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CERTAIN CLASSES RELATED TO FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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ABSTRACT. A general class of analytic functions in the unit disc, containing the functions of bounded boundary rotation and closely related to the functions of bounded argument rotation, is defined; and for this class of functions the distortion and coefficient theorems are proved and the radius of convexity is obtained.

1. INTRODUCTION

Let p be an analytic function in $E = \{z : |z| < 1\}$ given in the form $p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$. Then p is said to be in the class $P[A, B], -1 \leq B < A \leq 1$ if and only if for $z \in E$

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}$$

where $w(0) = 0$ and $|w(z)| < 1$. If $f(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is analytic in E, then $f \in S^*[A, B]$ if and only if $\dfrac{zf'(z)}{f(z)} \in P[A, B]$ for all $z \in E$.

These classes were introduced by Janawski [1].

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2. DEFINITIONS AND SOME PRELIMINARY RESULTS

Definition 1. Let $P_k[A,B,C,D]$ denote the class of functions p that are analytic in E and represented by

$$p(z) = \frac{k+2}{4}p_1(z) - \frac{k-2}{4}p_2(z)$$

where $p_1(z) \in P[A,B]$, $p_2(z) \in P[C,D]$ where $-1 \le B < A \le 1$, $-1 \le D < C \le 1$, $k \ge 2$. It is clear that

$$P_2[A,B,C,D] \equiv P[A,B]$$

and $P_k[1,-1,1,-1] = P_k$, the well known class introduced by Pinchuk [5].

Definition 2. Let f be an analytic function given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then $f \in R_k[A,B,C,D]$ if and only if there exist $s_1 \in S^*[A,B]$ and $s_2 \in S^*[C,D]$ such that for $z \in E$,

$$f(z) = \frac{(s_1(z))^{\frac{k+2}{4}}}{(s_2(z))^{\frac{k-2}{4}}}, \quad k \ge 2.$$

Clearly $R_2[A,B,C,D] = S^*[A,B]$ and $R_k[1,-1,1,-1] = R_k$ the class of bounded argument rotation introduced in [8]. When $A = C = 1-2\rho$ and $B = D = -1$ we have the class $R_k(\rho)$ which was introduced in [4].

Definition 3. Let f be an analytic function given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $f'(z) \neq 0$, $z \in E$. Then $f \in V_k[A,B,C,D]$ if and only if there exist $s_1 \in S^*[A,B]$, $s_2 \in S^*[C,D]$ where $-1 \le B < A \le 1$ and $-1 \le D < C \le 1$ such that

$$f'(z) = \frac{\left(\frac{s_1(z)}{z}\right)^{\frac{k}{4}+\frac{1}{2}}}{\left(\frac{s_2(z)}{z}\right)^{\frac{k}{4}-\frac{1}{2}}} \tag{1}$$

where $V_2[A,B,C,D] = C[A,B]$ the class introduced by Silvia [7] and $V_k[1,-1,1,-1] = V_k$ the well known class of bounded boundary rotation defined by Paatero [3]. When $A = C = 1 - 2\rho$ and $B = D = -1$ we have the class $V_k(\rho)$ which was introduced in [4]. It is clear that $f \in V_k[A, B, C, D]$ if and only if $zf' \in R_k[A, B, C, D]$.

Taking logarithmic differentiation of (1) we have

Lemma 1. *Let $f \in V_k[A, B, C, D]$. Then*

$$\frac{(zf'(z))'}{f'(z)} \in P_k[A, B, C, D]$$

3. DISTORTION AND COEFFICIENT THEOREMS

Lemma 2 [1]. *Let $f \in S^*[A, B]$. Then for $|z| < r,\ 0 < r < 1$,*

$$r(1 - Br)^{\frac{A-B}{B}} \leq |f(z)| \leq r(1 + Br)^{\frac{A-B}{B}} \quad \text{if } B \neq 0, \text{ and}$$

$$re^{-Ar} \leq |f(z)| \leq re^{Ar} \text{ if } B = 0.$$

Theorem 1. *Let $f \in V_k[A, B, C, D]$. Then*

$$\frac{(1 - Br)^{\frac{A-B}{B}\frac{k+2}{4}}}{(1 + Dr)^{\frac{C-D}{D}\frac{k-2}{4}}} \leq |f'(z)| \leq \frac{(1 + Br)^{\frac{A-B}{B}\frac{k+2}{4}}}{(1 - Dr)^{\frac{C-D}{D}\frac{k-2}{4}}}$$

when $B \neq 0$ and $D \neq 0$.

$$\frac{r\exp(-Ar(k+2)/4)}{(1 + Dr)^{\frac{C-D}{D}\frac{k-2}{4}}} \leq |f'(z)| \leq \frac{r\exp(Ar(k+2)/4)}{(1 - Dr)^{\frac{C-D}{D}\frac{k-2}{4}}}$$

when $B = 0$ and $D \neq 0$.

$$\begin{aligned} (1 - Br)^{\frac{A-B}{B}\frac{k+2}{4}} e^{Cr\frac{k-2}{4}} &\leq |f'(z)| \\ &\leq (1 + Br)^{\frac{A-B}{B}\frac{k+2}{4}} e^{Cr\frac{k-2}{4}} \end{aligned}$$

when $D = 0$ and $B \neq 0$.

$$\begin{aligned}\exp-\left\{\frac{k}{4}(A+C)r+\frac{A-C}{2}r\right\} &\le |f'(z)| \\ &\le \exp\left\{\frac{k}{4}(A+C)r+\frac{A-C}{2}r\right\}\end{aligned}$$

when $B=D=0$. *All these bounds are sharp, as can be seen from* f_0' *given by the equations below:*

$$f_0'(z)=\begin{cases}\dfrac{(1+B\delta_1 z)^{\frac{A-B}{B}\frac{k+2}{4}}}{(1-D\delta_2 z)^{\frac{C-D}{D}\frac{k-2}{4}}} & |\delta_1|=|\delta_2|=1, \\ & B\neq 0, D\neq 0. \\ \dfrac{\exp A\delta_1 z(k+2)/4}{(1-D\delta_2 z)^{\frac{C-D}{D}\frac{k-2}{4}}} & \text{when } B=0 \text{ and } D\neq 0. \\ (1+B\delta_1 z)^{\frac{A-B}{B}\frac{k+2}{4}}\exp\dfrac{k-2}{2}C\delta_2 z & \text{when } D=0 \text{ and } B\neq 0. \\ \exp\left\{\dfrac{k}{4}(A+C)\delta_1 z+\dfrac{(A-C)}{2}\delta_2 z\right\} & B=D=0.\end{cases}$$

Proof. The proof follows immediately from Lemma 2 and Definition 3.

Lemma 3 [1] *Let* $p\in P[A,B]$ *be given by* $p(z)=1+\sum_{n=1}^{\infty}b_n z^n$. *Then* $|b_n|\le A-B$ *for all* n.

Lemma 4. *Let* $F\in P_k[A,B,C,D]$, $-1\le B<A\le 1$ *and* $-1\le D<C\le 1$, *be given by* $F(z)=1+\sum_{n=1}^{\infty}c_n z^n$. *Then*

$$\text{(i)}\ \frac{1}{2\pi}\int_0^{2\pi}|F(re^{i\theta})|^2 d\theta\le\frac{16+(((k+2)(A-B)-(k-2)(C-D))^2-16)r^2}{16(1-r^2)}$$

$$\text{(ii)}\ \frac{1}{2\pi}\int_0^{2\pi}|F'(re^{i\theta})|d\theta\le\frac{k+2}{4}\frac{A-B}{1-B^2r^2}+\frac{k-2}{4}\frac{C-D}{1-D^2r^2}.$$

Proof.

(i) Let $F(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$. Then using Parseval's identity we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2 d\theta &= \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \\ &\leq 1 + \frac{((k+2)(A-B)-(k-2)(C-D))^2}{16}\sum_{n=1}^{\infty} r^{2n},\end{aligned}$$

where we use Lemma 3. Hence the result.

(ii) Now

$$F'(z) = \frac{k+2}{4}p_1'(z) - \frac{k-2}{4}p_2'(z),$$

where $p_1 \in P[A,B]$ and $p_2 \in P[C,D]$.

Since, for $p \in P[A,B]$,

$$p'(z) = \frac{(A-B)w'(z)}{(1+Bw(z))^2},$$

$$\frac{1}{2\pi}\int_0^{2\pi} |p'(re^{i\theta})| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{(A-B)|w'(re^{i\theta})|d\theta}{|1+Bw(re^{i\theta})|^2} \leq \frac{A-B}{1-B^2r^2},$$

hence $\frac{1}{2\pi}\int_0^{2\pi} |F'(re^{i\theta})| d\theta \leq \frac{k+2}{4}\frac{A-B}{1-B^2r^2} + \frac{k-2}{4}\frac{C-D}{1-D^2r^2}$.

This proves the second part of the lemma.

A Special Case: When $k = 2$, $F \in P[A,B]$and

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2 d\theta &\leq \frac{1+((A-B)^2-1)r^2}{1-r^2} \\ \frac{1}{2\pi}\int_0^{2\pi} |F'(re^{i\theta})| d\theta &\leq \frac{(A-B)}{1-B^2r^2}.\end{aligned}$$

Theorem 2. *Let $f \in V_k[A,B,C,D]$ be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then for all n,*

$$|a_n| \leq \begin{cases} C_1(k,A,B,C,D) n^{\frac{C-D}{D}\frac{k-2}{4}-2} & for\ D>0, B\geq 0 \\ C_2(k,A,C) n^{-2} & if\ D=B=0 \\ C_3(k,A,B,C,D) n^{\frac{A-B}{-B}\frac{k+2}{4}-2} & if\ B<0, D\leq 0 \\ C_4(k,A,B,C,D) n^{-2} & if\ B>0, D<0 \\ C_5(k,A,B,C,D) n^{\frac{A-B}{-B}\frac{k+2}{4}+\frac{C-D}{D}\frac{k-2}{4}-2} & if\ B<0, D>0, \end{cases}$$

where the C_i's are constants depending on A, B, C, D, k.

The function $f_0(z)$ given in Theorem 1 shows that the exponent of n is the best possible.

Proof. Using Lemma 1 we have

$$(zf'(z))' = f'(z)p(z)$$

where $p \in P_k[A,B,C,D]$. Let $F = (z(zf'(z))')'$. Then

$$F(z) = f'(z)[p^2(z) + zp'(z)]$$

and

$$n^3|a_n| \leq \frac{1}{2\pi r^{n-1}} \int_0^{2\pi} |f'(z)||p^2(z) + zp'(z)|d\theta, \text{ for } z = re^{i\theta}.$$

Using Theorem 1 and Lemma 4 we have the following cases:

(i) If $D > 0, B \geq 0$, then

$$\frac{1}{1-Dr} < \frac{1}{1-r} \quad \text{and} \quad \frac{1}{1-Br} < \frac{1}{1-r}$$

$$n^3|a_n| \leq \frac{C_1(k,A,B,C,D)}{(1-r)^{\frac{C-D}{D}\frac{k-2}{4}+1}}.$$

(ii) If $B = D = 0$, we have

$$n^3|a_n| \le \frac{C_2(k, A, C)}{(1-r)}.$$

(iii) If $B < 0, D \le 0$, we have

$$\frac{1}{1+Br} \le \frac{1}{1-r}.$$

Then, $n^3|a_n| \le \dfrac{C_3(k, A, C, B, D)}{(1-r)^{\frac{A-B}{-B}\frac{k+2}{4}+1}}$.

(iv) If $B > 0, D < 0$, then

$$|a_n| \le C_4(k, A, B, C, D)n^{-2}.$$

(v) If $B < 0, D > 0$, then $\dfrac{1}{1+Br} \le \dfrac{1}{1-r}$ and $\dfrac{1}{1-Dr} < \dfrac{1}{1-r}$.

$$n^3|a_n| \le \frac{C_5(k, A, B, C, D)}{(1-r)^{\frac{A-B}{-B}\frac{k+2}{4}+\frac{C-D}{D}\frac{k-2}{4}+1}}.$$

In the above inequalities, setting $r = 1 - \dfrac{1}{n}$ we obtain the upper bounds for $|a_n|$ as stated in the theorem.

4. INTEGRAL OPERATOR AND RADIUS PROBLEM

Lemma 5 [1]. *Let $p(z) \in P[A, B]$ be given by $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$. Then*

$$\frac{1-Ar}{1-Br} \le Re\, p(z) \le |p(z)| \le \frac{1+Ar}{1+Br}.$$

Theorem 3. *Let $f \in V_k[A, B, C, D]$. Then f maps $|z| < r_0$ onto a convex domain, where*

$$
\begin{aligned}
(2) \qquad r_0 &= 8/\{(A-D)(k+2) + (C-B)(k-2) \\
&\quad + \Big[((A-D)(k+2) + (C-B)(k-2))^2 \\
&\quad + 16((k+2)AD - (k-2)CB)\Big]^{\frac{1}{2}}\Big\}.
\end{aligned}
$$

Proof. Since $f \in V_k[A,B,C,D]$ we have

$$\frac{(zf'(z))'}{f'(z)} = p(z), \quad p(z) \in P_k[A,B,C,D].$$

Using Lemma 5 we have

$$\begin{aligned}\text{Re}\frac{(zf'(z))'}{f'(z)} &\geq \left(\frac{k+2}{4}\right)\frac{1-Ar}{1-Br} - \left(\frac{k-2}{4}\right)\frac{1+Cr}{1+Dr} \\ &= \frac{4-((A-D)(k+2)+(k-2)(C-B))r}{4(1-Br)(1+Dr)} \\ &\quad -\frac{((k+2)AD-(k-2)CB)r^2}{4(1-Br)(1+Dr)}\end{aligned}$$

and hence $\text{Re}\frac{(zf'(z))'}{f'(z)} > 0$ for $|z| < r_0$, where r_0 is given by (2). The sharpness of the result follows from the function f_0, where

$$\frac{(zf_0'(z))'}{f_0'(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1-Az}{1-Bz} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1+Cz}{1+Dz}$$

We note here that for the class $V_k = V_k[1,-1,1,-1]$ this result agrees with a result of Robertson [6].

Remark 1. The above proof shows that if $p \in P_k[A,B,C,D]$, then Re $p(z) > 0$ for all $|z| < r_0$ where r_0 is given by (2).

Using the relation between $R_k[A,B,C,D]$ and $V_k[A,B,C,D]$ we have

Remark 2. Let $f \in R_k[A,B,C,D]$. Then f maps $|z| < r_0$ onto a star-like domain where r_0 is given by (2).

Lemma 6 [2]. *Let N and D be analytic in E with $N(0) = D(0) = 0$ and D be a p-valent starlike function. Suppose that $\frac{N'(z)}{D'(z)} \in P_k$. Then $\frac{N(z)}{D(z)} \in P_k$.*

Theorem 4. *Let $g, f \in R_k[A,B,C,D]$, and*

$$F(z) = \left\{\frac{\gamma+\alpha+1-\eta}{z^\gamma}\int_0^z f^\alpha(t)g(t)t^{\gamma-1-\eta}dt\right\}^{\frac{1}{\alpha+1-\eta}}$$

where $\alpha \geq -1,\ \gamma > 0,\ 0 \leq \dfrac{\eta}{\eta - \alpha - 1} < 1,\ \eta \leq 0.$

Then $F \in R_k\left(\dfrac{\eta}{\eta - 1 - \alpha}\right).$

Proof.

$$(\alpha + 1 - \eta)\frac{zF'(z)}{F(z)} = \frac{(f(z))^\alpha g(z) z^{\gamma-\eta} - \gamma \int_0^z (f(t))^\alpha g(t) t^{\gamma-1-\eta} dt}{\int_0^z (f(t))^\alpha g(t) t^{\gamma-1-\eta} dt}$$

We define the functions

$$M(z) = f^\alpha(z) g(z) z^{\gamma-\eta} - \gamma \int_0^z (f(t))^\alpha g(t) t^{\gamma-1-\eta} dt$$

and

$$N(z) = (\alpha + 1 - \eta) \int_0^z (f(t))^\alpha g(t) t^{\gamma-1-\eta} dt.$$

Then we see that $M(0) = N(0) = 0$ and $N(z)$ is $(\alpha + \gamma + 1 - \eta)$-valent starlike for $z \in E$.

Also

$$\frac{M'(z)}{N'(z)} = \frac{\alpha}{\alpha + 1 - \eta}\frac{zf'(z)}{f(z)} + \frac{1}{\alpha + 1 - \eta}\frac{zg'(z)}{g(z)} - \frac{\eta}{\alpha + 1 - \eta}.$$

But

$$\frac{zf'(z)}{f(z)} \in P_k[A, B, C, D] \subseteq P_k$$

and

$$\frac{zg'(z)}{g(z)} \in P_k[A, B, C, D] \subseteq P_k.$$

Hence,

$$\frac{M'(z)}{N'(z)} = \frac{\alpha}{\alpha + 1 - \eta} p_1(z) + \frac{1}{\alpha + 1 - \eta} p_2(z) + \left[1 - \frac{\alpha + 1}{\alpha + 1 - \eta}\right].$$

where $p_i \in P_k \quad i = 1, 2.$

So

$$\begin{aligned}
\frac{M'(z)}{N'(z)} &= \frac{\alpha}{\alpha+1-\eta}\left[\frac{k+2}{4}h_1(z) - \frac{k-2}{4}h_2(z)\right] \\
&\quad + \frac{1}{\alpha+1-\eta}\left[\frac{k+2}{4}h_3(z) - \frac{k-2}{4}h_4(z)\right] \\
&\quad + \left[1 - \frac{\alpha+1}{\alpha+1-\eta}\right] \quad \text{where } h_i \in P,\ i = 1,2,3,4 \\
&= \left[\frac{k+2}{4}\left(\frac{\alpha}{\alpha+1-\eta}h_1(z) + \frac{1}{\alpha+1-\eta}h_3(z)\right)\right. \\
&\quad \left. - \frac{k-2}{4}\left(\frac{\alpha}{\alpha+1-\eta}h_2(z) + \frac{1}{\alpha+1-\eta}h_4(z)\right)\right] \\
&\quad + \left[1 - \frac{\alpha+1}{\alpha+1-\eta}\right] \\
&= \frac{k+2}{4}\left[\frac{\alpha+1}{\alpha+1-\eta}H_1(z)\right] - \frac{k-2}{4}\left[\frac{\alpha+1}{\alpha+1-\eta}H_2(z)\right] \\
&\quad + \left[1 - \frac{\alpha+1}{\alpha+1-\eta}\right]
\end{aligned}$$

where $H_i \in P[1,-1],\ i = 1,2.$

Hence

$$\begin{aligned}
\frac{M'(z)}{N'(z)} &= \frac{k+2}{4}\left[\frac{\alpha+1}{\alpha+1-\eta}H_1(z) + \left(1 - \frac{\alpha+1}{\alpha+1-\eta}\right)\right] \\
&\quad - \frac{k-2}{4}\left[\frac{\alpha+1}{\alpha+1-\eta}H_2(z) + \left(1 - \frac{\alpha+1}{\alpha+1-\eta}\right)\right] \\
&= \frac{k+2}{4}f_1(z) - \frac{k-2}{4}f_2(z)
\end{aligned}$$

where $f_i \in P\left(1 - \frac{2\eta}{\eta-1-\alpha}, -1\right),\ \ i = 1,2.$

So

$$\frac{zF'(z)}{F(z)} \in P_k\left(\frac{\eta}{\eta-1-\alpha}\right) = P_k\left[1 - \frac{2\eta}{\eta-1-\alpha}, -1, 1 - \frac{2\eta}{\eta-1-\alpha}, -1\right]$$

Hence

$$F \in R_k\left(\frac{\eta}{\eta-1-\alpha}\right) = R_k\left[1-\frac{2\eta}{\eta-1-\alpha}, -1, 1-\frac{2\eta}{\eta-1-\alpha}, -1\right].$$

Using similar techniques, we can also prove the following

Theorem 5. *Let $f, g \in V_k[A,B,C,D]$, and*

$$F(z) = \left\{\frac{\gamma+\alpha-1-\eta}{z^\gamma}\int_0^z f^\alpha(t)g(t)t^{\gamma-1-\eta}dt\right\}^{\frac{1}{\alpha+1-\eta}}$$

where $\alpha \geq -1,\ \gamma > 0,\ 0 \leq \dfrac{\eta}{\eta-\alpha-1} < 1,\ \eta \leq 0.$

Then $F \in V_k\left(\dfrac{\eta}{\eta-1-\alpha}\right).$

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