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BIHARMONIC FUNCTIONS IN A LOCALLY COMPACT SPACE

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ABSTRACT. In this article, we study complex biharmonic functions on $\mathbf{R}^2$. We also discuss the Riquier problem on a locally compact space using Brelot's axioms.

1. INTRODUCTION

Biharmonic functions play an important role in the study of elastic plates where the governing equation for the bending deflection u is given by

$$k\Delta^2 u = T - \frac{h^2}{b(1-\mu)}\Delta T,$$

k being the flexural rigidity of the plate and T being the transverse loading [10]. Thus, in the open sets where $T = 0$, we have $\Delta^2 u = 0$. The functions u satisfying the condition $\Delta^2 u = 0$ are called *biharmonic* functions. In this article we give some properties of such functions and prove that such functions can be represented as difference of two subharmonic functions. We take up this result, later, in the general setup of a harmonic space where a sheaf of harmonic functions is defined on a locally compact space. The general intent of this article is to study the properties of biharmonic functions in a domain Ω in $\mathbf{R}^n$ or in a harmonic space with a view to define the biharmonic compactification $\overline{\Omega}$ of Ω so that the Riquier problem can be properly solved in Ω for the boundary values on $\overline{\Omega} \setminus \Omega$.

Key words and pharases: Biharmonic functions, Brelot's axioms, Riquier problem.
MSC 2000: 31B30

2. BENDING OF AN ELASTIC PLATE

In this section we consider another application of biharmonic functions that is related to concentrated transverse load. We shall again consider an elastic plate with the governing equation $k\Delta^2 u = T$, where the transverse loading is 0 everywhere except at one point of the plate. What this amounts to is that u is a biharmonic function with point singularity. Recall that a harmonic function $g(x)$ outside a compact set K in $\mathbf{R}^2$ is of the form $g(x) = G(x)+\alpha \log|x|+h(x)$, where $G(x)$ is harmonic in $\mathbf{R}^2$ and $h(x)$ is harmonic outside K tending to 0 at infinity.

Theorem 2.1. *Let $u(x)$ be a biharmonic function defined on the punctured unit disk $0 < |x| < 1$ in $\mathbf{R}^2$. Suppose $\lim_{x\to 0} \frac{\Delta u(x)}{\log|x|} = \alpha$. Then $u(x)$ can be represented uniquely as $u(x) = B(x) + \frac{\alpha}{4}|x|^2 \log|x| + \beta \log|x| + h(x)$, where $B(x)$ is biharmonic on the whole disc $|x| < 1$ and $h(x)$ is harmonic on $|x| > 0$ tending to 0 at infinity.*

Proof. Since u is biharmonic on the punctured unit disk, Δu is harmonic and hence, by [4, pp. 172-173], there exist a harmonic functions h_1 on $|x| > 0$ and a harmonic function h_2 in $|x| < 1$ such that $\Delta u = h_1 - h_2$. We can find biharmonic functions u_1 and u_2 in $|x| > 0$ and $|x| < 1$, respectively such that $h_1 = \Delta u_1$ and $h_2 = \Delta u_2$. Now $\Delta(u - (u_1 - u_2)) = 0$ on $0 < |x| < 1$. This means that $u-(u_1-u_2)$ is harmonic on the punctured unit disk. Therefore, by [4], there exist a harmonic function h_1' in $|x| > 0$ and a harmonic function h_2' in $|x| < 1$ such that $u-(u_1-u_2) = h_1'-h_2'$ and hence $u = (u_1+h_1')-(u_2+h_2')$. Set $p = u_1 + h_1'$ and $q = u_2 + h_2'$. Then p is a biharmonic function in $|x| > 0$ and q is a biharmonic function in $|x| < 1$ and $u = p - q$ in $0 < |x| < 1$.
Δq being harmonic in $|x| < 1$, $\lim_{x\to 0} \frac{\Delta q(x)}{\log|x|} = 0$. Moreover, since Δp is harmonic in $|x| > 0$ and since by hypothesis,

$$\lim_{x\to 0} \frac{\Delta p(x)}{\log|x|} = \lim_{x\to 0} \frac{\Delta u(x)}{\log|x|} = \alpha,$$

we have $\Delta p(x) = \alpha \log|x| + v(x)$ in $|x| > 0$ where $v(x)$ is harmonic tending to 0 at infinity. Hence, $p(x) = \frac{\alpha}{4}|x|^2 \log|x| + g(x)$ in $|x| > 0$, where $g(x)$ is harmonic. Hence $g(x)$ is of the form $G(x) + \beta \log|x| + h(x)$, where $G(x)$ is harmonic on $\mathbf{R}^2$ and $h(x)$ is harmonic tending to 0 at infinity. Now set

$B(x) = G(x) - q(x)$ to obtain the representation of u as stated in the theorem. The uniqueness part is a routine verification. □

Note that it is not always possible to apply this theorem directly to the case of an infinite plate under a concentrated load, since the deflection $u(x)$ may tend to infinity when x tends to 0. This is so, because the assumption of a point action of the concentrated load T on the middle plane of the plate considered is practically unrealizable. Hence to accord with a more accurate picture of reality, all lateral loads must be treated as continuously distributed over a certain portion of the plate having finite dimensions.

Corollary 2.1 (see[6]). *For a thin isotropic infinite elastic plate E, let the governing equation for the deflection u be $k\Delta^2 u = T$. Let $T = A\delta$ be a concentrated transverse load applied at the origin 0, where δ is the Dirac function. If the action of the load T produces a bounded deflection near 0, then $u(x) = B(x) + \frac{A}{8\pi k}|x|^2 \log|x|$ where $B(x)$ is biharmonic every where on E.*

Proof. Since $k\Delta^2 u = T = A\delta$ can be written in the form $\Delta(\frac{2\pi k}{A}\Delta u) = 2\pi\delta$, it is immediate that $\frac{2\pi k}{A}\Delta u = \log|x| + H(x)$, where $H(x)$ is a harmonic function on E. Consequently

$$\lim_{x\to 0} \frac{\Delta u}{\log|x|} = \frac{A}{2\pi k}.$$

Therefore, from the above theorem, $u(x)$ is of the form

$$u(x) = B(x) + \frac{A}{8\pi k}|x|^2 \log|x| + \beta \log|x| + h(x).$$

But by hypothesis, $u(x)$ is bounded near 0. This implies that $\beta \log|x| + h(x)$ is bounded near 0 and consequently can be extended harmonically to a neighborhood of 0. In other words, there exists a harmonic function v on E such that $v(x) = \beta \log|x| + h(x)$ when $|x| > 0$. Since h approaches 0 at infinity, its flux at infinity is 0; that is, for large R, $\int_{|x|=R} \frac{\partial h}{\partial n} ds = 0$ where n is the outward normal. Also $\int_{|x|=R} \frac{\partial v}{\partial n} ds = 0$, because v is harmonic on E. Hence, we conclude that $\beta = 0$. For $\int_{|x|=R} \frac{\partial}{\partial n}(\log|x|)ds = \int_0^{2\pi} \frac{d}{dr}(\log r) r d\theta = 2\pi$ and we have $2\pi\beta = \int_{|x|=R} \frac{\partial}{\partial n}(\beta \log|x|)ds = \int_{|x|=R} \frac{\partial}{\partial n}(v(x) - h(x))ds = 0$. Moreover, since v and h approach 0 at infinity, $v \equiv h \equiv 0$. Hence, the corollary. □

3. BIHARMONIC FUNCTIONS

Noting the importance of biharmonic functions in the study of the bending deflection of an elastic plate, we give in this section certain basic features of the biharmonic functions on $\mathbf{R}^n$, $n \geq 2$. An important observation we make is that if u is a biharmonic function on an open set ω in $\mathbf{R}^n$, u is the difference of two subharmonic functions (that is, a δ-subharmonic function) on ω. Since the Newtonian potential is the negative of a subharmonic function, the above observation leads to the possibility of applying potential-theoretic techniques to investigate the properties of biharmonic functions.

Definition. Let D be an open subset of $\mathbf{R}^n$, $n \geq 2$. A *biharmonic* function u is a C^4-function such that $\Delta^2 u = 0$, where

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_1^n} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

Remark. We shall take a bounded open set $D \subset \mathbf{R}^2$ and the results will be similar for $\mathbf{R}^n$, $n \geq 3$. In this case we write

$$\Delta^2 u = \Delta(\Delta u) = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}.$$

Lemma 3.1. *If Φ is a C^1-function on $D(0,R)$ and if*

$$x\frac{\partial \Phi}{\partial x} + y\frac{\partial \Phi}{\partial y} + 2P\Phi + c = 0 \tag{3.1}$$

in $D(0,R)$, where P is any positive integer and c is a constant, then Φ is a constant in $D(0,R)$ and $x\frac{\partial \Phi}{\partial x} + y\frac{\partial \Phi}{\partial y} = 0$ there.

Proof. Let $x = r\cos\theta$ and $y = r\sin\theta$. Then we have

$$\begin{aligned} \frac{\partial \Phi}{\partial r} &= \frac{\partial \Phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \Phi}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{x}{r}\frac{\partial \Phi}{\partial x} + \frac{y}{r}\frac{\partial \Phi}{\partial y}, \end{aligned}$$

and hence $x\frac{\partial \Phi}{\partial x} + y\frac{\partial \Phi}{\partial y} = r\frac{\partial \Phi}{\partial r}$. So the Equation (3.1) can be written in the form

$$\frac{\partial \Phi}{\partial r} + \frac{2P}{r}\Phi = \frac{-c}{r}. \tag{3.2}$$

This differential equation can be solved easily using the integral factor $\mu(r) = r^{2P}$ and so $r^{2P}\Phi = \frac{-c}{2P}r^{2P}$. Hence, $\Phi = \frac{-c}{2P}$ and so it is constant. Therefore, $x\frac{\partial \Phi}{\partial x} + y\frac{\partial \Phi}{\partial y} = 0$. □

Note that, using elementary calculation, one can easily show that

$$\Delta(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}) = 2\Delta u + x\frac{\partial}{\partial x}\Delta u + y\frac{\partial}{\partial y}\Delta u \tag{3.3}$$

$$\Delta^2(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}) = 4\Delta^2 u + x\frac{\partial}{\partial x}\Delta^2 u + y\frac{\partial}{\partial y}\Delta^2 u. \tag{3.4}$$

Proposition 3.1. *Let D be a domain containing 0. A function u is biharmonic on D , if and only if $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is biharmonic.*

Proof. Let $D(0, R) \subseteq D$ and u is biharmonic. Then the result follows immediately using the Equation (3.4). Conversely, suppose that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is biharmonic. Then we have $\Delta^2(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}) = 0$. So, using the Equation (3.4), $4\Delta^2 u + x\frac{\partial}{\partial x}\Delta^2 u + y\frac{\partial}{\partial y}\Delta^2 u = 0$. Taking $\Phi = \Delta^2 u$, $P = 2$ and $c = 0$ in the Equation (3.1), we get (by Lemma 3.1) $\Phi = \frac{-c}{2P} = 0$ and $\Delta^2 u = 0$. Therefore, u is biharmonic. □

Proposition 3.2. *For any harmonic function h on a domain D, the function $r^2 h$ is biharmonic.*

Proof. Using the basic differentiation, one can easily show that

$$\begin{aligned} \frac{\partial^2}{\partial x^2}(r^2 h) &= 2h + 4x\frac{\partial h}{\partial x} + r^2\frac{\partial^2 h}{\partial x^2} \\ \frac{\partial^2}{\partial y^2}(r^2 h) &= 2h + 4y\frac{\partial h}{\partial y} + r^2\frac{\partial^2 h}{\partial y^2}. \end{aligned}$$

So $\Delta(r^2 h) = 4h + 4(x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y})$. Since h is harmonic, by the Equation (3.3), $x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y}$ is also harmonic. Therefore, $r^2 h$ is biharmonic. □

The following result is known as the Finite Almansi Expansion [9], for the uniqueness of such an expansion see also [11].

Theorem 3.1. *For any biharmonic function u on a star domain D with centre 0, there exist unique harmonic functions h and H such that $u(x) = |x|^2 h(x) + H(x)$.*

Proof. Let u be a biharmonic function. Then $f = \Delta u$ is harmonic. For any harmonic function H, $r^2 H = |x|^2 H(x)$ is biharmonic, by Proposition 3.2. Moreover, $\Delta(r^2 H) = 4H + 4r\frac{dH}{dr}$. So the differential equation

$$4H + 4r\frac{\partial H}{\partial r} = f$$

can be solved on the star domain D. Its solution is given by the equation

$$H = \frac{1}{4r}\int_0^r f(\rho, \varphi) d\varphi.$$

Therefore, we can find a harmonic function H such that $\Delta u = \Delta(r^2 H)$ and so $\Delta(u - r^2 H) = 0$. Therefore, $h = u - r^2 H$ is harmonic and so $u = h + r^2 H$. Now we show that this representation is unique. Suppose that there exist harmonic functions h_1, h_2 such that $u = r^2 H + h = r^2 h_1 + h_2$. Now $r^2(H - h_1) = h_2 - h$ is harmonic. So $\Delta(h_2 - h) = 0 = \Delta(r^2(H - h_1))$. Let $h' = H - h_1$. Then, by Proposition 3.2,

$$\Delta(r^2 h') = 4h' + 4(x\frac{\partial h'}{\partial x} + y\frac{\partial h'}{\partial y}) = 4h' + 4r\frac{\partial h'}{\partial r} = 0.$$

So $h' = -r\frac{\partial h'}{\partial r}$. The solution of this differential equation takes the form $h' = \frac{c_1}{r} + c$, for some constants c_1, c. But as r approaches 0, the function h' approaches $\pm\infty$ if $c_1 \neq 0$, a contradiction. Hence, $c_1 = 0$. Therefore, h' is a constant and so $H - h_1 = c$. But $\Delta(h_2 - h) = 0 = \Delta(r^2(H - h_1)) = \Delta(r^2 c) = 4c$. So $c = 0$ and hence $H = h_1$, $h = h_2$. □

4. COMPLEX BIHARMONIC FUNCTIONS ON $\mathbf{R}^2$

Definition. A complex function ϕ on a domain ω in $\mathbf{R}^2$ is said to be a *complex biharmonic* function on ω if ϕ is of the form $\phi(z) = |z|^2 f_1(z) + f_2(z)$, where f_1 and f_2 are analytic on ω.

Proposition 4.1. *Given a real biharmonic function b on a simply connected, star domain Ω with center 0 in $\mathbf{R}^2$, there exists a complex biharmonic function ϕ on Ω such that the real part of $\phi = \Re(\phi) = b$.*

Proof. By Theorem 3.1, the function b is of the form $b = r^2 h_1 + h_2$ on Ω, where h_1 and h_2 are harmonic on Ω. Let H_1 and H_2 be the harmonic conjugates of h_1 and h_2 respectively on Ω. Then $B = r^2 H_1 + H_2$ is biharmonic on Ω, by

Proposition 3.2, Define $\phi(z) = |z|^2 f_1(z) + f_2(z)$ on Ω, where $f_1 = h_1 + iH_1$ and $f_2 = h_2 + iH_2$ are analytic on Ω and $\Re(\phi) = r^2 h_1 + h_2 = b$. □

Theorem 4.1. *Let ϕ be a complex biharmonic function on $\mathbf{R}^2$ such that $|\phi(z)| = \mathbf{o}(|z|)$ when $|z|$ approaches infinity. Then ϕ is constant.*

Proof. Since ϕ is complex biharmonic, there exist analytic functions f_1 and f_2 on $\mathbf{R}^2$ such that $\phi(z) = |z|^2 f_1(z) + f_2(z)$. Let $\xi \in \mathbf{C}$ be fixed and let $|\xi| < r = |z|$ and $|\xi| \le |z|/2$. Choose z to be large so that $|z - \xi| \ge |z| - |\xi| \ge |z|/2$. Now

$$\begin{aligned} |r^2 f_1(\xi) + f_2(\xi)| &= \left| r^2 \frac{1}{2\pi i} \int_{|z|=r} \frac{f_1(z)}{z - \xi} dz + \frac{1}{2\pi i} \int_{|z|=r} \frac{f_2(z)}{z - \xi} dz \right| \\ &= \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{|z|^2 f_1(z)}{z - \xi} dz + \frac{1}{2\pi i} \int_{|z|=r} \frac{f_2(z)}{z - \xi} dz \right| \\ &= \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{\phi(z)}{z - \xi} dz \right| \\ &\le \frac{1}{2\pi} \int_{|z|=r} \frac{|\phi(z)|}{|z|/2} |dz| \\ &= \frac{1}{2\pi} \frac{2\mathbf{o}(r)}{r} 2\pi r = 2\mathbf{o}(r). \end{aligned}$$

Divide by r^2 and allow r approach infinity. Then we have $|f_1(\xi)| = 0$. Since ξ is arbitrary, we have $f_1 \equiv 0$. Hence, $|f_2(z)| = \mathbf{o}(|z|)$. This implies that f_2 is constant and hence ϕ is a constant. □

The following equation is called the *integral representation of biharmonic functions* [2].

Lemma 4.1. *Let D be a smooth domain and let u, v be biharmonic functions on D . Then*

$$(4.1) \qquad \int_{\partial D} (\Delta u \frac{\partial v}{\partial \nu} - \Delta v \frac{\partial u}{\partial \nu}) ds + \int_{\partial D} (u \frac{\partial \Delta v}{\partial \nu} - v \frac{\partial \Delta u}{\partial \nu}) ds = 0,$$

where $\frac{\partial}{\partial \nu}$ is the inner derivative and ds is the arc length.

Proof. Consider the Green formula on D

$$\int_{\partial D}(u\frac{\partial v}{\partial \nu}-v\frac{\partial u}{\partial \nu})ds+\int_D(u\Delta v-v\Delta u)d\sigma=0.$$

Now if we replace v by Δv and u by Δu in this equation, then we have two equations. Taking the sum of these two equations proves the result. □

Remark. For any open ball $D(a,R)$, the function that is defined by

$$G(x,y)=\log\frac{|y-a||x-y^*|}{R|x-y|}=\log\frac{|x-a||y-x^*|}{R|x-y|},$$

where $x\in D$, $y\in\overline{D}$ and $x^*=\frac{R^2}{|x-a|^2}(x-a)+a$, is called the *Green function* in $D(a,R)$. $G(x,y)=G(y,x)$, $G(x,y)$ is defined except when $x=y$ and $x^*=y$, $G(a,y)=\log\frac{R}{|y-a|}$ if $y\neq a$, and also if y is on the boundary ($y=y^*$) and x in the interior, then $G(x,y)=0$. Moreover, G is harmonic in x ($\neq y$) and $G\geq 0$. Now

$$\begin{aligned}\frac{\partial G(x,y)}{\partial \nu_x} &= \frac{\partial G(x,y)}{\partial x}\frac{\partial x}{\partial \nu_x}+\frac{\partial G(x,y)}{\partial y}\frac{\partial y}{\partial \nu_x}\\ &= \frac{x}{r}\frac{\partial G(x,y)}{\partial x}+\frac{y}{r}\frac{\partial G(x,y)}{\partial y}\\ &= \frac{1}{r}(\frac{R^2-|y-a|^2}{|x-y|^2})\end{aligned}$$

Replacing v by $G(x,y)$ in the Green formula, we see that

$$\int_{\partial D}(u\frac{\partial G}{\partial \nu_x}-G\frac{\partial u}{\partial \nu_x})ds(x)+\int_D(u\Delta G-G\Delta u)\,dx=0.$$

But on the boundary $\partial D(a,R)$, we have $G=0$ and on $D(a,R)$, $\Delta_x G(x,y)=-2\pi\delta_y(x)$. So we have $\int_{\partial D}u\frac{\partial G}{\partial \nu_x}ds(x)-\int_D G\Delta u dx-2\pi u(y)=0$. Therefore, u can be written in the form

$$u(y)=\frac{1}{2\pi}\int_{\partial D}\left(u(x)\frac{\partial G(x,y)}{\partial \nu_x}\right)ds(x)-\frac{1}{2\pi}\int_D G(x,y)\Delta u(x)dx.$$

For $x\in\partial D$, $\frac{\partial G(x,y)}{\partial \nu_x}=\frac{R^2-|y-a|^2}{R|x-y|^2}$. So

$$u(y)=\frac{1}{2\pi}\int_{\partial D}\frac{R^2-|y-a|^2}{R|x-y|^2}u(x)ds(x)-\frac{1}{2\pi}\int_D G(x,y)\Delta u(x)dx. \tag{4.2}$$

The following result is known as Montel Theorem [9].

Theorem 4.2. *Let $u_1, u_2, \ldots$ be a sequence of bounded biharmonic functions on a domain D. If this sequence converges pointwise to a function u on D, then it converges uniformly on D and u will be biharmonic. Moreover, the partial derivatives of u_i converge to the partial derivatives of u.*

Now we state our main result.

Theorem 4.3. *Any real biharmonic function u on an open set ω in $\mathbf{R}^n$, $n \geq 2$, can be written as the difference of two subharmonic functions.*

Proof. For any biharmonic function u on a domain ω, the function Δu is harmonic and so it can be written in the form $\Delta u = h^+ - h^-$, where h^+ and h^- are positive subharmonic functions. Now given a Radon measure μ in Ω, it is possible to define $u = -G \star \mu$, where G is the Green function provided μ has compact support, with the property $\Delta u = (-\Delta G) \star \mu = \delta \star \mu = \mu$; that is, u is a superharmonic function in Ω with μ as its associated measure. In case, μ does not have compact support, we restrict μ in a neighborhood of each point and then patch up the resulting functions to obtain a subharmonic function u (now not necessarily positive) so that $\Delta u = \mu$ in Ω (see [5] and [3]).
Consequently, it is possible to find subharmonic functions u_1 and v_1 such that $\Delta u_1 = h^+$ and $\Delta v_1 = h^-$. Therefore, $\Delta u = \Delta u_1 - \Delta v_1 = \Delta(u_1 - v_1)$ and so $h = u - (u_1 - v_1)$ is harmonic; that is, $u = u_1 - u_2$. □

5. RIQUIER PROBLEM

In the study of harmonic functions, one of the important problems is the Dirichlet problem, namely: If ω is a bounded open set in $\mathbf{R}^n$, $n \geq 2$, and if f is a continuous function on $\partial\omega$, to find a harmonic function h on ω such that h approaches f on the boundary $\partial\omega$. The Dirichlet problem does not always have a solution for an arbitrary ω. If ω is a bounded open set such that for every continuous function f on $\partial\omega$, there exists a harmonic function h on ω (which is necessarily unique) tending to f at every point on the boundary $\partial\omega$, we say that ω is a *regular open set*.

In this section, we treat a similar problem for biharmonic functions, which is known as the *Riquier Problem*, [9], namely: If ω is a bounded open set in

$\mathbf{R}^n$ and if f and g are two continuous functions on $\partial\omega$, to find a biharmonic function b on ω such that b and Δb tend to f and g on $\partial\omega$, respectively. Let us say that b is a solution of the Riquier problem (ω, f, g).

Theorem 5.1. *The Riquier problem for a bounded open set Ω (if it is solvable) has a unique solution.*

Proof. Let u_1, u_2 be two solutions of (Ω, f, g). Then Δu_1 and Δu_2 are harmonic functions on Ω and $\Delta(u_1 - u_2)$ approaches 0 on the boundary $\partial\Omega$. Therefore, by the maximum principle, $\Delta(u_1 - u_2) = 0$ and so $u_1 - u_2$ is a harmonic function on Ω. But $u_1, u_2 \to g$ on $\partial\Omega$, so $u_1 - u_2 \to 0$ on $\partial\Omega$. So, again by the maximum principle, $u_1 - u_2 = 0$ on Ω. Therefore, the solution of the Riquier problem is unique. □

Theorem 5.2. *The Riquier problem $(\Omega, f, g)$is solvable when Ω is a bounded, regular open set for the classical Dirichlet problem.*

Proof. Since f is a continuous function on $\partial\Omega$, the classical Dirichlet problem (Ω, f) is solvable. That is, we can find a unique harmonic function h that tends to f on $\partial\Omega$. Now consider the function v on $\overline{\Omega}$ as

$$v = \begin{cases} h & \text{on } \Omega \\ f & \text{on } \partial\Omega. \end{cases}$$

This function can be extended as a continuous function with compact support on $\mathbf{R}^n$, using Tietze extension theorem. Let us denote the extension also by v. Let us define, following the method indicated in Theorem 4.3, a function u_0 on $\mathbf{R}^n$ such that $\Delta u_0 = v$. Since v is harmonic on Ω, u_0 is biharmonic on Ω and continuous on $\mathbf{R}^n$. Now take the harmonic function H on Ω which tends to $g - u_0$ on $\partial\Omega$ and define the function $u = u_0 + H$ on Ω. Then u is biharmonic on Ω and tends to $u_0 + (g - u_0)$ on $\partial\Omega$; and $\Delta u = \Delta u_0 = h$ (on Ω) tends to f on $\partial\Omega$. □

What we want to find now is a method to extend these results to Riemann surfaces and Riemannian manifolds (hyperbolic or parabolic). For that, we place ourselves more generally on a locally compact space Ω called the *harmonic space.* ($\mathbf{R}^n$, $n \geq 1$, Riemann surfaces and Riemannian manifolds are examples of harmonic spaces.)

Definition. Axiomatic Potential Theory (Brelot's Axioms)
Let Ω be a locally compact, non-compact, connected space. For each ω open subset of Ω, let $K(\omega)$ denotes the family of continuous functions on ω such that $K(\omega)$ is a vector space satisfying the following axioms, see [6] and [7].

- **Axiom 1 (Sheaf Property)**
  a. If $f \in K(\omega)$ and ω_1 open $\subset \omega$, then $f \in K(\omega_1)$.
  b. If $f \in K(\omega_i)$, $i \in I$, then $f \in K(\omega)$, where $\omega = \cup_{i\in I}\omega_i$.
  For example let $\Omega = \mathbf{C}$ and for any ω open, $H(\omega) = \{h \in C^2(\omega), \Delta h = 0\}$. Then $H(\omega)$ satisfies the sheaf property.
  Recall that a relatively compact domain ω is said to be *regular* if for any continuous function f on $\partial\omega$, there exists $u \in K(\omega)$ such that $u \to f$ on $\partial\omega$ (Denote $u = H_f^\omega$) and if $f \geq 0$, then $H_f^\omega \geq 0$.
- **Axiom 2.** Ω has a base of regular domains.
- **Axiom 3.** If ω is a domain and if $h_i \in K(\omega)$ is an increasingly ordered filter (that is, if $h_i, h_j \in K(\omega)$ then there exists $h_k \in K(\omega)$ such that $h_k \geq \sup(h_i, h_j)$), then $h = \sup_i h_i$ is either an element in $K(\omega)$ or $h \equiv \infty$.

Remark. If the space Ω satisfies these axioms, then Ω is called a *Brelot harmonic space.* Now let Ω be a Brelot space with a countable base and let $\mu \geq 0$ be a Radon measure on Ω. Then we can find a superharmonic function u on Ω associated with μ [1], whether there exists a potential $s > 0$ on Ω or not in the following sense.

Theorem 5.3. *Let Ω be a Brelot space with a countable base and let $\mu \geq 0$ be a Radon measure defined on Ω. Then there exists a superharmonic function u on Ω such that for any relatively compact open subset ω of Ω, we have*

$$u(x) = \int_\omega P_y^\omega(x)d\mu(y) + h(x),$$

where h is harmonic on ω and $P_y^\omega(x)$ is a potential on ω and harmonic outside the point y.

Definition. Let μ be a fixed non-negative Radon measure on a Brelot space Ω with a countable base and let h be a harmonic function. Then we can define a new measure $d\nu(x) = h(x)d\mu(x)$ and a function $v(x)$ associated with the signed measure μ using Theorem 5.3. This will be called *biharmonic* on Ω, generated by h.

Example 5.1. If $\Omega = \mathbf{R}^n$, the biharmonic functions are exactly the classical biharmonic functions on $\mathbf{R}^n$ when μ is the Lebesgue measure.

Theorem 5.4. *Every biharmonic function on a Brelot harmonic space Ω is the difference of two subharmonic functions in Ω.*

Proof. Let u be a biharmonic function on Ω with the associated measure $hd\mu$, where h is a harmonic function in Ω. Now, every harmonic function can be written as $h = h^+ - h^-$, where $h^+ = \sup(h, 0)$ and $h^- = \sup(-h, 0)$ are subharmonic functions. So $d\nu = h^+ d\mu - h^- d\mu = d\nu_1 - d\nu_2$. We can then find v_1, v_2 two superharmonic functions with associated measures $d\nu_1$ and $d\nu_2$ respectively. Then $u - (v_1 - v_2)$ is harmonic on every relatively compact open set in ω; that is, $u - (v_1 - v_2)$ is a harmonic function H on Ω. Write $u = (v_1 + H) - v_2$. Hence every biharmonic function is the difference of two subharmonic functions. □

Theorem 5.5. ***(Riquier problem for harmonic spaces)***
Let Ω be a Brelot space with the property that if $r(x)$ is continuous on Ω, then the δ-superharmonic function generated by $r(x)d\mu(x)$ (as in Theorem 5.3) is continuous. Consider the system (ω, f, g), where ω is a regular domain in Ω (ω is relatively compact in Ω). Then there exists a unique biharmonic function $b(x)$ on ω with associate measure $kd\mu$ (k harmonic on ω) such that $\lim_{x\to y\in\partial\omega} b(x) = g(y)$ *and* $lim_{x\to y\in\partial\omega} k(x) = f(y)$.

Proof. For $f \in C(\partial\omega)$, we can find a harmonic function $H_f^\omega(x)$ that tends to f on the boundary. Extend H_f^ω as a continuous function k on Ω. Then we can construct a function v biharmonic on ω with associated measure $kd\mu = H_f^\omega d\mu$, and continuous on $\overline{\omega}$. Now, since the Dirichlet problem (ω, g) is solvable, we can find a harmonic function $H_{g-v}^\omega(x)$ on ω that tends to $g - v$ on the boundary. Therefore, taking $h(x) = H_{g-v}^\omega(x)$, we find that $b(x) = v(x) + h(x)$ is a biharmonic function on ω that tends to g on $\partial\omega$, and its associate measure is $kd\mu$ where k tend to f on $\partial\omega$. Hence, this function is the solution of the Riquier problem. For the uniqueness: let b_1 be another such biharmonic function on ω. Let $B = b - b_1$ on ω. Let s be the harmonic function on ω such that $sd\mu$ be the measure associated with B. Since s tends to 0 on $\partial\omega$, $s \equiv 0$. Consequently, B is harmonic on ω. Since B tends to 0 on $\partial\omega$, $B \equiv 0$. □

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