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COMPLETELY SUPERHARMONIC POLYHARMONIC FUNCTIONS ON A RIEMANNIAN MANIFOLD

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ABSTRACT. A polyharmonic function $u(x)$ on a domain in $\mathbb{R}^n$, $n \geq 2$, is said to be completely superharmonic of order m if $(-\Delta)^i u \geq 0$ for $0 \leq i \leq m-1$ and $(-\Delta)^m u = 0$. We generalize this notion on a Riemannian manifold R and show that if such a function exists on R, then there exists an $m-$harmonic Green function $g(x,y)$ (i.e. for any $y \in R$, $(-\Delta)^m g_y(x) = \delta_y(x)$ the Dirac measure) exists on R.

1. INTRODUCTION

M. Nicolesco [4, pp. 16-17] calls a polyharmonic function $u(x)$ defined on a domain ω in $\mathbb{R}^n$, $n \geq 2$, a completely superharmonic function of order m if $(-\Delta)^i u(x) \geq 0$ for $0 \leq i \leq m$ and $\Delta^m u(x) = 0$ and gives a representation for such a function in a star domain about its centre. In this note we generalize such functions on the domains ω in a Riemannian manifold R and obtain some important properties of such functions. We show that the existence of such a function on R implies that the symmetric $m-$harmonic Green function $g(x,y) = g_y(x)$ (i.e. $(-\Delta)^m g_y(x) = \delta_y(x)$) exists on R.

2. PRELIMINARIES

Let R be an oriented Riemannian manifold of dimension $n \geq 2$ with local coordinates $x = (x^1, \cdots, x^n)$ and a $C^\infty-$metric tensor g_{ij} such that $g_{ij}x^ix^j$ is positive definite. If α is the determinant of g_{ij}, denote the volume element

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by $dx = \alpha^{1/2} dx^1 \cdots dx^n$; let $-\Delta = d\delta + \delta d$ be the Laplace-Beltrami operator acting on R in the sense of distributions. If $R = \mathbb{R}^n$ $(n \geq 2)$, the above Δ reduces to the form $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$.

To every open set ω in R, let $H(\omega)$ denote the class of harmonic functions on ω (i.e. $u \in H(\omega)$ if and only if u is a C^2-function on ω such that $\Delta u = 0$). It is well-known that these harmonic functions have the sheaf property, solve locally the Dirichlet problem and possess the Harnack property (namely, if u_n is an increasing sequence of harmonic functions on ω in R, and if $u = \lim_{n \to \infty} u_n$, then u is a harmonic function on ω or $u \equiv \infty$ on ω). Consequently, the sheaf of harmonic functions satisfies the axioms 1, 2, 3 of M. Brelot [3, pp. 13-14] in the axiomatic potential theory and hence we can use all the notions and the results of this axiomatic theory in the context of the Riemannian manifold R. Some of these are the following:

(1) A positive superharmonic function s on a domain ω in R (i.e. $s \geq 0$ and $\Delta s \leq 0$) is said to be a Green potential if and only if any harmonic function h majorized by s is nonpositive. We then state the Riesz decomposition theorem: if $s \geq 0$ is superharmonic on ω, then s is the unique sum of a potential $p \geq 0$ and a harmonic function $h \geq 0$.

(2) If there is a potential > 0 on the domain ω in R, then we can define the Green function $G(x, y) = G_y(x)$ with pole $y \in \omega$ such that $G(x, y)$ is symmetric on ω, and $G_y(x)$ as a function of x is a potential on ω such that $(-\Delta) G_y(x) = \delta_y(x)$, the Dirac measure. Then for any potential $p(x)$ on ω, $(-\Delta) p = \mu$ is a Radon measure on ω and $p(x) = \int_\omega G_y(x) d\mu(y)$. Conversely, if $\lambda \geq 0$ is any Radon measure on ω, and if $q(x) = \int_\omega G_y(x) d\lambda(y)$, then $q(x) \equiv \infty$ or $q(x)$ is a potential on ω.

(3) More generally we have the following result by V. Anandam [2, Section 2]: Let $\mu \geq 0$ be a Radon measure on an open set $\omega \subset R$. Then there exists a superharmonic function s (not necessarily a potential) on ω such that $(-\Delta) s = \mu$.

(4) Consequently, if $f(x) \geq 0$ is any locally $dx-$integrable function on ω and if $d\mu(x) = f(x) dx$, then μ defines a non-negative Radon measure and

there exists a superharmonic function $s(x)$ on ω such that $(-\Delta)s = \mu$. We shall rather denote this relation by the equation $(-\Delta)s = f$.

3. COMPLETELY SUPERHARMONIC POLYHARMONIC FUNCTIONS ON R

Let us fix a domain Ω in a Riemannian manifold R which may be hyperbolic (i.e. there exists the Green function on R) or parabolic (i.e. there is no Green function on R).

Let h be a harmonic function on Ω. As indicated in the preliminaries, we can choose two superharmonic functions s_1 and s_2 on Ω such that $(-\Delta)s_1 = h^+$ and $(-\Delta)s_2 = h^-$. Let $s = s_1 - s_2$. Then s is a δ-superharmonic function (i.e. the difference of two superharmonic functions) on Ω such that $(-\Delta)s = h$.

Since every superharmonic function on ω is locally $dx-$integrable, s is locally $dx-$integrable. Hence, we can find a $\delta-$superharmonic function u on Ω such that $(-\Delta)u = s$. We can continue this process.

Definition 3.1. We say that $h = (h_m, h_{m-1}, \cdots, h_2, h_1)$ is an $m-$harmonic function on Ω if and only if h_1 is harmonic on Ω and $(-\Delta)h_{i+1} = h_i$, $1 \leq i \leq m-1$.

Definition 3.2. An $m-$harmonic function $(h_i)_{m \geq i \geq 1}$ on Ω is said to be completely superharmonic if $h_i \geq 0$.

Remarks

1) If $R = \mathbb{R}^n$, the above definition of a completely superharmonic function of order m coincides with the one given by M. Nicolesco [4, p. 160]. For, if $u = h_m$, then $(-\Delta)^i u \geq 0$, $1 \leq i \leq m-1$ and $(-\Delta)^m u = 0$.

2) If $h = (h_i)$ is an $m-$harmonic function that is completely superharmonic, then each h_i is positive superharmonic for $2 \leq i \leq m$ and h_1 is positive harmonic. For, by definition $(-\Delta)h_2 = h_1 \geq 0$. Hence h_2 is superharmonic; similarly $(-\Delta)h_3 = h_2 \geq 0$ and hence h_3 is superharmonic and so on.

3) If R is parabolic, the only $m-$harmonic function on R that is completely superharmonic is of the form $(c,0,0,0,\cdots,0)$ where c is a positive constant. For, let $h=(h_i)_{m\geq i\geq 1}$ be an $m-$harmonic function that is completely superharmonic. Then each h_i is positive superharmonic on R. But R being parabolic, the only positive superharmonic function on R is a constant so that h is of the form $h=(c_m,c_{m-1},\cdots,c_2,c_1)$ where all the c_i's are positive constants. But $(-\Delta)\,c_{i+1}=c_i$ for $1\leq i\leq m-1$, so that $c_i=0$ except possibly for c_m.

If R is hyperbolic, we have the following.

Proposition 3.1. *Let R be a hyperbolic Riemannian manifold with the Green function $G(x,y)$ defined on it. Suppose*

$$\int G(x,y_m)G(y_m,y_{m-1})\cdots G(y_3,y_2)\,dy_2dy_3\cdots dy_m$$

is finite for some $x\in R$. Then there exists an $m-$harmonic function $h>0$ on R that is completely superharmonic.

Proof. By hypothesis, $\int G(y_3,y_2)\,dy_2\not\equiv\infty$. Hence if we write $p_2(x)=\int G(x,y_2)\,dy_2$, then $p_2(x)$ is a potential on R such that $(-\Delta)\,p_2=1$. Again $\int G(y_4,y_3)\,p_2(y_3)\,dy_3\not\equiv\infty$ by hypothesis. Hence $p_3(x)=\int G(x,y_3)\,p_2(y_3)dy_3$ is a potential on R such that $(-\Delta)\,p_3=p_2$. Proceeding thus, we see by induction, if $p_{m-1}(x)=\int G(x,y_{m-1})\,p_{m-2}(y_{m-1})\,dy_{m-1}$, then $p_{m-1}(x)$ is a potential on R such that $(-\Delta)\,p_{m-1}=p_{m-2}$.
Finally, let $p_m(x)=\int G(x,y_m)\,p_{m-1}(y_m)\,dy_m$. Since $p_m(x)\not\equiv\infty$, $p_m(x)$ is a potential and $(-\Delta)\,p_m=p_{m-1}$. Hence, if $h=(p_m,p_{m-1},\cdots,p_2,1)$, each p_i is a potential and $(-\Delta)\,p_{i+1}=p_i$ for $2\leq i\leq m-1$, so that h is an $m-$harmonic function on R that is completely superharmonic. □

Theorem 3.1. *Let R be a hyperbolic Riemannian manifold. Suppose there exists an $m-$harmonic function $h>0$ on R that is completely superharmonic. Then there exists an $m-$harmonic Green function (i.e. a positive symmetric function $g(x,y)=g_y(x)$ which considered as a function of x is a potential satisfying the condition $(-\Delta)^m\,g_y(x)=\delta_y(x)$, the Dirac measure) on R.*

Proof. Proof. Let $h = (h_m, \cdots, h_1) > 0$ be an $m-$harmonic function that is completely superharmonic on R. That is, $h_i > 0$ is superharmonic for all $i, h_1$is harmonic and $(-\Delta) h_{i+1} = h_i$ for $1 \leq i \leq m-1$.

Let $y \in R$ be fixed and $p_1(x) = G(x, y)$ be the Green function with pole at y. Let D be a parametric disc centred at y. Given any continuous function $f(x)$ on ∂D, let Bf denote the Dirichlet solution on $R \backslash D$ with boundary values $f(x)$ on ∂D and 0 at the (Alexandroff) point at infinity of R. Then note that for some $\lambda > 0$, $p_1(x) \leq \lambda h_1(x)$ on $R \backslash D$. For, choose $\lambda > 0$ such that $p_1 \leq \lambda h_1$ on ∂D. Since, $p_1 = Bp_1$ on $R \backslash D$, we have $p_1 = Bp_1 \leq \lambda B h_1$ on $R \backslash D$.

Now, $p_1(x) > 0$ being locally $dx-$integrable on R, there exists a superharmonic function $s(x)$ on R such that $(-\Delta) s = p_1$. Let $u(x)$ be a function on R such that $(-\Delta) u = \lambda h_1 - p_1$. Note that $u(x)$ is not superharmonic on R but it is certainly superharmonic on $R \backslash \bar{D}$ since $\lambda h_1 - p_1 \geq 0$ on $R \backslash D$. Then $(-\Delta)(s+u) = \lambda h_1 = \lambda(-\Delta) h_2$ on R so that $s + u = \lambda h_2+$(a harmonic function H) on R. Since $h_2 > 0$ and u is superharmonic on $R \backslash \bar{D}$, $s(x)$ majorizes a subharmonic function on $R \backslash \bar{D}$. This means that $s(x)$ is an admissible superharmonic function on R (see V. Anandam [1]). Consequently, $s(x) = p_2(x) + v(x)$ on R where $p_2(x)$ is a potential and $v(x)$ is a harmonic function which is not necessarily positive on R. This implies that $(-\Delta) p_2 = (-\Delta) s = p_1$. Moreover $p_2 \leq \lambda h_2$ on $R \backslash \bar{D}$.

To see the last inequality, we proceed as follows: Let $\omega = R \backslash \bar{D}$. On ω, the functions s and u are superharmonic and $s + u = \lambda h_2 + H$. Since $h_2 > 0$, we have $u > H - s$ on ω. That is u has a subharmonic minorant on ω so that by the Riesz decomposition $u = q + u_1$ on ω where q is a potential on ω and u_1 is harmonic on ω. Moreover, since $h_2 > 0$ is superharmonic, $h_2 = Q + u_2$ on ω where Q is a potential on ω and $u_2 \geq 0$ is harmonic on ω. Consequently, on ω, $(p_2 + v) + (q + u_1) = \lambda(Q + u_2) + H$. This leads to the equality of the potentials $p_2 + q = \lambda Q$ by the uniqueness of Riesz decomposition. In particular, we deduce that $p_2 \leq \lambda Q$. Since $Q \leq h_2$ on ω, we see that $p_2 \leq \lambda h_2$ on $\omega = R \backslash \bar{D}$.

Note $G(x,y)$ is symmetric. Now $p_2(x)$ which is a potential on R, dependent on y, can be written as (since $(-\Delta)p_2 = p_1$)

$$\begin{aligned} p_2(x,y) &= \int G(x,z)\, p_1(z)\, dz \\ &= \int G(x,z) G(z,y)\, dz \\ &= \int G(y,z)\, G(z,x)\, dz \\ &= p_2(y,x) \end{aligned}$$

Hence $p_2(x,y)$ is a symmetric function which, for fixed y is a potential on R as a function of x. □

We now continue this process to construct symmetric potentials $p_3(x,y)$ and then $p_4(x,y)$ and so on to finally arrive at $p = (p_m, \cdots, p_2, p_1)$ where for each i, $p_i(x,y)$ is symmetric and for fixed y, is a potential on R as a function of x; further $(-\Delta)\, p_{i+1} = p_i$ for $1 \leq i \leq m-1$.

Let now $g(x,y) = p_m(x,y)$. Then $g(x,y)$ is a symmetric function in x and y; moreover, for fixed y, $g(x,y)$ is a potential > 0 on R such that $(-\Delta)^{m-1} g_y(x) = p_1(x,y) = G(x,y)$. Consequently $(-\Delta)^m g_y(x) = \delta_y(x)$ on R.

Remarks

1) The existence of an $m-$harmonic function $h > 0$ on R that is completely superharmonic is only a sufficient condition for the existence of an $m-$harmonic Green potential $g(x,y)$ on R, as shown in the above theorem. This condition is not necessary. For example, take $R = \mathbb{R}^{2m+1}$. Let $g(x,y) = |x-y|^{-1}$ for x, y in $\mathbb{R}^{2m+1}$. Then $(-\Delta)^m g_y(x) = c\delta_y(x)$ for some constant c. Hence the $m-$harmonic Green potential can be defined on $\mathbb{R}^{2m+1}$. But there is no $m-$harmonic function $h > 0$ on $\mathbb{R}^{2m+1}$ that is completely superharmonic. For, if possible, let $h = (h_i)_{m \geq i \geq 1}$ be one such function. Since $h_1 > 0$ is harmonic on $\mathbb{R}^{2m+1}$, it is a constant c. Consequently the superharmonic function $h_2 > 0$ satisfies the condition $(-\Delta)\, h_2 = c$.

Since $h_2 > 0$ is superharmonic on $\mathbb{R}^{2m+1}$, by Riesz representation $h_2(x) = \int_{\mathbb{R}^{2m+1}} G(x,y)\, [(-\Delta)\, h_2(y)]\, dy +$(a harmonic function $H(x)$) on $\mathbb{R}^{2m+1}$. This implies that $\int_{\mathbb{R}^{2m+1}} G(x,y)\left[(-\Delta) h_2(y)\right] dy \not\equiv \infty$ as function of x. But $G(x,y) = \frac{1}{|x-y|^{(2m+1)-2}}$ and $(-\Delta) h_2(y) = c$ so that we should have $\int \frac{1}{|x-y|^{2m-1}} dy \not\equiv \infty$. This means that (using the spherical polar coordinates) $\int_1^\infty \frac{1}{r^{2m-1}} r^{2m} dr < \infty$, which is a contradiction.

2) Using the method of proof given in Proposition 3.1, we can show that a necessary and sufficient condition for the existence of the $m-$harmonic Green function on a hyperbolic Riemannian manifold R is the following: Let R be a hyperbolic Riemannian manifold with the Green function $G(x,y)$. Then there exists an $m-$harmonic Green funtion $g(x,y)$ on R if and only if

$$Q(x) = \int G(x, y_m)\, G(y_m, y_{m-1}) \cdots G(y_2, y_1)\, dy_1.dy_2 \cdots dy_m$$

is finite for some $x \in R$.

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