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ON THE RANGE OF SOME LINEAR PARTIAL DIFFERENTIAL OPERATORS

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ABSTRACT. Some reglarity theorems for linear partial differential operators with C^{∞}-coefficients are considered.

1. INTRODUCTION

If Ω is a convex domain in $\mathbb{R}^n$ and if A is a hypoelliptic operator with constant coefficients, then we have $A(D'(\Omega)) = D'(\Omega)$, where $D'(\Omega)$ denotes the space of distributions (Hörmander [8]).

Since $D'(\Omega)$ contains many important subclasses like the $L^p(\Omega)$ spaces, the Schwartz space $S'(\Omega)$, and the Sobolev spaces $H^s(\Omega)$, it is interesting to ask the following question: If A is a differential operator defined on an open set ω in $\mathbb{R}^n$, then given a distribution $T \in D'(\omega)$ is there a necessary and sufficient condition so that we could find a solution f in any prescribed subclass of $D'(\omega)$ to the equation $Af = T$ in the sense of distributions? In this note, we obtain such a condition by proving certain properties on the ranges of linear operators defined on locally convex spaces.

2. A THEOREM OF BRELOT

M. Brelot proved in [2] that if ω is any open set in $\mathbb{R}^n$ and if $\mu \in D'(\omega)$ is a Radon measure on ω then there exists a locally integrable function (more precisely a subharmonic function) u in w such that $\Delta u = \mu$. We can prove a slight generalization of this result. Let L be a second order elliptic differential

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operator defined on a domain Ω in $\mathbb{R}^n$ with the restrictions given below.

$$L = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial}{\partial x_i} + c(x),$$

where $a_{ij} = a_{ji}$, the quadratic form $\sum_{i,j} a_{ij}(x)\xi_i\xi_j$ is positive definite for any $x \in \Omega$, and $c \leq 0$. Finally, the coefficients a_{ij}, b_i and c are assumed to be locally Lipschitzian functions.

Theorem 2.1. *For any Radon measure μ on Ω, there exists a locally Lebesgue integrable function u in Ω such that $Lu = \mu$.*

Proof. The class of C^2 - functions h such that $Lh = 0$ satisfies the axioms 1,2,3 of M. Brelot [3]; these functions are called the L-harmonic functions (see Hervé [7] Chapter VII). If $c < 0$ at a point in Ω, there exists an L-potential > 0 in Ω, and if $c \equiv 0$ the constants are harmonic in Ω.

If λ is a Radon measure with compact support in Ω, then there exists an L-subharmonic function v in Ω such that $Lv = l$. We remark that by (2) in Proposition 35.1, Herv´e [7], any L-subharmonic function in Ω is locally Lebesgue integrable.

Now, if μ is any Radon measure in Ω, we write as in Theorem 4.2, Anandam [1], μ as the sum of a countable number of measures with compact support, $\mu = \sum_n \mu_n$, and then using a harmonic approximation lemma of De la Pradelle [6] in the context of the Brelot axiomatic potential theory, we deduce as in the construction of Mittag-Leffler's theorem, an L-subharmonic function u in Ω such that $Lu = \mu$ in Ω. As remarked earlier, u is locally Lebesgue integrable and hence defines a distribution.

The theorem above motivated the question posed in the Introduction.

3. TWO LEMMAS

Let E be a locally convex space and V a vector space. Let $T : E \to V$ be a linear transformation. Let E^* and V^* denote the algebraic duals of E and V. Let E' denote the topological dual of E, with the strong topology.

For a subspace $H \subset V^*$, define T^* so that for each $f \in H$ and $u \in E$, $\langle T^* f, u \rangle = \langle f, Tu \rangle$. Then $T^* : H \to E^*$ is a linear transformation.

Definition 3.1. Let $T : E \to V$ be a linear transformation and H be a subspace of V^*. T is said to be H-regular if $T^*(H) \subset E'$.

Lemma 3.1. *Let E be a semi-reflexive locally convex space and V be a vector space. Let $T : E \to V$ be a linear transformation. Suppose that for a subspace $H \subset V^*$, T is H-regular. Then given $v \in V$, there exists $x \in E$ such that $\langle Tx, f \rangle = \langle v, f \rangle$ for any $f \in H$ if and only if $|\langle v, f \rangle| \leq p(T^* f)$ for some continuous semi-norm p on E', and every $f \in H$.*

Proof. Suppose there exists $x \in E$ such that $\langle Tx, f \rangle = \langle v, f \rangle$ for every $f \in H$. Recall that if L is a continuous linear functional on a locally convex space X, then there exists a continuous semi-norm p on X such that $|L(y)| \leq p(y)$ for every $y \in X$. Consequently, since $x \in E$ defines a continuous linear functional on E', we have

$$|\langle v, f \rangle| = |\langle x, T^* f \rangle| \leq p(T^* f) \qquad \text{for a continuous semi-norm } p \text{ on } E'.$$

Conversely, suppose $|\langle v, f \rangle| \leq p(T^* f)$. Let $F = \{g \in E' : g = T^* f$ for some $f \in H\}$. On the subspace $F \subset E'$ define the linear functional S so that

$$Sg = \langle v, f \rangle \qquad \text{where } g = T^* f.$$

This linear functional is well-defined. For, if $g = T^* f_1$, then

$$|\langle v, f \rangle - \langle v, f_1 \rangle| = |\langle v, f - f_1 \rangle| \leq p(T^*(f - f_1)) = 0.$$

Also, S is continuous on F, since $|Sg| = |\langle v, f \rangle| \leq p(T^* f) = p(g)$. Hence, by the Hahn-Banach theorem we can assume that S is a continuous linear functional on $E'$and $|S(e)| \leq p(e)$ for every $e \in E'$. Then $S \in E''$ and since E is assumed to be semi-reflexive, there exists $x \in E$ such that $S(e) = \langle x, e \rangle$ for every $e \in E'$ (H.H. Schaefer [9]). In particular, if $e = T^* f$, $f \in H$ we have $\langle x, T^* f \rangle = S(T^* f) = \langle v, f \rangle$. Thus, for any $f \in H$, $\langle v, f \rangle = \langle x, T^* f \rangle = \langle Tx, f \rangle$.

A remark on the surjectivity. Let E be a semi-reflexive locally convex space and X be a topological vector space whose topological dual X' separates the points of X. (Examples: $X = l^p$, $0 < p < \infty$, or X is any locally convex space). Let $T : E \to X$ be a linear transformation and suppose that H is a $\sigma(X', X)$ dense subspace in X' for which $T^*(H) \subset E'$. Supose for some $v \in X$, there exists a continuous semi-norm p on E' such that $|\langle v, f\rangle| \leq p(T^*f)$ for every $f \in H$. Then there exists $x \in E$ such that $Tx = v$. For, by Lemma 3.1 there exists $x \in E$ such that $\langle v, f\rangle = \langle Tx, f\rangle$ for every $f \in H$. Since $Tx - v \in X$ vanishes on a dense set of X', $\langle Tx - v, y\rangle = 0$ for every $y \in X'$; and since X' separates points of X, $Tx - v = 0$ in E.

Uniqueness. When a solution as above to the equation $Tx = v$ exists, it is unique if and only if $T^*(H)$ is dense in E'. For, if $T^*(H)$ is not dense in E', there exists $h \in E$, $h \neq 0$, with $\langle h, T^*f\rangle = 0$ for $f \in H$. This means that $\langle Th, f\rangle = 0$ and hence $Th = 0$. Then $x_1 = x + h \neq x$ but $Tx_1 = Tx = v$. Conversely, suppose $Tx_1 = v = Tx_2$. Then $\langle T(x_1 - x_2), h\rangle = 0$ and $\langle x_1 - x_2, T^*h\rangle = 0$ for every $h \in H$. Then the fact that $T^*(H)$ is dense in E' implies that $x_1 - x_2 = 0$.

We can give a proof to the following lemma along the same lines as for the earlier case, if E is not semi-reflexive.

Lemma 3.2. *Let $\mathcal{E}$ be a locally convex space and V be a vector space. Let $T : \mathcal{E} \to V$ be a linear transformation. Let a locally convex subspace X of $\mathcal{E}$ have a strong dual $X' \subset \mathcal{E}$. Suppose that there exists a subspace $H \subset V^*$ and a linear transformation $T^* : H \to X$ such that $\langle u, T^*f\rangle = \langle Tu, f\rangle$ for every $f \in H$ and $u \in X'$. Then given $v \in V$, there exists $x \in X'$ such that $\langle Tx, f\rangle = \langle v, f\rangle$ for every $f \in H$ if and only if $|\langle v, f\rangle| \leq p(T^*f)$ for every $f \in H$ and a continuous semi-norm p on X.*

4. SOME CONSEQUENCES OF THE TWO LEMMAS

Let Ω be an open set in $\mathbb{R}^n$. Let $D(\Omega)$ be the family of C^∞-functions with compact support in Ω. Let $D'(\Omega)$ be the space of distributions in Ω and $S'(\Omega)$ the space of Schwartz distributions. Now, endowed with their respective strong topologies, both $D'(\Omega)$ and $S'(\Omega)$ are Montel spaces and consequently they are locally convex reflexive spaces. Also $L^p(\Omega)$ for $1 < p < \infty$ is a reflexive Banach space, among other important classes of locally convex reflexive spaces.

We can cite $D'(\Omega)$, $S'(\Omega)$, $L^p(\Omega)$ $(1 < p < \infty)$ and $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ (where $H^s(\mathbb{R}^n)$ is a Sobolev space) as some of the important classes which can take the place of E in the Lemma 3.1. For the vector space V we shall take $D'(\Omega)$, which is an all-inclusive class if we consider the linear transformation as a differential operator A with the following restrictions: Let $A = \sum_{|k|\le m} a_k(x)\partial^k$ be a linear partial differential operator of order m, with $a_k(x) \in C^\infty(\Omega)$. In this case $A : D'(\Omega) \to D'(\Omega)$. Let $A^*u = \sum_{|k|\le m}(-1)^{|k|}\partial^k(a_k(x)u)$ be the adjoint operator. The subspace $H \in V^*$ in Lemma 3.1 will be taken as $D(\Omega)$, so that A is $D(\Omega)$-regular.

The following results in [4] and [5] have rapport with Lemma 3.1.

1) Let B be a reflexive Banach space, X a locally convex space and $T : B \to X$ (not necessarily bounded) a linear transformation. In [4] a necessary and sufficient condition is obtained for a given $v \in X$ to have a solution to the equation $Tu = v$. This result is used to discuss the existence of an L^p-weak solution of $Du = v$ where D is a differential operator with smooth coefficients and $v \in L^p$.

2) Let $P(D)$ be a hypoelliptic operator with constant coefficients, having a fundamental solution that is locally integrable in $\mathbb{R}^n$. Let u be a distribution defined on an open set Ω in $\mathbb{R}^n$ such that $Pu = f$. It is proved in [5] that if $f \in L^1_{loc}(\Omega)$ then $u \in L^1_{loc}(\Omega)$ and if f is in $C^m(\Omega)$, so is u.

Now we exhibit three other examples as applications of Lemma 3.1.

Consequence 4.1. Let $T \in D'(\Omega)$ be a given distribution. Then there exists $u \in L^p(\Omega)$, $1 < p < \infty$, such that $Au = T$ in the sense of distributions if and only if $|T(\varphi)| \le c\|A^*\varphi\|_q$ for every

$$\varphi \in D(\Omega) \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

For, in Lemma 3.1, we take $E = L^p$. Then E' with its strong topology is L^q. Hence a semi-norm in E' is of the form $c\|\cdot\|_q$ for some constant $c > 0$. Notice that $\langle Au, \varphi\rangle = \langle T, \varphi\rangle$ for every $\varphi \in D(\Omega)$ is the same as saying that $Au = T$ in the sense of distributions.

Consequence 4.2. Let $T \in D'(\mathbb{R}^n)$ be a given distribution. Suppose the coefficients of the differential operator A are either constants or from the Schwartz space (i.e. the rapidly decreasing C^∞-functions). Suppose $|T(\varphi)| \leq c\|A^*\varphi\|_{H^{-s}}$ for all $\varphi \in D'(\mathbf{R}^n)$. Then

i) There exists $u \in H^s$ such that $Au = T$ in the sense of distributions.

ii) Moreover, if $s > \frac{n}{2} + m$, then T is a continuous function and $Au = T$ in the classical sense.

For, in Lemma 3.1, we take $E = H^s$. This is a Hilbert space and its topological dual is $E' = H^{-s}$. Hence i) follows from Lemma 3.1.

Now, if the coefficients of A are restricted as in the statement, then $T = Au \in H^{s-m}$. Since, for any non-negative integer k, $H^s \subset C^k$ if $s > \frac{n}{2} + k$, we conclude that T is equal to a continuous function a.e. and u is a C^m-function. Consequently the weak solution $T = Au$ is indeed a strong solution.

Consequence 4.3. Suppose A is as before a partial differential operator with C^∞- coefficients defined on an open set Ω in $\mathbb{R}^n$. Suppose T is a distribution in $D'(\Omega)$ and that we look for a bounded solution $f \in L^\infty(\Omega)$ satisfying the equation $Af = T$. In this case, we are not in a position to use Lemma 3.1 directly, since $L^\infty(\Omega)$ is not semi-reflexive. (A normed space is semi-reflexive if and only if it is reflexive). However, since A satisfies the relation $\langle AT, \varphi\rangle = \langle T, A^*\varphi\rangle$ for every $T \in D'(\Omega)$ and $\varphi \in D(\Omega)$ and since $L^1(\Omega) \subset D'(\Omega)$ and $(L^1(\Omega))' = L^\infty(\Omega)$, we have a special situation to take advantage of. In this context, we can use Lemma 3.2.

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