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FOUR-DIMENSIONAL ALMOST KÄHLER MANIFOLDS OF POINTWISE CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

J.T. CHO AND K.SEKIGAWA

ABSTRACT. In this paper we prove that the theorem of Schur holds for the class of 4-dimensional almost Kähler manifolds.

1. INTRODUCTION

Let $M = (M, J, g)$ be an almost Hermitian manifold and $U(M)$ the unit tangent bundle of M. Then the holomorphic sectional curvature $H = H(x)$ $(x \in U(M))$ can be regarded as a differentiable function on $U(M)$. If the function H is constant along each fiber, then M is called a space of pointwise constant holomorphic sectional curvature. Especially, if H is constant on the whole of $U(M)$, then M is called a space of constant holomorphic sectional curvature. An almost Hermitian manifold $M = (M, J, g)$ is called a Hermitian manifold if the almost complex structure J is integrable. On one hand, an almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if the Kähler form is closed (or equivalently, $\mathcal{S}_{X,Y,Z}g((\nabla_X J)Y, Z) = 0)$). A Hermitian manifold $M = (M, J, g)$ with the closed Kähler form is called a Kähler manifold. A Kähler manifold is characterized by an almost Hermitian manifold with the parallel almost complex structure with respect to the Levi-Civita connection (cf. [6]). It is well-known that the theorem of Schur holds for the class of Kähler manifolds, namely that a real $2n(\geq 4)$-dimensional Kähler manifold of pointwise constant holomorphic sectional curvature is of constant holomorphic sectional curvature. However, A. Gray and L. Vanhecke ([2]) have proved that the theorem of Schur does

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not hold anymore for the class of Hermitian manifolds by constructing examples of Hermitian manifolds of pointwise constant holomorphic sectional curvature which are not of constant holomorphic sectional curvature. It is a natural question to consider whether the theorem of Schur holds for a given class of almost Hermitian manifolds.

In the present paper, we show that the theorem of Schur holds for the class of four- dimensional almost Kähler manifolds. More precisely, we shall prove the following

Main Theorem. *Let $M = (M, J, g)$ be a four-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature. Then M is a Kähler manifold of constant holomorphic sectional curvature.*

The above Theorem is an improvement of the result by T.Sato ([4], Corollary 3.5).

2. PRELIMINARIES

Let $M = (M, J, g)$ be an almost Hermitian manifold. We denote by Ω and N the Kähler form and the Nijenhuis tensor of M defined respectively by $\Omega(X,Y) = g(X, JY)$ and $N(X,Y) = [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY]$ for $X, Y \in \mathfrak{X}(M)$, ($\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M). We note the the Nihenhuis tensor satisfies the following equalities.

$$(2.1) \qquad N(X,Y) = -N(Y,X), \quad N(JX,Y) = -JN(X,Y).$$

for $X, Y \in \mathfrak{X}(M)$. Further, we denote by ∇, R, ρ and τ the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. Here the curvature tensor, R, is defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

for $X, Y, Z \in \mathfrak{X}(M)$. Further, we denote by ρ^* and τ^* the Ricci $*$-tensor and the $*$-scalar curvature defined respectively by

$$(2.2) \qquad \rho^*(x,y) = g(Q^*x, y) = \text{trace}\,(z \mapsto R(x, Jz)Jy),$$

$$\tau^* = \text{trace } Q^*, \tag{2.3}$$

for $x, y, z \in T_pM,\ p \in M$.

Now, we put

$$G(x,y,z,w) = R(x,y,z,w) - R(x,y,Jz,Jw),$$

where $R(x,y,z,w) = g(R(x,y)z,w)$, for $x,y,z,w \in T_pM(p \in M)$. T. Sato ([3]) has proved the following

Proposition 2.1. *Let $M = (M, J, g)$ be an almost Hermitian manifold of pointwise constant holomorphic sectional curvature $c = c(p),\ p \in M$. Then we have*

$$R(x,y,z,w) = \frac{c(p)}{4} R_0(x,y,z,w) + P(x,y,z,w), \tag{2.4}$$

where

$$\begin{aligned} R_0(x,y,z,w) \quad = \quad & g(x,w)g(y,z) - g(x,z)g(y,w) \\ & +g(x,Jw)g(y,Jz) - g(x,Jz)g(y,Jw) \\ & -2g(x,Jy)g(z,Jw), \end{aligned}$$

and

$$\begin{aligned} P(x,y,z,w) \quad = \quad & \frac{1}{96}[26\{G(x,y,z,w) + G(z,w,x,y)\} \\ & -6\{G(Jx,Jy,Jz,Jw) + G(Jz,Jw,Jx,Jy)\} \\ & +13\{G(x,z,y,w) + G(y,w,x,z)\} \\ & -G(x,w,y,z) - G(y,z,x,w)\} \\ & -3\{G(Jx,Jz,Jy,Jw) + G(Jy,Jw,Jx,Jz) \\ & -G(Jx,Jw,Jy,Jz) - G(Jy,Jz,Jx,Jw)\} \\ & +4\{G(x,Jy,z,Jw) + G(Jx,y,Jz,w)\} \\ & +2\{G(x,Jz,y,Jw) + G(Jx,z,Jy,w) \\ & -G(x,Jw,y,Jz) - G(Jx,w,Jy,z)\}], \end{aligned}$$

for $x,y,z,w \in T_pM,\ p \in M$.

It is also well-known that the following identities hold for an almost Kähler manifold $M=(M,J,g)$ (cf. [5]):

$$2g((\nabla_x J)y,z)=g(Jx,N(y,z)), \tag{2.5}$$

for $x,y,z,w\in T_pM(p\in M)$, and

$$\|\nabla J\|^2=\frac{1}{4}\|N\|^2=2(\tau^*-\tau). \tag{2.6}$$

By (2.5) and the Ricci identity, we have further

$$\begin{aligned} G(x,y,z,w) &= \frac{1}{4}\{g(x,N(N(z,w),y))-g(y,N(N(z,w),x))\} \\ &\quad -\frac{1}{2}\{g(x,J(\nabla_y N)(Jz,w))-g(y,J(\nabla_x N)(Jz,w))\} \end{aligned} \tag{2.7}$$

for $x,y,z,w\in T_pM,\ \ p\in M$ (cf. [5]). Taking account of (2.7), Proposition 2.1 reduces to the following

Proposition 2.2. *Let $M=(M,J,g)$ be an almost Kähler manifold of pointwise constant holomorphic sectional curvature $c=c(p)(p\in M)$. Then we have*

$$R(x,y,z,w)=\frac{c(p)}{4}R_0(x,y,z,w)+P(x,y,z,w),$$

where

$$\begin{aligned} R_0(x,y,z,w) &= g(x,w)g(y,z)-g(x,z)g(y,w) \\ &\quad +g(x,Jw)g(y,Jz)-g(x,Jz)g(y,Jw) \\ &\quad -2g(x,Jy)g(z,Jw), \end{aligned}$$

and

$$\begin{aligned} P(x,y,z,w) &= \frac{1}{4\cdot 96}[28\{g(x,N(N(z,w),y))-g(y,N(N(z,w),x))\} \\ &\quad +20\{g(z,N(N(x,y),w))-g(w,N(N(x,y),z))\} \\ &\quad +14\{g(x,N(N(y,w),z))-g(z,N(N(y,w),x))\} \\ &\quad +10\{g(y,N(N(x,z),w))-g(w,N(N(x,z),y))\} \end{aligned}$$

$$
\begin{aligned}
&+14\{g(w,N(N(y,z),x))-g(x,N(N(y,z),w))\}\\
&+10\{g(z,N(N(x,w),y))-g(y,N(N(x,w),z))\}]\\
&+\frac{1}{2\cdot 96}[26\{g(Jy,(\nabla_x N)(Jz,w))-g(Jx,(\nabla_y N)(Jz,w))\\
&+g(Jw,(\nabla_z N)(Jx,y))-g(Jz,(\nabla_w N)(Jx,y))\}\\
&-6\{g(y,(\nabla_{Jx}N)(z,Jw))-g(x,(\nabla_{Jy}N)(z,Jw))\\
&+g(w,(\nabla_{Jz}N)(x,Jy))-g(z,(\nabla_{Jw}N)(x,Jy))\}\\
&+13\{g(Jz,(\nabla_x N)(Jy,w))-g(Jx,(\nabla_z N)(Jy,w))\\
&+g(Jw,(\nabla_y N)(Jx,z))-g(Jy,(\nabla_w N)(Jx,z))\\
&+g(Jx,(\nabla_w N)(Jy,z))-g(Jw,(\nabla_x N)(Jy,z))\\
&+g(Jy,(\nabla_z N)(Jx,w))-g(Jz,(\nabla_y N)(Jx,w))\}\\
&-3\{g(z,(\nabla_{Jx}N)(y,Jw))-g(x,(\nabla_{Jz}N)(y,Jw))\\
&+g(w,(\nabla_{Jy}N)(x,Jz))-g(y,(\nabla_{Jw}N)(x,Jz))\\
&+g(x,(\nabla_{Jw}N)(y,Jz))-g(w,(\nabla_{Jx}N)(y,Jz))\\
&+g(y,(\nabla_{Jz}N)(x,Jw))-g(z,(\nabla_{Jy}N)(x,Jw))\}\\
&+4\{-g(y,(\nabla_x N)(Jz,Jw))-g(Jx,(\nabla_{Jy}N)(Jz,Jw))\\
&-g(x,(\nabla_y N)(Jw,Jz))-g(Jy,(\nabla_{Jx}N)(Jw,Jz))\}\\
&+2\{g(w,(\nabla_x N)(Jy,Jz))+g(Jx,(\nabla_{Jw}N)(Jy,Jz))\\
&+g(x,(\nabla_w N)(Jz,Jy))+g(Jw,(\nabla_{Jx}N)(Jz,Jy))\\
&-g(z,(\nabla_x N)(Jy,Jw))-g(Jx,(\nabla_{Jz}N)(Jy,Jw))\\
&-g(x,(\nabla_z N)(Jw,Jy))-g(Jz,(\nabla_{Jx}N)(Jw,Jy))\}],
\end{aligned}
$$

for $x,y,z,w\in T_pM, p\in M$.

From (2.1), we have also

$$
(2.8)\qquad (\nabla_Z N)(JX,Y) = -(\nabla_Z J)N(X,Y)-J(\nabla_X N)(X,Y) -N((\nabla_Z J)X,Y)
$$

for $X,Y,Z\in\mathfrak{X}(M)$.

In the sequel, we shall consider the case dim $M=4$. Then, from Proposition 2.2, taking account of (2.1)-(2.8) by long and tedius calculations, we have

Proposition 2.3. *Let $M = (M, J, g)$ be a four-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)(p \in M)$. Then we have*

$$
\begin{aligned}
R_{1212} &= R_{3434} = -c(p),\\
R_{1234} &= -\frac{c(p)}{2} - \frac{1}{16}(\tau^* - \tau),\\
R_{1324} &= -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) - \frac{1}{8}(A_{31} - A_{13} + A_{24} - A_{42}),\\
R_{1432} &= -\frac{c(p)}{4} - \frac{3}{32}(\tau^* - \tau) + \frac{1}{8}(A_{31} - A_{13} + A_{24} - A_{42}),\\
R_{1313} &= -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) - \frac{3}{8}(A_{13} - A_{31}) - \frac{1}{8}(A_{24} - A_{42}),\\
R_{1414} &= -\frac{c(p)}{4} + \frac{5}{32}(\tau^* - \tau) - \frac{1}{8}(A_{13} + A_{42}) - \frac{3}{8}(A_{31} + A_{24}),\\
R_{2323} &= -\frac{c(p)}{4} + \frac{5}{32}(\tau^* - \tau) + \frac{1}{8}(A_{31} + A_{24}) + \frac{3}{8}(A_{42} + A_{13}),\\
R_{2424} &= -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) + \frac{3}{8}(A_{24} - A_{42}) + \frac{1}{8}(A_{13} - A_{31}),\\
R_{1334} &= -R_{2434} = -\frac{1}{4}(A_{34} - A_{43}),\\
R_{1213} &= -R_{1224} = -\frac{1}{4}(A_{12} - A_{21}),\\
R_{1434} &= R_{2334} = -\frac{1}{4}(A_{33} + A_{44}),\\
R_{1214} &= R_{1223} = -\frac{1}{4}(A_{11} + A_{22}),\\
R_{1323} &= \frac{1}{8}(A_{41} + A_{14} + A_{32} - 3A_{23}),\\
R_{2324} &= \frac{1}{8}(A_{41} + A_{14} + A_{22} - 3A_{32}),\\
R_{1314} &= -\frac{1}{8}(A_{23} + A_{32} + A_{34} - 3A_{41}),\\
R_{1424} &= -\frac{1}{8}(A_{41} + A_{32} + A_{23} - 3A_{14}),
\end{aligned}
$$

where we set $A_{ij} = g(e_i, (\nabla_{e_j} N)(e_1, e_3)), R_{ijkl} = R(e_i, e_j, e_k, e_l)$ for a unitary basis $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ of $T_pM,\ p \in M$.

From the above Proposition 2.3, we have easily

$$\tau + 3\tau^* = 24c. \tag{2.9}$$

3. PROOF OF THE MAIN THEOREM

Let $M = (M, J, g)$ be a four-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)(p \in M)$.

First, we shall calculate the square norm $\|R\|^2$ of curvature tensor at an arbitrary point p of M. Let $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ be any unitary basis of T_pM. By direct calculation, we have the following general formula

$$\begin{aligned} \|R\|^2 = \; & 4\sum_{a,b}(R^2_{1a1b} + R^2_{2a2b} + R^2_{3a3b} + R^2_{4a4b}) \\ & -4(R^2_{1212} + R^2_{1313} + R^2_{1414} + R^2_{2323} + R^2_{2424} + R^2_{3434}) \\ & +8(R^2_{1234} + R^2_{1324} + R^2_{1432}). \end{aligned} \tag{3.1}$$

Taking account of Proposition 2.3, we have

$$\begin{aligned} \sum_{a,b} R^2_{1a1b} = \; & \frac{9}{8}c^2 + \frac{1}{8}(A_{12} - A_{21})^2 + \frac{1}{8}(A_{11} + A_{22})^2 + \frac{26}{32^2}(\tau^* - \tau)^2 \\ & +\frac{9}{64}(A_{13} - A_{31})^2 + \frac{1}{64}(A_{24} - A_{42})^2 \\ & +\frac{1}{64}(A_{13} + A_{42})^2 + \frac{9}{64}(A_{31} - A_{24})^2 \\ & -\frac{3c}{32}(\tau^* - \tau) + \frac{c}{4}(A_{13} + A_{24}) \\ & -\frac{1}{32}(\tau^* - \tau)(A_{42} + 2A_{13} + 3A_{31} + 4A_{24}) \\ & +\frac{3}{32}(A_{13} - A_{31})(A_{24} - A_{42}) + \frac{3}{32}(A_{13} + A_{42})(A_{31} + A_{24}) \\ & +\frac{1}{32}(A_{23} + A_{32} + A_{14} - 3A_{41})^2. \end{aligned} \tag{3.2}$$

Similarly, we calculate R^2_{2a2b}, R^2_{3a3b} and R^2_{4a4b} and have

$$4\sum_{a,b}(R^2_{1a1b} + R^2_{2a2b} + R^2_{3a3b} + R^2_{4a4b}) \tag{3.3}$$

$$\begin{aligned}
= \;& 18c^2 + \frac{13}{32}(\tau^* - \tau)^2 - \frac{3c}{2}(\tau^* - \tau) + \frac{\tau^* - \tau}{2}(A_{13} - A_{31} - A_{24} + A_{42}) \\
& +(A_{12} - A_{21})^2 + (A_{11} + A_{22})^2 + (A_{34} - A_{43})^2 + (A_{33} + A_{44})^2 \\
& +\frac{5}{4}\{(A_{13} - A_{31})^2 + (A_{24} - A_{42})^2 + (A_{13} + A_{42})^2 + (A_{31} + A_{24})^2\} \\
& +\frac{3}{2}\{(A_{13} - A_{31})(A_{24} - A_{42}) + (A_{13} + A_{42})(A_{31} + A_{24})\} \\
& +\frac{1}{8}\{(A_{23} + A_{32} + A_{14} - 3A_{41})^2 + (A_{14}) + A_{41} + A_{23} - 3A_{32})^2 \\
& +(A_{14} + A_{41} + A_{32} - 3A_{23})^2 + (A_{23} + A_{32} + A_{41} - 3A_{14})^2\}
\end{aligned}$$

Next, we calculate the second term of (3.1). Taking account of Proposition 2.3, we have

$$(3.4) \qquad -4(R^2_{1212} + R^2_{1313} + R^2_{1414} + R^2_{2323} + R^2_{2424} + R^2_{3434})$$

$$\begin{aligned}
= \;& -9c^2 - \frac{4 \cdot 52^2}{32}(\tau^* - \tau)^2 + \frac{3c}{4}(\tau^* - \tau) \\
& -\frac{\tau^* - \tau}{4}(A_{13} - A_{31} - A_{24} + A_{42}) \\
& -\frac{5}{8}\{(A_{13} - A_{31})^2 + (A_{24} - A_{42})^2 + (A_{13} + A_{42})^2 + (A_{31} + A_{24})^2\} \\
& -\frac{3}{4}\{(A_{13} - A_{31})(A_{24} - A_{42}) + (A_{13} + A_{42})(A_{31} + A_{24})\}.
\end{aligned}$$

Lastly, we calculate the last term of (3.1). Taking account of Proposition 2.3, we have

$$\begin{aligned}
(3.5) \qquad & 8(R^2_{1234} + R^2_{1324} + R^2_{1423}) \\
& = 3c^2 + \frac{1}{16}(\tau^* - \tau)^2 + \frac{3c}{8}(\tau^* - \tau) + \frac{\tau^* - \tau}{4}(A_{13} - A_{31} - A_{24} + A_{42}).
\end{aligned}$$

Therefore by (3.1) and (3.3)-(3.5), we have

Lemma 3.1. *Let $M = (M, J, g)$ be a four-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)$ $(p \in M)$. Then we have*

$$\|R\|^2 \;=\; 12c^2 + \frac{17}{64}(\tau^* - *)^2 + \frac{\tau^* - \tau}{2}(A_{13} - A_{31} - A_{24} + A_{42})$$

$$
\begin{aligned}
&+(A_{12}-A_{21})^2+(A_{11}+A_{22})^2+(A_{34}-A_{43})^2+(A_{33}+A_{44})^2 \\
(3.6)\quad &+\frac{5}{8}\{(A_{13}-A_{31})^2+(A_{24}-A_{42})^2+(A_{13}+A_{42})^2+(A_{31}+A_{24})^2\} \\
&+\frac{3}{4}\{(A_{13}-A_{31})(A_{24}-A_{42})+(A_{13}+A_{42})(A_{31}+A_{24})\} \\
&+\frac{1}{8}\{(A_{23}+A_{32}+(A_{14}-3A_{41})^2+(A_{14}+A_{41}+A_{23}-3A_{32})^2 \\
&+(A_{14}-A_{41}+A_{32}-3A_{23})^2+(A_{23}+A_{32}+A_{41}-3A_{14})^2\}
\end{aligned}
$$

for any unitary basis $\{e_i\}=\{e_1,e_2=Je_1,e_3,e_4=Je_3\}$ *of* $T_pM, p\in M$.

Now we replace $\{e_3,e_4\}$ by $\{e_4,-e_3\}$. Then by virtue of (2.8), we see that A_{13}, A_{31}, A_{24} and A_{42} is changed respectively into

$$
\begin{aligned}
(3.7)\quad A_{13} &\longmapsto g(e_1,(\nabla_{e_4}N)(e_1,e_4)) \\
&= A_{24}-\frac{1}{2}\{g(N(e_1,e_3),e_3)^2+g(N(e_1,e_3),e_4)^2\} \\
A_{31} &\longmapsto g(e_4,(\nabla_{e_1}N)(e_1,e_4)) \\
&= -A_{31}+\frac{1}{2}\{g(N(e_1,e_3),e_1)^2+g(N(e_1,e_3),e_2)^2\} \\
A_{24} &\longmapsto -g(e_2,(\nabla_{e_3}N)(e_1,e_4)) \\
&= -A_{13}+\frac{1}{2}\{g(N(e_1,e_3),e_3)^2+g(N(e_1,e_3),e_4)^2\} \\
A_{42} &\longmapsto -g(e_3,(\nabla_{e_2}N)(e_1,e_4)) \\
&= -A_{42}-\frac{1}{2}\{g(N(e_1,e_3),e_1)^2+g(N(e_1,e_3),e_2)^2\}
\end{aligned}
$$

Thus from (2.6) and (3.7) we have

$$
\begin{aligned}
(3.8)\quad A_{13}-A_{31} &\longmapsto A_{24}+A_{31}-\frac{1}{2}(\tau^*-\tau), \\
A_{24}-A_{42} &\longmapsto A_{13}+A_{42}+\frac{1}{2}(\tau^*-\tau), \\
A_{13}+A_{42} &\longmapsto A_{24}-A_{42}-\frac{1}{2}(\tau^*-\tau), \\
A_{31}+A_{24} &\longmapsto A_{13}-A_{31}+\frac{1}{2}(\tau^*-\tau).
\end{aligned}
$$

Replacing $\{e_3,e_4\}$ by $\{e_4,-e_3\}$, we have further

$$
(3.9)\quad A_{12}-A_{21}\longmapsto A_{11}+A_{22},
$$

$$A_{11}+A_{22}\longmapsto -A_{12}+A_{21},$$
$$A_{34}-A_{43}\longmapsto A_{33}+A_{44},$$
$$A_{33}+A_{44}\longmapsto -A_{34}+A_{43},$$

$$(3.10)\qquad A_{23}\longmapsto -A_{14},\quad A_{32}\longmapsto -A_{32},\quad A_{14}\longmapsto -A_{23},\quad A_{41}\longmapsto -A_{41}.$$

Taking account of (3.9) and (3.10), we see that
$\frac{1}{8}\{(A_{23}+A_{32}+A_{14}-3A_{41})^2+(A_{14}+A_{41}+A_{23}-3A_{32})^2+(A_{14}+A_{41}+A_{32}-3A_{23})^2+(A_{23}+A_{32}+A_{41}-3A_{14})^2\}$

and
$(A_{12}-A_{21})^2+(A_{11}+A_{22})^2+(A_{34}-A_{43})^2+(A_{33}+A_{44})^2$ are both invariant by the change $\{e_3,e_4\}$ into $\{e_4,-e_3\}$. On the other hand, from (3.8) we see that

$$(3.11)\quad A_{13}-A_{31}-A_{24}+A_{42}\longmapsto -(A_{13}-A_{31}-A_{24}+A_{42})(\tau^*-\tau),$$

$$(3.12)\qquad (A_{13}-A_{31})^2+(A_{24}-A_{42})^2+(A_{13}+A_{42})^2+(A_{31}+A_{24})^2$$
$$\longmapsto (A_{13}-A_{31})^2+(A_{24}-A_{42})^2+(A_{13}+A_{42})^2+(A_{31}+A_{24})^2$$
$$+2(\tau^*-\tau)(A_{13}-A_{31}-A_{24}+A_{42})+(\tau^*-\tau)^2,$$

$$(3.13)\qquad (A_{13}-A_{31})(A_{24}-A_{42})+(A_{13}+A_{42})(A_{31}+A_{24})$$
$$\longmapsto (A_{13}-A_{31})(A_{24}-A_{42})+(A_{13}+A_{42})(A_{31}+A_{24})$$
$$-(\tau^*-\tau)(A_{13}-A_{31}-A_{24}+A_{42})-\frac{1}{2}(\tau^*-\tau)^2.$$

Since $\|R\|^2$ is invariant by the change $\{e_3,e_4\}$ into $\{e_4,-e_3\}$, taking account of Lemma 3.1 and (3.11)-(3.13), we have

$$(3.14)\qquad (\tau^*-\tau)\{(A_{13}-A_{31}-A_{24}+A_{42})+\frac{1}{2}(\tau^*-\tau)\}=0.$$

Suppose that M is not Kählerian. Then there exists a point $p \in M$ such that $\tau^* - \tau > 0$ at p. From (3.14) it follows that

$$A_{13} - A_{31} - A_{24} + A_{42} = -\frac{1}{2}(\tau^* - \tau) \tag{3.15}$$

at p. From (3.2) and (3.15) we calculate at p

$$\begin{aligned}\sum_{a,b} R^2_{1a1b} &= \frac{9}{8}c^2 + \frac{1}{8}(A_{12} - A_{21})^2 + \frac{1}{8}(A_{11} + A_{22})^2 \\ &\quad + \frac{26}{32^2}(\tau^* - \tau)^2 - \frac{3c}{32}(\tau^* - \tau) + \frac{c}{4}(A_{13} + A_{24}) \\ &\quad + \frac{1}{4}(A_{13} - A_{31})^2 + \frac{1}{4}(A_{13} + A_{42})^2 \\ &\quad + \frac{1}{16}(\tau^* - \tau)(A_{13} - A_{31}) + \frac{3}{16}(\tau^* - \tau)(A_{13} + A_{42}) \\ &\quad + \frac{5}{128}(\tau^* - \tau)^2 + \frac{1}{32}(A_{23} + A_{32} + A_{14} - 3A_{41})^2.\end{aligned} \tag{3.16}$$

Since $\sum_{a,b} R^2_{1a1b}$ depends only on e_1 (invariant by the change $\{e_3, e_4\}$ into $\{e_4, -e_3\}$), taking account of (3.7) - (3.10) and (3.15) we have

$$(\tau^* - \tau)(A_{42} + A_{31}) = 0 \tag{3.17}$$

at p. Thus from (3.17) we have

$$A_{42} + A_{31} = 0 \tag{3.18}$$

at p. But by changing $\{e_1, e_2, e_3, e_4\}$ into $\{e_3, e_4, e_1, e_2\}$, we see that

$$A_{42} \longmapsto -A_{24}, \quad A_{31} \longmapsto -A_{13}. \tag{3.19}$$

So, (3.18) and (3.19) yield

$$A_{13} + A_{24} = 0 \tag{3.20}$$

at p. On the other hand, Proposition 2.3 yields

$$\begin{aligned}\rho_{11} &= \frac{\tau}{4} + \frac{1}{2}(A_{13} + A_{24}), \\ \rho_{22} &= \frac{\tau}{4} - \frac{1}{2}(A_{13} + A_{24}), \\ \rho_{33} &= \frac{\tau}{4} - \frac{1}{2}(A_{31} + A_{42}), \\ \rho_{44} &= \frac{\tau}{4} + \frac{1}{2}(A_{31} + A_{42}),\end{aligned} \tag{3.21}$$

where we set $\rho_{ij} = \rho(e_i, e_j)$. From (3.18), (3.20) and (3.21) we see that

$$\rho_{ii} = \frac{\tau}{4} \quad (1 \leq i \leq 4). \tag{3.22}$$

Thus we have

Lemma 3.2. *Under the same hypothesis as in Lemma 3.1, if M is not Kählerian, then the open subspace $M_0 = \{p \in M | \tau^* - \tau > 0$ at $p\}$ is an Einstein manifold.*

We put $M_1 = \{p \in M | \tau^* - \tau = 0 \text{ at } p\}$. Then $M = M_0 \cup M_1$. If the interior M_1' of M_1 is non-empty, then M_0' is a Kählerian and has constant holomorphic sectional curvature, and is Einsteinian. Thus we have

Proposition 3.3. *Let $M = (M, J, g)$ be a four-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature, then M is an Einstein manifold.*

Now, we suppose that M_0 is non-empty, and we discuss on M_0. Then (3.15), (3.18) and (3.20) yield

$$\begin{aligned} A_{13} - A_{31} &= -\frac{1}{4}(\tau^* - \tau), \\ A_{24} - A_{42} &= \frac{1}{4}(\tau^* - \tau). \end{aligned} \tag{3.23}$$

From Proposition 2.3 and the Einstein condition we have

$$\begin{aligned} A_{11} + A_{22} &= A_{33} + A_{44}, \\ A_{12} - A_{21} &= A_{34} - A_{43}, \end{aligned} \tag{3.24}$$

$$\begin{aligned} A_{14} &= A_{23}, \\ A_{41} &= A_{32}. \end{aligned} \tag{3.25}$$

Also, from Proposition 2.3, (3.23) and the definition of ρ^* we have

$$\begin{aligned} \rho^*_{14} &= \rho^*_{23} = -\rho^*_{41} = -\frac{1}{2}(A_{12} - A_{21}), \\ \rho^*_{13} &= \rho^*_{24} = -\rho^*_{31} = -\frac{1}{2}(A_{11} + A_{22}). \end{aligned} \tag{3.26}$$

Here, we recall the following general formula which holds in a four-dimensional almost Hermitian manifold (cf. [1]);

$$\rho(x,y)+\rho(Jx,Jy)-\rho^*(x,y)-\rho^*(Jx,Jy)=\frac{\tau^*-\tau}{2}g(x,y),$$

for any tangent vector x, y on the manifold. So, since M_0 is an Einstein manifold, we have

$$\rho^*(x,y)+\rho^*(y,x)=\frac{\tau^*}{2}g(x,y), \tag{3.27}$$

for any $x, y \in T_pM_0$ and any $p \in M_0$. From (3.16), taking account of (3.15), (3.18), (3.20), (3.23), (3.24), (3.25), (3.26) and (3.27) we have

$$\begin{aligned}\sum_{a,b} R_{1a1b}^2 &= \frac{9}{8}c^2+\frac{1}{8}\|\rho^*\|^2-\frac{1}{32}\tau^{*2}-\frac{3c}{32}(\tau^*-\tau)\\ &\quad +\frac{18}{32^2}(\tau^*-\tau)^2+\frac{1}{8}(A_{14}-A_{41})^2.\end{aligned} \tag{3.28}$$

Also, from Lemma 3.1 together with (3.15), (3.18), (3.20), (3.23), (3.24), (3.25), (3.26) and (3.27) we have

$$\|R\|^2=12c^2+\frac{5}{64}(\tau^*-\tau)^2+2(A_{14}-A_{41})^2+2(\|\rho^*\|^2-\frac{1}{4}\tau^{*2}). \tag{3.29}$$

From (3.29) we see that $(A_{14}-A_{41})^2$ is a function on M_0, and hence from (3.28), $\sum_{a,b} R_{1a1b}^2$ is independent with the choice of e_1. Thus we see that M_0 is a 2-stein space. Further, since $(A_{14}-A_{41})^2$ is a function on M_0, $(A_{14}-A_{41})^2$ is independent with the choice of a unitary basis $\{e_i\}=\{e_1, e_2=Je_1, e_3, e_4=Je_3\}$. Thus, from (2.8), (3.18), (3.20) and (3.23) we have

$$\begin{aligned}(A_{14}-A_{41})^2 &= \{g(\cos te_1+\sin te_2, (\nabla_{e_4}N)(\cos te_1+\sin te_2, e_3))\\ &\quad -g(e_4, (\nabla_{\cos te_1+\sin te_2}N)(\cos te_1+\sin te_2, e_3))\}^2\\ &= (A_{14}-A_{41})^2\cos^2 2t\end{aligned}$$

for all real number t. So, we see that

$$A_{14}-A_{41}=0,$$

and hence from (3.28) and (3.29) respectively we have

$$(3.30)\qquad \sum_{a,b} R_{1a1b}^2 = \frac{9}{8}c^2 + \frac{1}{8}\|\rho^*\|^2 - \frac{1}{32}\tau^{*2} - \frac{3c}{32}(\tau^* - \tau) + \frac{18}{32^2}(\tau^* - \tau)^2$$

and

$$(3.31)\qquad \|R\|^2 = 12c^2 + \frac{5}{64}(\tau^* - \tau)^2 + 2(\|\rho^*\|^2 - \frac{1}{4}\tau^{*2}).$$

Now, let f denote the smooth function on M_0 defined by

$$f = \sum_{a,b} R_{1a1b}^2.$$

Then for any tangent vector $x \in T_pM_0$ $(p \in M_0)$ we have

$$(3.32)\qquad \sum_{a,b} R_{xaxb}^2 = f(p)g(x,x)g(x,x),$$

where we set $R_{xaxb} = g(R(x,e_a)x, e_b)$. By taking twice Euclidean Laplacian in T_pM in both members of (3.32), we have

$$(3.33)\qquad f(p) = \frac{1}{96}\tau^2 + \frac{1}{16}\|R\|^2.$$

Thus from (3.30), (3.31) and (3.33) we have

$$(3.34)\qquad \frac{3}{8}c^2 - \frac{1}{96}\tau^2 + \frac{13}{32^2}(\tau^* - \tau)^2 - \frac{3c}{32}(\tau^* - \tau) = 0.$$

At last, from (2.9) and (3.34) we have

$$\frac{7}{32^2}(\tau^* - \tau)^2 = 0,$$

and hence $\tau^* - \tau = 0$ on M_0. This is a contradiction. Therefore we conclude that M_0 is empty, and we have proved our Main Theorem.

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Department of Mathematics, Niigata University, Niigata 950-21, Japan

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