^k\end{aligned}$$

where $0 \leq \gamma < 1$. By using Theorem 2.1 we observe that

$$\frac{1}{2}\left[(1+\lambda)^{n}\left(\lambda\left(1-\beta+2\beta\xi\right)+2\beta\xi\left(1-\mu\right)\right)\right]\sum_{k=2}^{\infty}ka_{k}$$
$$\leq \sum_{k=2}^{\infty}c_{k}\left(n,\lambda\right)\left[\left(k-1\right)\lambda\left(1-\beta+2\beta\xi\right)+2\beta\xi\left(1-\mu\right)\right]$$
$$\leq 2\beta\xi\left(1-\mu\right),$$

which implies

$$\sum_{k=2}^{\infty}ka_{k} \leq \frac{4\beta\xi\left(1-\mu\right)}{(1+\lambda)^{n}\left[\lambda\left(1-\beta+2\beta\xi\right)+2\beta\xi\left(1-\mu\right)\right]}$$

Furthere we note that

$$1 < \frac{\Gamma\left(k+1\right)\Gamma\left(2-\gamma\right)}{\Gamma\left(k+1-\gamma\right)} < k$$

for $0 \leq \gamma < 1$, $k \geq 2$.

Hence we have

$$\begin{aligned}
|G(z)| &\geq |z| - |z|^{2}\sum_{k=2}^{\infty}\frac{\Gamma\left(k+1\right)\Gamma\left(2-\gamma\right)}{\Gamma\left(k+1-\gamma\right)}a_{k} \\
&\geq |z| - |z|^{2}\sum_{k=2}^{\infty}ka_{k} \\
|G(z)| &\geq |z| - \frac{4\beta\xi\left(1-\mu\right)}{(1+\lambda)^{n}\left[\lambda\left(1-\beta+2\beta\xi\right)+2\beta\xi\left(1-\mu\right)\right]}|z|^{2}
\end{aligned}$$

which proves (7.8); and

$$\begin{aligned}
|G(z)| &\leq |z| + |z|^{2}\sum_{k=2}^{\infty}\frac{\Gamma\left(k+1\right)\Gamma\left(2-\gamma\right)}{\Gamma\left(k+1-\gamma\right)}a_{k} \\
&\leq |z| + |z|^{2}\sum_{k=2}^{\infty}na_{n} \\
&\leq |z| + \frac{4\beta\xi\left(1-\mu\right)}{(1+\lambda)^{n}\left[\lambda\left(1-\beta+2\beta\xi\right)+2\beta\xi\left(1-\mu\right)\right]}|z|^{2}
\end{aligned}$$

which gives (7.9).

Finally, The bounds of (7.8) and (7.9) are sharp, the extremal function being

$$D_z^\gamma f(z) = \frac{z^{1-\gamma}}{\Gamma(2-\gamma)}\left(1 - \frac{4\beta\xi(1-\mu)}{(1+\lambda)^n\left[\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)\right]}\right).$$

□

Corollary 7.2. *Let f be given by (1.2) and be in the class $T_\lambda^n(\mu,\beta,\xi)$. Then $D_z^\gamma f(z)$ is included in the disk with center at origin and radius r_5 given by,*

$$r_5 = \frac{1}{\Gamma(2-\gamma)}\left(1 + \frac{4\beta\xi(1-\mu)}{(1+\lambda)^n\left[\lambda(1-\beta+2\beta\xi)+2\beta\xi(1-\mu)\right]}\right)$$

where $0 \le \gamma < 1$.

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N. ALDihan: Mathematics Dept., Sci. Sections, Girls College of Education,
Sitteen st., Malaz, Riyadh 11417, P.O. 27104,
Saudi Arabia.

e-mail: N_ALDihan@hotmail.com.

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