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Schrödinger matrices on a finite network

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Abstract. Potential theory associated with the Schrödinger equation on a finite graph is studied, dealing with the Dirichlet problem, the Green's function, the Poisson kernel, the Balayage etc. The basic tool used is a minimum principle which guarantees the existence of the inverse of any symmetric submatrix of the matrix which represents the Schrödinger operator on the finite graph.

Keywords. Finite networks, Schrödinger equation, Minimum principle.
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1. Introduction

A finite graph X provided with specific transition indices on the edges is here referred to as a network. Important examples of networks are random walks and finite electrical networks. The Laplacian operator on the finite network is denoted by Δ and the Schrödinger operator is denoted by Δ_q, given by $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$; here $q(x) \geq 0$ is defined on the vertices of the network; when we write Δ_q, it is assumed that $q(x) > 0$ for at least one vertex. We identify Δ_q as a symmetric $n \times n$ matrix where n is the number of vertices in X. Let Δ_q^l be a square matrix of order l, $1 \leq l \leq n$ formed by l rows and the corresponding l columns of the matrix Δ_q. By using a minimum principle on X, we prove that Δ_q^l is a non-singular matrix which fact is fundamental in solving a generalized version of the classical Dirichlet problem in a network. Then, by characterizing the Green's function, the Poisson kernel, the condenser

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principle, the balayage and a general version of the equilibrium principle on the network as Dirichlet problems with suitable boundary conditions, we are able to deduce immediately these potential-theoretic results in a network. Thus, these results are obtained effectively by calculating the inverses of suitable submatrices Δ_q^l.

Bendito et al. [1] prove many of these results in a finite network with symmetric conductance, by computing appropriate equilibrium measures: the equilibrium measure for a proper subset F of X is a function $\mu \geq 0$ on X such that $\Delta_q \mu(x) = -1$ for $x \in F$ and $\{x : \mu(x) \neq 0\} \subset F$. (It is proved that μ is unique and $\{x : \mu(x) \neq 0\} = F$). The effective computation of such measures is accomplished by solving linear programming problems. In another direction, interpreting the above problems in the context of probability theory (random walks), solutions are obtained by Chung and Yau [4], Doyle and Snell [5], Tetali [7] and others.

2. Preliminaries

Let X be a finite digraph, connected and without self-loops, consisting of n vertices and m edges. $x \sim y$ signifies that x and y are neighbors, that is x and y are connected by an edge. Associated to each pair of distinct vertices x and y is a number, called *conductance*, $t(x,y) \geq 0$ such that $t(x,y) > 0$ if and only if $x \sim y$. We do not assume (as the earlier authors like Bendito et al. do in the context of electrical networks) that for every pair of vertices, $t(y,x) = t(x,y)$. (In electrical networks, if x and y are two nodes connected by a wire with resistance $c(x,y)$, then the conductance is $t(x,y) = [c(x,y)]^{-1}$. In the case of a random walk, provided with a transition probability structure, $t(x,y) = p(x,y)$ is the probability of transition from the state x to the state y.) Write $t(x) = \sum_{y \sim x} t(x,y)$ and note $t(x) > 0$ for every vertex x. The Laplacian operator Δ is defined as $\Delta u(x) = \sum_{y \sim x} t(x,y)[u(y) - u(x)]$. If $q(x) \geq 0$ is a function on X ($q(x) > 0$ for at least one vertex x), then $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$ is a Schrödinger equation.

Lemma 2.1. *Suppose $\Delta_q u \leq 0$ on X. Then $u \geq 0$ on X.*

Proof. For, suppose u takes negative values on X, let u attain its minimum value -m at z. Since $0 \geq \Delta_q u(z) = \sum_{y\sim\, z} t(z,y)[u(y)-u(z)] - q(z)u(z)$, if we assume $q(z) > 0$, then we find $\Delta_q u(z) > 0$, a contradiction. Hence $q(z) = 0$ and as consequence $u(y) = u(z)$ for $y \sim z$. Since q does not vanish everywhere, there is some vertex a where $q(a) > 0$. Let $\{z = x_0, x_1, ..., x_j = a\}$ be the path joining z and a. Since $x_1 \sim x_0 = z$, the $u(x_1) = -m$. Let i be the smallest index such that $q(x_i) = 0$ and $q(x_{i+1}) > 0$. Then, by the above argument we should have $u(x_i) = -m$ and $u(y) = -m$ if $y \sim x_i$ Since $x_{i+1} \sim x_i$, we should have $u(x_{i+1}) = -m$. Consequently,

$$\begin{aligned} 0 \geq \Delta_q u(x_{i+1}) &= \sum_{y\sim\, x_{i+1}} t(x_{i+1},y)[u(y) - u(x_{i+1})] - q(x_{i+1})u(x_{i+1}) \\ &= [\text{a non} - \text{negative quantity}] - [\text{a positive quantity}] \\ &\times [\text{a negative quantity}] \\ &> 0, \end{aligned}$$

which is a contradiction. Hence, $u \geq 0$ on X. □

Remark

(1) The above lemma implies that if $\Delta_q u = 0$ on X, then $u \equiv 0$; however this conclusion is not valid if $q \equiv 0$ but what is true in this case is that u will be a constant. That is, if $\Delta u = 0$ on X, then u is a constant. For, let M be the maximum value of u on X. Since X is a finite network, for some $z \in X$, $u(z) = M$. Then,

$$0 = \Delta u(z) = \sum_{y\sim z} t(z,y)[u(y) - u(z)] \leq 0$$

since $M \geq u(y)$ for all $y \in X$. Hence $u(y) = M$ if $y \sim z$. Thus, if u takes its maximum value M at a vertex z, then at all the neighbours of z also, u takes the value M. This fact together with the assumption that X is connected implies that at any vertex in X, the value of u is M. That is, u is the constant function M.

(2) Biggs [2, pp.46-48] has the proof using the matrix theory to show that all principal submatrices of the Laplacian matrix are invertible. The same method can be modified, in the case of symmetric conductance, to prove Theorems 2.1 and 2.2 below which are given for the

Schrödinger operator. However, we prove these two theorems using analytic methods more suitable for potential theory.

Theorem 2.1. *The matrix Δ_q is non-singular.*

Proof. If the vertices of X are denoted by $x_1, x_2, ..., x_n$, and if $u(x)$ is a function defined on X, then let us write $u = (u_1, u_2, ..., u_n)^t$ as a column vector where $u_i = u(x_i)$ Now, by the above Lemma 2.1, the equation $\Delta_q u = 0$ has the unique solution $u \equiv 0$ Hence, Δ_q is a non-singular matrix. □

The following result is very useful in proving that the solution to some important problems like the Equilibrium problem (Theorem 2.4), the Poisson problem (Corollary 2.1), the Classical Dirichlet problem (Corollary 3.1), Balayage (Theorem 4.1) and the Condenser problem (Theorem 4.2) is unique.

Lemma 2.2. *(Minimum Principle) Let F be a proper subset of X. Suppose u is a real-valued function on X such that $\Delta_q u \leq 0$ on F and $u \geq 0$ on $X \backslash F$. Then $u \geq 0$ on X. (In particular, if $\Delta_q v = 0$ on F and $v = 0$ on $X \backslash F$, then $v \equiv 0$).*

Proof. Suppose u takes a negative value on X. If $-m = \min_{x \in X} u(x)$, then $u(x_0) = -m$ for some $x_0 \in F$. Then

$$\begin{aligned} 0 \geq \Delta_q u(x_0) &= \sum_{y \sim x_0} t(x_0, y)[u(y) - u(x_0)] - q(x_0)u(x_0) \\ &= \sum_{y \sim x_0} t(x_0, y)[u(y) - u(x_0)] + mq(x_0). \end{aligned}$$

If $q(x_0) > 0$, then $\Delta_q u(x_0) > 0$, a contradiction. Hence $q(x_0) = 0$, which implies that $u(y) = u(x_0) = -m$ for all $y \sim x_0$.

Since F is a proper subset of X, there is some $a \in X \backslash F$. Let $\{x_0, x_1, ..., x_j = a\}$ be a path connecting x_0 and a. Then, $u(x_1) - m$. Again, as indicated above, if we have to avoid a contradiction, then we should have $q(y) = -m$ whenever $y \sim x_1$. In particular, $u(x_2) = -m$. Let i be the smallest index such that $x_i \in F$ and $x_{i+1} \in X \backslash F$. If $q(x_i) > 0$, we arrive at a contradiction. If $q(x_i) = 0$, then we should have $q(y) = q(x_i) = -m$ for all $y \sim x_i$ in particular, $q(x_{i+1}) = -m$. But $x_{i+1} \in X \backslash F$ so that $u(x_{i+1}) = 0$, again a contradiction.

Hence the assumption that u takes on negative values is not tenable. We conclude that $u \geq 0$ on X. $\square$

Remark When $F \neq X$ the above proof is valid even if $q \equiv 0$. That is, the above lemma is valid if Δ_q is replaced by Δ.

Theorem 2.2. *For any l, $1 \leq l \leq n$ the matrix Δ_q^l is non-singular.*

Proof. Since we have already proved the theorem when $l = n$ (Theorem 2.1), we shall suppose $l < n$. Let F be the subset $x'_1, x'_2, ..., x'_l$ corresponding to the l rows of Δ_q^l. Consider the equation $\Delta_q^l u' = 0$, where $u' = (u'_1, u'_2, ..., u'_l)^t$ is arbitrary. Let $u = (u_1, u_2, ..., u_n)^t$, obtained from u' by introducing $(n-1)$ extra zeros at the appropriate places. Note now $\Delta_q u = 0$ on F and $u = 0$ on $X \backslash F$. Hence, by the Minimum Principle (Lemma 2.2), $u \equiv 0$, and hence $u' \equiv 0$. This proves that the matrix Δ_q^l is non-singular. $\square$

Theorem 2.3. *(Domination principle) Let v and f be two real-valued functions on X. Let $\Delta_q v \leq 0$ on X and $A = \{x : \Delta_q f(x) < 0\}$. If $v \geq f$ on A, then $v \geq f$ on X .*

Proof. First remark that A cannot be the whole set X. For, suppose f takes its minimum value m at the vertex z, that is $f(y) \geq f(z)$ for every y in X. Then,

$$0 \geq \Delta_q f(z) = \sum_{y \sim z} t(z,y)[f(y) - f(z)] \geq 0,$$

which gives a contradiction. Hence $A \neq X$.

Let us suppose $A = \phi$. That is, $\Delta_q f \geq 0$ on X. Then, by Lemma 2.1, $f \leq 0$ on X while $v \geq 0$ on X and the condition $v \geq f$ on A is not significant in this case. But $v \geq f$ on X. We shall therefore assume that $A \neq \phi$, $A \neq X$, in which case the set $F = X \backslash A$ is a proper subset of X. Let $u = v - f$. Then, on $F = X \backslash A$, $\Delta_q u \leq 0$; on A, $u \geq 0$. Hence, by the above Lemma 2.2, $u \geq 0$ on X. $\square$

In the context of an electrical network X, an important problem is how to distribute a unit mass on a proper subset F so that the associated potential is constant on F. That is, the problem is to find $u \geq 0$ on X such that

$\sum_{x\in F}\Delta u(x) = -1$, $\Delta u(x) = 0$ if $x \in X\backslash F$ and for a constant α, $u(x) = \alpha$ if $x \in F$. The following result shows that a solution to this problem exists even when Δ is replaced by the operator Δ_q.

Theorem 2.4. *(Generalized Equilibrium Principle) Let F be a subset of X. Given two functions f on F and g on $X\backslash F$, there exists a unique function u on X such that $\Delta_q u = -f$ on F and $u = g$ on $X\backslash F$. Moreover, if f and g are non-negative, then $u \geq 0$ on F.*

Proof. If $X = F$ then Theorem 2.1 establishes the existence of the unique solution u and then Lemma 2.1 proves that $u \geq 0$ on X. Let us suppose that F is a proper subset of X. Let $u = (u_1, u_2, ..., u_n)^t$ and $v = (v_1, v_2, ..., v_n)^t$ be the column vectors such that $u(x) = g(x)$ if $x \in X\backslash F$ and $v(x) = -f(x)$ if $x \in F$. Now, if we write $\Delta_q u = v$ then (Δ_q being a non-singular matrix) we can first determine the value of $u(x)$ for $x \in F$. Consequently, we have all the values u takes on X such that $u = g$ on $X\backslash F$ and $\Delta_q u = -f$ on F. To prove the uniqueness of u, suppose s is another function such that $s = g$ on $X\backslash F$ and $\Delta_q s = -f$ on F. Then $\varphi = s - u$ satisfies the conditions $\varphi = 0$ on $X\backslash F$ and $\Delta_q \varphi = 0$ on F. The Minimum Principle shows that $\varphi \equiv 0$. □

Corollary 2.1. *(q-Poisson solution) There is always a unique solution to the Poisson equation: that is, given f on X, there exists a unique u on X such that $\Delta_q u = f$ on X.*

Note: In a finite network X, using the symmetry of the transition indices $t(x, y)$, it is seen that for any real-valued function v on X, $\sum_{x\in X}\Delta v(x) = 0$. Consequently, with respect to the Laplacian operator, we can only state that if f is a real-valued function on X, then there exists u such that $\Delta u = f$ on X if and only if $\sum_{x\in X} f(x) = 0$; the solution, when it exists, is unique up to an additive constant.

3. Dirichlet problem

Let F be a subset of X. Say that x is an interior vertex of F if $y \in F$ for every $y \sim x$. Let $\mathring{F}$ denote the set of all interior vertices of F. Write $\partial F = F\backslash\mathring{F}$ For any x, write $V(x) = \{y \in X, y \sim x\} \cup \{x\}$; and for any subset A, $V(A) = \cup_{x\in A} V(x)$. A real-valued function u on F is said to be q-superharmonic (respectively, q-harmonic, q-subharmonic) on F if and only if $\Delta_q u \leq 0$ (respectively, $\Delta_q u = 0$, $\Delta_q u \geq 0$) at every vertex of $\overset{\circ}{F}$. However, we use the expression that u is q-harmonic at a vertex a, if u is defined on $V(a)$ and $\Delta_q u(a) = 0$.

1. If u and v are q-superharmonic on F, then $s = \inf\{u, v\}$ is q-superharmonic on F. For, if $x \in \overset{\circ}{F}$, then

$$\begin{aligned}\Delta_q s(x) &= \sum_y t(x,y)s(y) - [t(x) + q(x)]s(x)\\ &= \sum_y t(x,y)s(y) - [t(x) + q(x)]v(x), \ \textit{assuming } s(x) = v(x)\\ &\leq \sum_y t(x,y)v(y) - [t(x) + q(x)]v(x) = \Delta_q v(x) \leq 0.\end{aligned}$$

2. If u_n is a sequence of q-superharmonic (respectively, q-harmonic) functions on F, and if $u(x) = \lim_{n\to\infty} u_n(x)$ exists and is finite for every $x \in F$ then u is q-superharmonic (respectively, q-harmonic) on F. For, if $x \in \overset{\circ}{F}$ then $\sum_y t(x,y)u_n(y) - [t(x) + q(x)]u_n(x) \leq 0$. Taking limits when $n \to \infty$, we obtain $\sum_y t(x,y)u(y) - [t(x) + q(x)]u(x) \leq 0$. That is, u is q-superharmonic on F, (Similar proof in the case of a sequence of q-harmonic functions.)

Perron family For a real-valued function u on a subset E of X, and an arbitrary vertex $a \in \overset{\circ}{E}$, let us write $P_a u(x) = u(x)$ if $x \neq a$ and $P_a u(a) = \sum_{z\neq a} \dfrac{t(a,z)}{t(a) + q(a)} u(z)$. ($P_a u$ is the discrete analogue of the Poisson integral. We call $P_a u$ as the Poisson modification of u at a).

Lemma 3.1. *Assume that u is q-superharmonic on E. Let $a \in \overset{\circ}{E}$ Then $P_a u \leq u$, $P_a u$ is q-superharmonic on E and q-harmonic at the vertex a.*

Proof. Since u is q-superharmonic at a, we have $P_a u(a) \leq u(a)$. For $x \in V(a)$, we have $P_a u = u$ on $E \cap V(x)$, so that $\Delta_q P_a u(x) = \Delta_q u(x) \leq 0$ at each $x \in \overset{\circ}{E} \backslash V(a)$.

In case, $x \in \overset{\circ}{E} \cap V(a)$,

(i) For $x \neq a$, we have

$$\begin{aligned}\Delta_q P_a u(x) &= -[t(x)+q(x)]P_a u(x) + \sum_z t(x,z) P_a u(z) \\ &\leq -[t(x)+q(x)]u(x) + \sum_z t(x,z)u(z) \\ &= \Delta_q u(x) \\ &\leq 0.\end{aligned}$$

(ii) For $x = a$, we have

$$\begin{aligned}\Delta_q P_a u(a) &= -[t(a)+q(a)]P_a u(a) + \sum_z t(a,z) P_a u(z) \\ &= -\sum_z t(a,z)u(z) + \sum_z t(a,z)u(z) \\ &= 0.\end{aligned}$$

Thus, for all $x \in \overset{\circ}{E}$, we have $\Delta_q P_a(x) \leq 0$. Hence $P_a u$ is q-superharmonic on E. Further, $[t(a)+q(a)]P_a u(a) = \sum_{z \neq a} t(a,z)u(z) = \sum_{z\neq a} t(a,z)P_a u(z)$. Hence, $P_a u(x)$ is q-harmonic at $x = a$. □

Let E be a subset of X. A non-empty family $\mathcal{F}$ of real-valued functions on $V(E)$ is said to be a *Perron family* on E if it satisfies the following conditions.

1. For any $v_1, v_2 \in \mathcal{F}$, there exists $v \in \mathcal{F}$ such that $v \leq \min(v_1, v_2)$.
2. $P_a v \in \mathcal{F}$, for every $v \in \mathcal{F}$ and $a \in E$.
3. There exists a real-valued function u_0 on E such that $v \geq u_0$ for all $v \in \mathcal{F}$.

Theorem 3.1. *If $\mathcal{F}$ is a Perron family on E, then $h(x) = \inf v(x)$, $x \in V(E)$, $v \in \mathcal{F}$, is q-harmonic at every vertex of E.*

Proof. Since each v in $\mathcal{F}$ majorizes u_0 , we have $h(x) \geq u_0(x)$ in $V(E)$. Let $a \in E$ be fixed arbitrarily. Since $V(a)$ is a finite set, we can find a sequence $\left\{v_x^{(n)}\right\}$ in $\mathcal{F}$ for every $x \in V(a)$ such that $v_x^{(n)} \to h(x)$ as $n \to \infty$. By (1), there exists u_n in $\mathcal{F}$ such that $u_n \leq \min\left\{v_x^{(n)}, x \in V(a)\right\}$. Then, $u_n(x) \to h(x)$ as $n \to \infty$ for every $x \in V(a)$. For, if $x \in V(a)$ then $u_n(x) \leq v_x^{(n)}$; since $u_n \in F$, $u_n(x) \geq h(x)$. Hence, for $x \in V(a)$, $h(x) \leq u_n(x) \leq v_x^{(n)}$. Since

$\lim v_x^{(n)} = h(x)$, we conclude $\lim u_n(x) = h(x)$. Let $v_n = P_a u_n$. Then, by (2), $v_n \in F$ and $v_n \leq u_n$. Since $h \leq v_n \leq u_n$, and since $\lim u_n(x) = h(x)$, then $v_n(x) \to h(x)$ as $n \to \infty$ for every $x \in V(a)$; also v_n is q-harmonic at a. Consequently, we have $\Delta_q h(a) = \lim_{n\to\infty} \Delta_q v_n(a) = 0$. That is, h is q-harmonic at a. □

Theorem 3.2. *(Greatest q-harmonic minorant, g.q-h.m.) Let u (respectively v) be q-superharmonic (respectively q-subharmonic) on a subset $\mathcal{F}$ such that $v \leq u$. Then, there exists a q-harmonic function h on F such that $v \leq h \leq u$; moreover, if H is any other q-harmonic function on F such that $v \leq H \leq u$ then $H \leq h$ (Hence, h is called the greatest q-harmonic minorant of u).*

Proof. Let $\mathcal{F}$ be the family of q-subharmonic functions t on F such that $t \leq u$ Then, $\mathcal{F}$ is a Perron family (of q-subharmonic functions). For, (i) if t_1, t_2 are in $\mathcal{F}$, then $t = \sup(t_1, t_2)$ is a q-subharmonic function in F such that $t \leq u$; (ii) if $t \in \mathcal{F}$, and $a \in \overset{\circ}{F}$ then $P_a(t)$ is q-subharmonic on F and $P_a(t) \geq t$ (proved analogous to Lemma 3.1); and (iii) For any t in $\mathcal{F}$, $t \leq u$ by the definition of $\mathcal{F}$. Consequently, a result analogous to Theorem 3.1 shows that $h(x) = \sup t(x)$ where $t \in \mathcal{F}$, is a q-harmonic function on F such that $h \leq u$. Clearly, if H is a q-harmonic function on F dominated by u, then $H \in \mathcal{F}$. Consequently, $H \leq h$ on F. □

Theorem 3.3. *(Generalized q-Dirichlet Problem) Let F be a subset of X and $E \subset \overset{\circ}{F}$. Let f be a real-valued function on $F \backslash E$ such that $v \leq f \leq u$ on $F \backslash E$ where u and v are real-valued functions on F such that $\Delta_q v \geq 0$ and $\Delta_q u \leq 0$ on E. Then there exists a function h on F such that $\Delta_q h = 0$ on E, $v \leq h \leq u$ on F and $h = f$ on $F \backslash E$. Moreover, if h_1 is any function on F such that $h_1 = f$ on $F \backslash E$ and $\Delta_q h_1 = 0$ on E, then $h = h_1$ on F.*

Proof. Let $u_1 = f$ on $F \backslash E$ and $u = u_1$ on E. Then $\Delta_q u_1 \leq 0$ on E. To see this, remark that if $x \in E$, then

$$\begin{aligned}
[t(x) + q(x)]u_1(x) &= [t(x) + q(x)]u(x) \\
&\geq \sum_y t(x,y)u(y), \text{ since } \Delta_q u(x) \leq 0, \text{ by hypothesis} \\
&\geq \sum_y t(x,y)u_1(y), \text{ since } u_1 \leq u \text{ on } V(E) \subset F.
\end{aligned}$$

Hence, $\Delta_q u_1(x) \leq 0$ for every $x \in E$.
Similarly, if $v_1 = f$ on $F \backslash E$ and $v = v_1$ on E, then $\Delta_q v_1 \geq 0$ at every vertex of E. Let $\mathcal{F}$ be the family of real-valued functions s on F such that $v \leq v_1 \leq s \leq u_1 \leq u$ on F and $\Delta_q s \geq 0$ on E. Note that if s_1, s_2 are in $\mathcal{F}$, then $\Delta_q \sup(s_1, s_2) \geq 0$ so that $\sup(s_1, s_2) \in \mathcal{F}$; if $a \in E$ then

$$P_a s(a) = \sum_z \frac{t(a,z)}{t(a)+q(a)} s(z) \geq \sum_z \frac{t(a,z)}{t(a)+q(a)} v_1(z) \geq v_1(a)$$

and

$$P_a s(a) = \sum_z \frac{t(a,z)}{t(a)+q(a)} s(z) \leq \sum_z \frac{t(a,z)}{t(a)+q(a)} u_1(z) \leq u_1(a).$$

Hence, $P_a s \in \mathcal{F}$; finally, if $s \in \mathcal{F}$, then $s \leq u$ on F. Consequently, $\mathcal{F}$ is a Perron family of q-subharmonic functions on E. Hence, if $h(x) = \sup s(x)$, $x \in F$, $s \in F$, then $v_1 \leq h \leq u_1$ on F and $\Delta_q h = 0$ at every vertex of E; note that $h = f$ on $F \backslash E$ since $v_1 = u_1 = f$ on $F \backslash E$. To prove the uniqueness of the solution h, suppose h_1 is another function such that $h_1 = f$ on $F \backslash E$ and $\Delta_q h_1 = 0$ at every vertex of E. Let $w = h - h_1$ on F. Then, w extended by 0 outside F satisfies the conditions: $\Delta_q w = 0$ on E and $w = 0$ on $X \backslash E$. Hence, by the minimum principle, $w \equiv 0$ on X. □

Corollary 3.1. *(Classical q-Dirichlet problem) Let F be a proper subset of X. Let f be a real-valued function on ∂F Then there exists a unique function h on F such that h is q-harmonic at each vertex of $\mathring{F}$ and $h = f$ on ∂F.*

Proof. Proof. Suppose $|f| \leq M$ Use the above theorem with $-M \leq f \leq M$ on ∂F and $E = \mathring{F}$ to get the unique solution h. □

Examples of the Dirichlet problem (with $q \equiv 0$)

In the next section, we shall give a few results connected with the Dirichlet problem in the context of potential theory associated with the operator Δ_q (with the restriction $q \geq 0$. Here we illustrate the relevance of the Dirichlet problem in the context of random walks and electrical networks (assuming $q \equiv 0$); for further developments on the following three examples, see Biggs [3], Doyle and Snell [5], Soardi [6, pp.14-31], and Tetali [7].

(i) Consider the following problem in random walks: Let X be the state space provided with the transition probabilities $p(x, y)$. Given two vertices a and b in X, what is the probability of the walker reaching a before b? To solve this, let us denote by $\varphi(x)$ the probability of the walker starting at the vertex x and reaching a before b. Then $\varphi(a) = 1$ and $\varphi(b) = 0$. If x is different from a and b, then the walker starting at x takes a first step to a neighboring vertex y and then from y he should reach a before reaching b. Consequently, $\varphi(x) = \sum_y p(x, y)\varphi(y)$. Since $p(x, y)$ are probabilities, $\sum_y p(x, y) = 1$ for any fixed x. Hence,

$$\sum_y p(x, y)\varphi(x) = \varphi(x) = \sum_y p(x, y)\varphi(y),$$

that is, $\sum_y p(x, y)[\varphi(y) - \varphi(x)] = 0$; that is, $\Delta\varphi(x) = 0$ if x is different from a and b. Thus, this is a Dirichlet problem with $F = X$ and $E = X \backslash \{a, b\}$.

(ii) In the above random walk, what is the escape probability $p_{esc}(a, b)$ which is defined as the probability of starting at a and visiting b before returning to a ? As above, let $\varphi(x)$ represent the probability of the walker starting at x and visiting a before reaching b. Let λ represent the walker leaving a and arriving at a without visiting b, so that $1-\lambda = p_{esc}(a, b)$. Since $\lambda = \sum_y p(a, y)\varphi(y)$ we have

$$\begin{aligned} p_{esc}(a, b) &= 1 - \lambda \\ &= \varphi(a) - \sum_y p(a, y)\varphi(y) \\ &= -\sum_y p(a, y)[\varphi(y) - \varphi(a)] \\ &= (-\Delta)\varphi(a). \end{aligned}$$

Thus, the escape probability is related to the solution of a Dirichlet problem.

(iii) In the context of an electrical network X with symmetric conductance, if a and b are two nodes, then the effective resistance $r(a, b)$ between a and b is the voltage when a unit current enters a and leaves b. What is the value of $r(a, b)$? The effective resistance problem reduces to finding the quantity $r(a, b) = \Psi(a) - \Psi(b)$, where $\Psi(x)$ is a real-valued function on X such that $(-\Delta)\Psi(a) = 1$, $(-\Delta)\Psi(b) = -1$ and $\Delta\Psi(x) = 0$ for all x other than a and b. For this, let us consider the function φ defined as above, with $\varphi(a) = 1$ and $\varphi(b) = 0$, and $\Delta\varphi(x) = 0$ if $x \neq a, b$. Since for any function f on X, $\sum_x \Delta f(x) = 0$, we should have $(-\Delta)\varphi(a) = \alpha$ and $(-\Delta)\varphi(b) = -\alpha$ for some α. By the minimum principle, we know that $0 \leq \varphi \leq 1$ on X. Hence, $\alpha \geq 0$. But the assumption that $\alpha = 0$ will lead to the conclusion φ is a constant, not true. Hence, $\alpha > 0$. Consequently, $\Delta\Big[\Psi(x) - \frac{1}{\alpha}\,\varphi(x)\Big] = 0$ for all $x \in X$. This means, $\Psi(x) = \frac{1}{\alpha}\,\varphi(x) + c$ where c is a constant. Then, $\Psi(b) = \frac{1}{\alpha}\,\varphi(b) + c = 0 + c$ so that $\Psi(x) - \Psi(b) = \frac{1}{\alpha}\,\varphi(x)$. Hence,

$$r(a,b) = \big[\Psi(a) - \Psi(b)\big] = \frac{1}{\alpha}\,\varphi(a) = \frac{1}{\alpha} = \frac{1}{(-\Delta)\varphi(a)}\,.$$

Let us now consider the electrical network X as a random walk and give a relation between the escape probability and the effective resistance (see Biggs [3] and Tetali [7]). Let us provide X with transition probabilities defined by $p(x,y) = \dfrac{t(x,y)}{t(x)}$ for each vertex $x \in X$ in which case the Laplacian $\tilde{\Delta} f$ of a function f is defined as $\tilde{\Delta} f(x) = \sum_y \dfrac{t(x,y)}{t(x)}\big[f(y) - f(x)\big]$. Note then, $t(x)\Delta f(x) = \Delta f(x)$, and $p_{esc}(a,b) = (-\tilde{\Delta}) f(a)$ where $f(a) = 1$, $f(b) = 0$, and $(-\tilde{\Delta}) f(x) = 0$ for $x \neq a, b$. Now, by the uniqueness of the solution, $f \equiv \varphi$. Hence,

$$\begin{aligned} p_{esc}(a,b) &= (-\tilde{\Delta})\varphi(a) \ \big(from\,(ii)\,above\big) \\ &= \frac{1}{t(a)}(-\Delta)\varphi(a) \\ &= \frac{1}{t(a) r(a,b)}. \end{aligned}$$

Thus, $t(a) \cdot p_{esc}(a,b) \cdot r(a,b) = 1$.

4. Some potential-theoretic problems with Dirichlet solutions

Let us consider now the operator Δ_q, with $q \geq 0$. An important problem in an electrical network is: Given a potential function u on X and a subset F of X, is it possible to sweep out the charges associated with u onto F so that the potential function v associated with the new distribution of charges preserves the same values on F ?

Theorem 4.1. *(q-Balayage) Let F be a proper subset of X and let u be a real-valued function on X such that $\Delta_q u \leq 0$ on $X \backslash F$. Then, there exists a unique function v on X such that $v \leq u$ on X, $v = u$ on F , and $\Delta_q v = 0$ on $X \backslash F$.*

Proof. By Theorem 2.4, there exists a unique function φ on X such that $\Delta_q \varphi = \Delta_q u$ on $X \backslash F$ and $\varphi = 0$ on F. Since by hypothesis, $\Delta_q \varphi \leq 0$ on $X \backslash F$ and $\varphi = 0$ on F, we conclude by using the Minimum Principle, that $\varphi \geq 0$ on X. Then, $v = u - \varphi$ on X has the stated properties. Thus, v is the q-Dirichlet solution on $X \backslash F$ with values u on F. □

Theorem 4.2. *(Generalized q-Condenser Principle) Let A and B be two non-empty disjoint subsets of X. Let $F = X \backslash (A \cup B) \neq \phi$. Let a and b be two real numbers such that $a \leq 0 \leq b$. Then there exists a unique φ on X such that*
$a \leq \varphi(x) \leq b$ for every $x \in X$,
$\varphi(x) = a$ and $\Delta_q \varphi(x) \geq 0$ for $x \in A$,
$\varphi(x) = b$ and $\Delta_q \varphi(x) \leq 0$ for $x \in B$,
and $\Delta_q \varphi(x) = 0$ for $x \in F$.

Proof. In Theorem 2.4, take $f = 0$, $g(x) = a$ if $x \in A$ and $g(x) = b$ if $x \in B$. Then there exists a unique function φ on X such that $\varphi = a$ on A, $\varphi = b$ on B, and $\Delta_q \varphi(x) = 0$ on F (If $a = b$ then $\varphi \equiv 0$). Note that by the Minimum Principle, $a \leq \varphi(x) \leq b$ on X. Consequently, at any vertex z where φ attains its minimum value a which is non-positive, $\Delta_q \varphi(z) \geq 0$; similarly, at every vertex y where φ attains its maximum value b which is non-negative, $\Delta_q \varphi(y) \leq 0$. Hence, $\Delta_q \varphi \geq 0$ on A and $\Delta_q \varphi \leq 0$ on B. Thus, φ is the q-Dirichlet solution as in Theorem 3.3, with F in the place of E and $V(F)$ in the place of F; $f = a$ on $A \cap \{V(F) \backslash F\}$, and $f = b$ on $B \cap \{V(F) \backslash F\}$; φ is extended by a on A, and by b on B. □

Theorem 4.3. *(q-Green function) Let F be a subset of X. If $y \in \mathring{F}$, then there exists a unique function $G_y^F(x) \geq 0$ on X such that $\Delta_q G_y^F(x) = -\delta_y(x)$ if $x \in \mathring{F}$, $G_y^F(s) = 0$ if $s \in X \backslash \mathring{F}$ and $G_y^F(x) \leq G_y^F(y)$ for any $x \in X$.*

Proof. Let $f(x) = -\delta_y(x)$ if $x \in \mathring{F}$. Then, by Theorem 2.4, there exists a unique function on X, denoted by $G_y^F(x)$ such that $\Delta_q G_y^F(x) = f(x)$ if $x \in \mathring{F}$ and $G_y^F(s) = 0$ if $s \in X \backslash \mathring{F}$. Since $\Delta_q G_y^F(x) \leq 0$ on $\mathring{F}$ and $G_y^F = 0$ on $X \backslash \mathring{F}$, we conclude that $G_y^F \geq 0$ on X. (It can be seen that $G_y^F > 0$ on $\mathring{F}$ if $\mathring{F}$ is connected). To prove the last assertion, note that $G_y^F(x) = G_y^F(y)$ if $x = y$; if $x \neq y$ and $x \in \mathring{F}$, then $\Delta_q G_y^F(x) = 0$; and if $s \in X \backslash \mathring{F}$ then $G_y^F(s) = 0$ so that $\Delta_q G_y^F(s) \geq 0$, since $G_y^F \geq 0$ on X. Thus, applying the Domination Principle (Theorem 2.3, with $v(x) = G_y^F(y)$ for all $x \in X$ and $f(x) = G_y^F(y)$ for $x \in X$), we conclude that $G_y^F(x) \leq G_y^F(y)$ for all $x \in X$. Thus, $G_y^F(x)$ is the solution to the following Dirichlet problem: Let f be a function defined on $\left(X \backslash \mathring{F}\right) \cup \{y\}$ such that $f(y) = 1$ and $f(x) = 0$ if $x \in X \backslash \mathring{F}$. Then, from Theorem 3.3 (by taking X in the place of F and $\mathring{F} \backslash \{y\}$ in the place of E), we extend f on X as the Dirichlet solution so that $\Delta_q f = 0$ on $\mathring{F} \backslash \{y\}$. By the Minimum Principle, $0 \leq f(x) \leq 1$ for all $x \in X$ and $\Delta_q f(y) < 0$. Then, $G_y^F(x) = \dfrac{f(x)}{(-\Delta_q) f(y)}$. □

Remark The above argument goes through even when $F = X$ so that for any $y \in X$, there exists a function $G_y(x) = G_y^F(x) > 0$ on X such that $(-\Delta_q) G_y(x) = \delta_y(x)$ and $G_y(x) \leq G_y(y)$ for all $x \in X$.

Proposition 4.1. *If $f(x)$ is a real-valued function on X, then it is of the form $f(x) = \sum_y (-\Delta_q) f(y) G_y(x)$.*

Proof. Let $u(x) = \sum_y (-\Delta_q) f(y) G_y(x)$. Then, $(-\Delta_q) u(x) = (-\Delta_q) f(x)$ for all $x \in X$. Hence $u \equiv f$. □

q-Poisson kernel Let F be a proper subset of X. $P(x, \xi)$ for $x \in F$ and $\xi \in \partial F$ is said to be the q-Poisson kernel of F, if it is the Dirichlet solution on F with boundary values $f(s) = \delta_\xi(s)$ for all $s \in \partial F$; note $P(s, \xi) = \delta_s(\xi)$ for s, ξ in ∂F and $\Delta_q P(x, \xi) = 0$ if $x \in \mathring{F}$ and $\xi \in \partial F$.

Proposition 4.2. *Let F be a proper subset of X. Then, the Dirichlet solution on F with boundary values $f(\xi)$ on ∂F is given by $h(x) = \sum_\xi P(x, \xi) f(\xi)$.*

Proof. We have, $\Delta_q h(x) = 0$ if $x \in \mathring{F}$ and $h(s) = f(s)$ for any $s \in \partial F$. Hence, by the uniqueness of the q-Dirichlet solution, the proposition follows. □

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