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EXISTENCE RESULTS FOR NONCONVEX SWEEPING PROCESSES WITH PERTURBATIONS AND WITH DELAY: LIPSCHITZ CASE

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ABSTRACT. In this paper we prove the existence of solutions for nonconvex sweeping processes with perturbations and with delay of the form : $\dot{u}(t) \in -N(C(t); u(t)) + F(t, u_t)$ when C is a nonconvex Lipschitz set-valued mapping, and F is a set-valued mapping with convex compact values taking their values in a general Hilbert space H.

1. INTRODUCTION

This paper is concerned with some existence results of solutions for the following type of first order functional differential inclusions:

$$\text{(NSPPD)} \qquad \begin{cases} \dot{u}(t) \in -N^P(C(t); u(t)) + F(t, u_t) \text{ a.e. on } I; \\ u(t) \in C(t), \qquad \forall t \in I; \\ u(s) = \mathcal{T}(0)u(s) = \varphi(s), \qquad \forall\, s \in [-\tau, 0]. \end{cases}$$

We will call it *a Nonconvex Sweeping Process with Perturbation and with Delay*. Such problems have been studied by many authors (see for example [1], [7], [8], [9], [10], [12], [13], [14] [15], and the references therein). In [9], some topological properties of solution sets for the NSPPD problem in the convex case are established, and in [10], the compactness of solution sets in $\mathbb{R}$ is obtained in the nonconvex case. Using some new properties and characterizations of uniformly r-prox regular sets, obtained in [3, 4, 5, 6, 10, 16], we establish, in

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the present paper, the existence of Lipschitz solutions to the NSPPD problem in the nonconvex case and in the infinite dimensional setting.

2. PRELIMINARIES AND FUNDAMENTAL RESULTS

Throughout the paper, H will denote a real separable Hilbert space. Let S be a nonempty closed subset of H, we denote by $d_S(\cdot)$ the usual distance function to the set S. We recall (see for example [6]) that *the proximal normal cone* to S at $x \in S$ is defined by

$$N^P(S;x) = \{\xi \in H : \exists\, \alpha > 0 : x \in Proj(x + \alpha\xi, S)\}$$

where $Proj(u,S) = \{y \in S : d_S(u) = \|u - y\|\}$. Recall (see for example [6]) also that *the proximal subdifferential* $\partial^P f(x)$ of a Lipschitz continuous function $f : H \to I\!R$ at a point $x \in H$ is the set of all $\xi \in H$ for which there exist $\delta, \sigma > 0$ such that for all $x' \in x + \delta I\!B$

$$\langle \xi, x' - x \rangle \leq f(x') - f(x) + \sigma \|x' - x\|^2.$$

Here $I\!B$ denotes the closed unit ball centered at the origin of H. Recall now that for a given $r \in]0, +\infty]$, a subset S is uniformly r-prox-regular if and only if every nonzero proximal to S can be realized by an r-ball, this means that for all $y \in S$ and all $\xi \in N^P(S;y), \xi \neq 0$ one has

$$\langle \frac{\xi}{\|\xi\|}, x - y \rangle \leq \frac{1}{2r} \|x - y\|^2,$$

for all $x \in S$(see [16]). We make the convention $\frac{1}{r} = 0$ for $r = +\infty$ (in this case, the uniform r-prox-regularity is equivalent to the convexity of S).

Let $C : I \rightrightarrows H$ be a set valued mapping. We will say that C is Lipschitz continuous with ratio λ if for any $y \in H$ and for any $t, s \in I$ one has

$$\left| d_{C(t)}(y) - d_{C(s)}(y) \right| \leq \lambda \left| t - s \right|.$$

Let $\Phi : X \rightrightarrows Y$ be a set-valued mapping defined between two topological vector spaces X and Y, we say that Φ is *upper semi-continuous* (in short u.s.c.) at $x \in dom(\Phi) = \{x' \in X : \Phi(x') \neq 0\}$ if for any open set O containing $\Phi(x)$ there exists a neighborhood V of x such that $\Phi(V) \subset O$.

Let $T > 0$. We will deal with a finite delay $\tau > 0$. If $u : [-\tau, T] \to H$, then for every $t \in I = [0, T]$, we define the function $u_t(s) = \mathcal{T}(t)u(s) = u(t+s)$, $s \in [-\tau, 0]$ and the Banach space $\mathcal{C}_T = \mathcal{C}([-\tau, T], H)$ (resp. $\mathcal{C}_0$) of all continuous mappings from $[-\tau, T]$ (resp.$[-\tau, 0]$) to H with the norm given by $\|\varphi\|_T = \max\{\|\varphi(s)\| : s \in [-\tau, T]\}$. Clearly, if $u \in \mathcal{C}_T$, then $u_t \in \mathcal{C}_0$, and the mapping $u \to u_t$ is continuous in the sense of uniform convergence.
The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel of the paper (see [5, 16]).

Proposition 2.1. *Let S be a nonempty closed subset of H and $x \in S$. The following assertions hold:*

(1) $\partial^P d_S(x) = N^P(S, x) \bigcap I\!B$;
(2) *If S is uniformly r-prox-regular, then $\partial^P d_S(x)$ is a closed convex set in H and for any $x \in H$ with $d_S(x) < r$ one has $Proj(x, S) \neq 0$.*

The following closure property of the proximal subdifferential is due to Bounkhel and Thibault [6] (see also [2, 3, 4] for different versions of this property and their applications). It is one of the powerful results used to prove our existence results in this paper.

Proposition 2.2. *Let $r \in]0, \infty]$, Ω be an open subset in $I\!R$, and $C : \Omega \rightrightarrows H$ be a Lipschitz continuous set-valued mapping with uniformly r-prox-regular values. Then the set-valued mapping: $(t, x) \to \partial^P d_{C(t)}(x)$ from $\Omega \times H$ (endowed with the strong topology) to H (endowed with the weak topology) is upper semicontinuous, which is equivalent to the upper semicontinuity of the function $(t, x) \to \sigma(\partial^P d_{C(t)}(x), p)$ for any $p \in H$. Here $\sigma(S, p)$ denotes the support function associated with S, i.e., $\sigma(S, p) = \sup_{s \in S}\langle s, p\rangle$.*

3. MAIN RESULTS

The following proposition provides an approximate solution for the NSPPD problem under consideration.

Theorem 3.1. *Let H be a separable Hilbert space, $T > 0$, and $r \in]0, +\infty]$. Assume that $C(t)$ is uniformly r-prox-regular for every $t \in I = [0, T]$ and Lipschitz continuous with ratio $\lambda > 0$. Let $F : I \times \mathcal{C}_0 \rightrightarrows H$ be a set-valued mapping with convex compact values in H such that $F(t, \cdot)$ is u.s.c. on $\mathcal{C}_0$ for any fixed $t \in I$ and $F(\cdot, \varphi)$ admits a measurable selection on I for any fixed $\varphi \in \mathcal{C}_0$. Assume that $F(t, \varphi) \subset l\mathbb{B}$ for all $(t, \varphi) \in I \times \mathcal{C}_0$, for some $l > 0$. Then, for any $\varphi \in \mathcal{C}_0$ with $\varphi(0) \in C(0)$ and for any n large enough there exists a continuous mapping $u_n : [-\tau, T] \to H$ which enjoys the following properties:*

1) $\dot{u}_n(t) \in -N^P(C(\theta_n(t)); u_n(\theta_n(t))) + F(\rho_n(t), \mathcal{T}(\rho_n(t))u_n)$, a.e $t \in I$, *where* $\theta_n, \rho_n : I \to I$ *with* $\lim_{n\to+\infty} \theta_n(t) = t$ *and* $\lim_{n\to+\infty} \rho_n(t) = t$ *for all* $t \in I$.
2) $\|\dot{u}_n(t)\| \leq (2l + \lambda)$, *for* a.e $t \in I$.

Proof. We prove the conclusion of our theorem when F is globally *u.s.c.* on $I \times \mathcal{C}_0$ and then as in [9] (see also[2]), we can proceed by approximation to prove it when $F(t, \cdot)$ is u.s.c. on $\mathcal{C}_0$ for any fixed $t \in I$ and $F(\cdot, \varphi)$ admits a measurable selection on I for any fixed $\varphi \in \mathcal{C}_0$. We construct via discretization the desired sequence of continuous mappings $\{u_n\}_n$ in $\mathcal{C}_T$. For every $n \in \mathbb{N}$, we consider the following partition of I:

$$t_{n,i} = \frac{iT}{2^n} \quad (0 \leq i \leq 2^n) \tag{3.1}$$

and

$$I_{n,i} =]t_{n,i}, t_{n,i+1}] \text{ if } 0 \leq i \leq 2^n - 1.$$

Put

$$\mu_n = \frac{T}{2^n}.$$

Fix $n_0 \geq 1$ satisfying for every $n \geq n_0$

$$(2l + 3\lambda)\mu_n < r. \tag{3.2}$$

First, we put $u_n(s) = \varphi(s)$, for all $s \in [-\tau, 0]$ and for all $n \geq n_0$.

For every $n \geq n_0$, we define by induction:

$$u_n(t_{n,i+1}) = u_{n,i+1} = Proj(u_{n,i} + \mu_n f_0(t_{n,i}, \mathcal{T}(t_{n,i})u_n), C(t_{n,i+1})), \tag{3.3}$$

where $f_0(t_{n,i}, \mathcal{T}(t_{n,i})u_n)$ is the minimal norm element of $F(t_{n,i}, \mathcal{T}(t_{n,i})u_n)$, i.e.,

$$\|f_0(t_{n,i}, \mathcal{T}(t_{n,i})u_n)\| = \min\{\|y\| : \quad y \in F(t_{n,i}, \mathcal{T}(t_{n,i})u_n)\} \leq l$$

and

$$\mathcal{T}(t_{n,i})u_n = (u_n)_{t_{n,i}}.$$

The above construction is possible in spite of the nonconvexity of the images of C. Indeed, we can show that for every $n \geq n_0$ we have

$$d_{C(t_{n,i+1})}(u_{n,i} + \mu_n f_0(t_{n,i}, \mathcal{T}(t_{n,i})u_n)) \leq l\mu_n + \lambda|t_{n,i+1} - t_{n,i}| \leq (l+\lambda)\mu_n \leq \frac{r}{2}$$

and hence as C has uniformly r-prox-regular values, by Proposition 2.1 one can choose a point $u_{n,i+1} = Proj(u_{n,i} - \mu_n f_0(t_{n,i}, \mathcal{T}(t_{n,i})u_n), C(t_{n,i+1}))$, for all $n \geq n_0$. Note that from (3.3) and (3.1) one deduces for every $0 \leq i < 2^n$

$$\|u_{n,i+1} - (u_{n,i} + \mu_n f_0(t_{n,i}, \mathcal{T}(t_{n,i})u_n))\| \leq (l+\lambda)\mu_n. \tag{3.4}$$

By construction we have $u_{n,i} \in C(t_{n,i})$ for all $0 \leq i < 2^n$.

For every $n \geq n_0$, these $(u_{n,i})_{0\leq i\leq 2^n}$ and $(f_0(t_{n,i}, \mathcal{T}(t_{n,i})u_n)_{0\leq i\leq 2^n}$ are used to construct two mappings u_n and f_n from I to H by defining their restrictions to each interval $I_{n,i}$ by linear interpolation as follows:

$$u_n(t) = u_{n,i} + \frac{t - t_{n,i}}{\mu_n}(u_{n,i+1} - u_{n,i}). \tag{3.5}$$

Hence for every t and t' in $I_{n,i}$ $(0 \leq i \leq 2^n)$ one has

$$u_n(t') - u_n(t) = \frac{t'-t}{\mu_n}(u_{n,i+1} - u_{n,i}).$$

Thus, in view of (3.4), if $t, t' \in I_{n,i}$ $(0 \leq i < 2^n)$ with $t \leq t'$, one obtains

$$\|u_n(t') - u_n(t)\| \leq (2l+\lambda)(t'-t), \tag{3.6}$$

and, by addition, this also holds for all $t, t' \in I$ with $t \leq t'$. This inequality entails that u_n is Lipschitz continuous with ratio $2l + \lambda$.

Coming back to the definition of u_n in (3.5), one observes that for $0 \leq i < 2^n$

$$\dot{u}_n(t) = \frac{u_{n,i+1} - u_{n,i}}{\mu_n} \text{ for a.e. } \quad t \in I_{n,i}.$$

Then one obtains, in view of (3.4), for almost all $t \in I$

$$\|\dot{u}_n(t) - f_n(t)\| \leq (3l+\lambda), \tag{3.7}$$

which proves the part (2) of the theorem.

Now, let θ_n, ρ_n be defined from I to I by $\theta_n(0) = 0$, $\rho_n(0) = 0$, and

$$\theta_n(t) = t_{n,i+1}, \rho_n(t) = t_{n,i} \text{ if } t \in I_{n,i} \ (0 \leq i < 2^n).$$

Then, by (3.3), the construction of u_n and f_n, and the properties of proximal normal cones to subsets, we have for almost all $t \in I$

$$f_n(t) \in F(\rho_n(t), \mathcal{T}(\rho_n(t))u_n)$$

and

$$\dot{u}_n(t) - f_n(t) \in -N^P(C(\theta_n(t)); u_n(\theta_n(t))). \tag{3.8}$$

These last inclusions ensure part (1) of the theorem, and hence the proof is complete. □

Now, we are able to state our first existence result for the NSPPD

Theorem 3.2. *Assume that the assumptions of Theorem* 3.1 *are satisfied. Assume that $C(t)$ is strongly compact for every $t \in I$. Then for every $\varphi \in \mathcal{C}_0$ with $\varphi(0) \in C(0)$, there exists a continuous mapping $u : [-\tau, T] \to H$ such that u is Lipschitz continuous on I and satisfies :*

$$\text{(NSPPD)} \qquad \begin{array}{ll} \dot{u}(t) \in -N^P(C(t); u(t)) + F(t, u_t) & a.e. \quad on \quad I; \\ u(t) \in C(t), & \forall\, t \in I; \\ u(s) = \mathcal{T}(0)u(s) = \varphi(s), & \forall\, s \in [-\tau, 0]; \end{array}$$

and

$$\|\dot{u}(t)\| \leq (2l + \lambda) \qquad a.\,e. \ on \quad I.$$

Proof. Let $\varphi \in \mathcal{C}_0$ with $\varphi(0) \in C(0)$. By Theorem 3.1 there exists a sequence of continuous mappings $\{u_n\}$ enjoying the properties 1) and 2) in Theorem 3.1. Let $n_0 \in I\!N$ satisfying (3.2). Then for any $n \geq n_0$ and any $t \in I$ we have

$$\begin{aligned} d(u_n(t), C(t)) &\leq \|u_n(t) - u_n(t_{n,i})\| + \lambda|t - t_{n,i}| \\ &\leq (2l + \lambda)|t - t_{n,i}| + \lambda|t - t_{n,i}| \\ &\leq 2(l + \lambda)|t - t_{n,i}| \leq 2(l + \lambda)|\mu_n \end{aligned} \tag{3.9}$$

Since $C(t)$ is strongly compact and $\mu_n \to 0$, (3.9) implies that the set $\{u_n(t) : n \geq n_0\}$ is relatively strongly compact in H for all $t \in I$. Thus, by Arzela-Ascoli's theorem we can extract a subsequence of the sequence $\{u_n\}_n$, still denoted by $\{u_n\}_n$, which converges uniformly on $[-\tau, T]$ to a Lipschitz continuous function u which clearly satisfies $u \equiv \varphi$ on $[-\tau, 0]$. Now by letting $n \to +\infty$ we get for all $t \in I$

$$u(t) \in C(t).$$

It follows from our construction in the proof of Theorem 3.1 and by the uniform convergence of $\{u_n\}_n$ to u over I we get

$$\|u_n(\theta_n(t)) - u(t)\| \leq \|u_n(\theta_n(t)) - u(\theta_n(t))\| + \|u(\theta_n(t)) - u(t)\| \to 0.$$

Now, using the same technique in [9], we obtain

$$\lim_{n\to\infty} \|\mathcal{T}(\rho_n(t))u_n - \mathcal{T}(t)u_n\| = 0 \quad \text{in} \quad \mathcal{C}_0.$$

Therefore, as the uniform convergence of u_n to u on $[-\tau, T]$ implies that $\mathcal{T}(t)u_n$ converges uniformly to $\mathcal{T}(t)u$ on $[-\tau, 0]$, we conclude that

$$\mathcal{T}(\rho_n(t))u_n \longrightarrow \mathcal{T}(t)u = u_t \quad \text{in} \quad \mathcal{C}_0. \tag{3.10}$$

On the other hand, from $f_n(t) \in F(\rho_n(t), \mathcal{T}(\rho_n(t))u_n)$ and (3.6), $\{f_n\}$ and $\{\dot{u}_n\}$ are bounded sequences in $L^\infty(I, H, dt)$, then by extracting subsequences (because $L^\infty(I, H, dt)$ is the dual space of the separable space $L^1(I, H, dt)$) we may suppose without loss of generality that f_n and $\dot{u}_n$ weakly-$\star$ converge in $L^\infty(I, H, dt)$ to some mappings f and ω respectively. Then, for all $t \in I$ one has

$$\begin{aligned}
\varphi(0) + \int_0^t \dot{u}(s)ds &= u(t) \\
&= \lim_{n\to\infty} u_n(t) \\
&= \varphi(0) + \lim_{n\to\infty} \int_0^t \dot{u}_n(s)ds \\
&= \varphi(0) + \int_0^t \omega(s)ds,
\end{aligned}$$

which proves that $\dot{u}(t) = \omega(t)$ for almost all $t \in I$.

Now, using Mazur's lemma, we obtain

$$\dot{u}(t) - f(t) \in \bigcap_n \overline{co}\{\dot{u}_k(t) - f_k(t) : k \geq n\} \qquad a.e. \quad \text{in } I.$$

For a fixed t in I and any ξ in H, the last relation yields

$$\langle \dot{u}(t) - f(t), \xi \rangle \leq \inf_n \sup_{k\geq n} \langle \dot{u}_k(t) - f_k(t), \xi \rangle.$$

By (3.7), (3.8), and Proposition 2.1 we have for almost all $t \in I$

$$\dot{u}_n(t) - f_n(t) \in -N^P(C(\theta_n(t)); u_n(\theta_n(t))) \cap \delta I\!B = -\delta \partial^P d_{C(\theta_n(t))}(u_n(\theta_n(t))),$$

where $\delta = (3l + \lambda)$. Hence, according to this last inclusion and Proposition 2.1 we get

$$\begin{aligned} \langle \dot{u}(t) - f(t), \xi \rangle &\leq \delta \limsup_n \sigma(-\partial^P d_{C(\theta_n(t))}(u_n(\theta_n(t)); \xi) \\ &\leq \delta \sigma(-\partial^P d_{C(t)}(u(t)); \xi). \end{aligned}$$

The last inequality follows from the upper semicontinuity of the proximal subdifferential in Proposition 2.2. Since $\partial^P d_{C(t)}(u(t))$ is closed and convex, we obtain

$$\dot{u}(t) - f(t) \in -\delta \partial^P d_{C(t)}(u(t)) \subset -N^P(C(t); u(t)),$$

and then

$$\dot{u}(t) \in -N^P(C(t); u(t)) + f(t),$$

because $u(t) \in C(t)$. Finally, from (3.10) and the global upper semicontinuity of F and the convexity of its values, and with the same techniques used above we can prove that

$$f(t) \in F(t, \mathcal{T}(t)u) = F(t, u_t) \qquad a.e. \quad t \in I.$$

Thus, the existence is proved. □

Under different assumptions another existence result for NSPPD is also proved in the following theorem.

Theorem 3.3. *Assume that the assumptions of Theorem* 3.1 *are satisfied. Assume also that* $F(t, \varphi) \subset \mathcal{K} \subset lI\!B$ *for every* $(t, \varphi) \in I \times \mathcal{C}_0$, *where* $\mathcal{K}$ *is a strongly compact set in* H. *Then for every* $\varphi \in \mathcal{C}_0$ *with* $\varphi(0) \in C(0)$, *there exists a continuous mapping* $u : [-\tau, T] \to H$ *such that* u *is Lipschitz continuous on* I *and satisfies :*

$$\begin{aligned} &\dot{u}(t) \in -N^P(C(t); u(t)) + F(t, u_t) \qquad a.e. \quad on \quad I; \\ &u(t) \in C(t), \qquad \forall\, t \in I; \\ &u(s) = \mathcal{T}(0)u(s) = \varphi(s), \qquad \forall\, s \in [-\tau, 0]; \end{aligned}$$

and

$$\|\dot{u}(t)\| \leq (2l + \lambda) \qquad a.e.\ on \quad I.$$

Proof. Let $\varphi \in \mathcal{C}_0$ with $\varphi(0) \in C(0)$. By Theorem 3.1 there exists a sequence of continuous mappings $\{u_n\}$ enjoying the properties (1) and (2) in Theorem 3.1. Let $n_0 \in I\!N$ satisfy (3.2). Let us show that the sequence $\{u_n\}_n$ satisfies the Cauchy property in the space of continuous mappings $\mathcal{C}(I, H)$ endowed with the norm of uniform convergence. Fix $m, n \in I\!N$ such that $m \geq n \geq n_0$ and also fix $t \in I$ with $t \neq t_{m,i}$ for $i = 0, ..., 2^m$ and $t \neq t_{n,j}$ for $j = 0, \ldots, 2^n$. Observe by (3.1), (3.4), and (3.6) that

$$\begin{aligned}(3.11) \qquad d_{C(\theta_n(t))}(u_m(t)) &= d_{C(\theta_n(t))}(u_m(t)) - d_{C(\theta_m(t))}(u_m(\theta_m(t)))\\ &\leq \lambda|\theta_n(t) - \theta_m(t)| + \|u_m(\theta_m(t)) - u_m(t)\|\\ &\leq \lambda(\mu_n + \mu_m) + (2l + \lambda)\mu_m = \lambda\mu_n + 2(l + \lambda)\mu_m\\ &\leq (2l + 3\lambda)\mu_{n_0},\end{aligned}$$

and so, by (3.2) we get $d_{C(\theta_n(t))}(u_m(t)) < r$. Set $\delta = (3l + \lambda)$. Then, the relations (3.8) and (3.11), and Proposition 2.2 entail

$$\begin{aligned}&\langle \dot{u}_n(t) - f_n(t), u_n(\theta_n(t)) - u_m(t) \rangle\\ &\quad\leq \frac{2\delta}{r}\|u_n(\theta_n(t)) - u_m(t)\|^2 + \delta d_{C(\theta_n(t))}(u_m(t))\\ &\quad\leq \frac{2\delta}{r}\Big[\|u_n(t) - u_m(t)\| + \|u_n(\theta_n(t)) - u_n(t)\|\Big]^2 + 2\delta(l + \lambda)\mu_{n,m},\end{aligned}$$

where $\mu_{n,m} = \mu_n + \mu_m$ and this yields, by (3.4) and (3.6),

$$\begin{aligned}\langle \dot{u}_n(t) - f_n(t), u_n(\theta_n(t)) - u_m(t) \rangle \leq& \frac{2\delta}{r}\Big[\|u_n(t) - u_m(t)\| + (2l + \lambda)\mu_n\Big]^2\\ (3.12) \qquad\qquad &+ 2\delta(l + \lambda)\mu_{n,m}.\end{aligned}$$

Now, let us define $g_n(t) = \int_0^t f_n(s)ds$ for all $t \in I$. Observe that for all $t \in I$ the set $\{g_n(t) : n \geq n_0\}$ is contained in the strong compact set $T\mathcal{K}$ and so it is relatively strongly compact in H. Then, as $\|f_n(t)\| \leq l$ a.e. on I, Arzela-Ascoli's theorem yields the relative strong compactness of the set $\{g_n : n \geq n_0\}$ with respect to the uniform convergence in $\mathcal{C}(I, H)$, and so we may assume without loss of generality that $\{g_n\}$ converges uniformly to some mapping g. Also, we may suppose that $\{f_n\}$ weakly converges in $L^1(I, H, dt)$

to some mapping f. Then, for all $t \in I$,

$$g(t) = \lim_n g_n(t) = \lim_n \int_0^t f_n(s)ds = \int_0^t f(s)ds,$$

which gives that g is absolutely continuous and $\dot{g} = f$ a.e. on I.

Put now $w_n(t) = u_n(t) - g_n(t)$ for all $n \geq n_0$ and all $t \in I$ and put $\eta_{n,m} = \max\{\mu_{n,m}, \lambda^{-1}[\|g_n - g\|_\infty + \|g_m - g\|_\infty]\}$. Then by (3.7) and (3.12) one gets

$$\begin{aligned}
\langle \dot{w}_n(t), w_n(\theta_n(t)) - w_m(t)\rangle &= \langle \dot{w}_n(t), u_n(\theta_n(t)) - u_m(t)\rangle \\
&\quad + \langle \dot{w}_n(t), g_n(\theta_n(t)) - g_m(t)\rangle \\
&\leq \frac{2\delta}{r}\Big[\|w_n(t) - w_m(t)\| + \|g_n(t) - g_m(t)\| + (2l+\lambda)\mu_n\Big]^2 \\
&\quad + 2\delta(l+\lambda)\mu_{n,m} + \delta\|g_n(\theta_n(t)) - g_m(t)\| \\
&\leq \frac{2\delta}{r}\Big[\|w_n(t) - w_m(t)\| + \lambda\eta_{n,m} + (2l+\lambda)\mu_n\Big]^2 + 3\delta(l+\lambda)\eta_{n,m}
\end{aligned}$$

This last inequality ensures, by (3.7), that

$$\begin{aligned}
\langle \dot{w}_n(t), w_n(t) - w_m(t)\rangle &\leq \langle \dot{w}_n(t), w_n(t) - w_n(\theta_n(t))\rangle + 3\delta(l+\lambda)\eta_{n,m} \\
&\quad + \frac{2\delta}{r}\Big[\|w_n(t) - w_m(t)\| + \lambda\eta_{n,m} + (2l+\lambda)\mu_n\Big]^2 \\
&\leq 4\delta^2\eta_{n,m} \\
&\quad + \frac{2\delta}{r}\Big[\|w_n(t) - w_m(t)\| + \lambda\eta_{n,m} + (2l+\lambda)\mu_n\Big]^2.
\end{aligned}$$

In the same way, we also have

$$\begin{aligned}
\langle \dot{w}_m(t), w_m(t) - w_n(t)\rangle &\leq 4\delta^2\eta_{n,m} \\
&\quad + \frac{2\delta}{r}\Big[\|w_n(t) - w_m(t)\| + \lambda\eta_{n,m} + (2l+\lambda)\mu_m\Big]^2.
\end{aligned}$$

It then follows from both last inequalities that we have for some positive constant α independent of m, n, and t (note that $\|w_n(t)\| \leq (l+\lambda)T + \|\varphi(0)\|$)

$$\langle \dot{w}_m(t) - \dot{w}_n(t), w_m(t) - w_n(t)\rangle \leq \alpha\delta\eta_{n,m} + 8\frac{\delta}{r}\|w_m(t) - w_n(t)\|^2,$$

and so, for some positive constants β and γ independent of m, n, and t

$$\frac{d}{dt}\Big(\|w_m(t) - w_n(t)\|^2\Big) \leq \beta\|w_m(t) - w_n(t)\|^2 + \gamma\eta_{n,m}.$$

As $\|w_m(0) - w_n(0)\|^2 = 0$, the Gronwall inequality yields for all $t \in I$

$$\|w_m(t) - w_n(t)\|^2 \leq \gamma\eta_{n,m}\int_0^t [\exp(\beta(t-s))]ds$$

and hence for some positive constant K independent of m, n, and t we have

$$\|w_m(t) - w_n(t)\|^2 \leq K\eta_{n,m}.$$

The Cauchy property in $\mathcal{C}(I, H)$ of the sequence $\{w_n\}_n = \{u_n - g_n\}_n$ is thus established and hence this sequence converges uniformly to some mapping w. Therefore the sequence $\{u_n\}_n$ constructed in Theorem 3.1 converges uniformly to $u = w + g$. Following the same arguments in the proof of Theorem 3.2 we prove the conclusion of the theorem. □

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