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A CONSEQUENCE OF THE LAURENT DECOMPOSITION

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ABSTRACT. Given an analytic function f outside a compact set in the complex plane, there exists a unique entire function g such that $g - f$ tends to 0 at the point at infinity.

In the complex plane, consider $f(z) = \frac{z^4+2}{z^2-1}$ which is analytic in $|z| > 1$. Let $g(z) = z^2 + 1$ which is an entire function. Remark that $|f(z) - g(z)| = \frac{3}{|z^2-1|} \to 0$ when $|z| \to \infty$. Another such example is $\frac{e^z}{z}$ which is analytic outside the origin. If $g(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}$, then $g(z)$ is an entire function such that $|f(z) - g(z)| = \frac{1}{|z|} \to 0$ when $|z| \to \infty$. These examples prompt the question: for any analytic function $f(z)$ defined outside a compact set in the complex plane, can we construct an entire function $g(z)$ such that $|f(z) - g(z)| \to 0$ when $|z| \to \infty$? We show that the answer is yes by making use of the following generalized version of the Laurent theorem. For the classical representation of this series development in an annular region, see for example [3, Section 2.4].

Theorem 0.1. *In the complex plane, let K be a compact set and ω be an open set such that $K \subset \omega$. Assume that there is an open disc D such that $K \subset D \subset \overline{D} \subset \omega$. Suppose f is analytic on $\omega \setminus K$. Then $f = f_1 + f_2$ on $\omega \setminus K$, where f_1 is analytic on ω and f_2 is analytic on K^c tending to 0 at the point at infinity. This decomposition is unique.*

Proof. Without loss of generality, let us assume that the disc $D = \{z : |z| < r\}$. Let r_1 and r_2 be numbers such that $r_2 < r < r_1$, $K \subset \{z : |z| < r_2\}$ and $\{z : |z| < r_1\} \subset \omega$. By Laurent series expansion, $f(z) = s_1(z) + s_2(z)$ for $r_2 < |z| < r_1$, where s_1 is analytic on $|z| < r_1$ and s_2 is analytic on $|z| > r_2$

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with a removable singularity at the point at infinity. Let $\lim_{z\to\infty} s_2(z) = \alpha$. Write $t_1 = s_1 + \alpha$ and $t_2 = s_2 - \alpha$. Then $f = t_1 + t_2$ for $r_2 <| z |< r_1$ and t_2 tends to 0 at the point at infinity.

Define

$$f_1 = \begin{cases} f - t_2 & \text{if } \mid z \mid > r_2 \text{ and } z \in \omega \\ t_1 & \text{if } \mid z \mid < r_1. \end{cases}$$

Then f_1 is a well-defined analytic function on ω.

Define

$$f_2 = \begin{cases} f - t_1 & \text{if } \mid z \mid < r_1 \text{ and } z \notin K \\ t_2 & \text{if } \mid z \mid > r_2. \end{cases}$$

Then f_2 is analytic on K^c, tending to 0 at the point at infinity, Moreover, $f = f_1 + f_2$ on $\omega \setminus K$.

For the uniqueness, note that if $f = h_1 + h_2$ is another such decomposition, then

$$\varphi = \begin{cases} f_1 - h_1 & \text{on } \omega \\ h_2 - f_2 & \text{on } K^c \end{cases}$$

is an entire function tending to 0 at the point at infinity. Hence $\varphi \equiv 0$. □

A consequence: Let f be and analytic function defined outside a compact set in the complex plane. Then there exists a unique entire function g such that $g - f$ tends to 0 at the point at infinity.

Proof. The uniqueness of g is easy to see. For, if g_1 and g_2 are two entire functions such that $g_1 - f$ and $g_2 - f$ tend to 0 at the point at infinity, then $g_1 - g_2$ tends to 0 at the point at infinity and hence $g_1 - g_2 \equiv 0$.

To prove the existence of g, let f be defined as an analytic function outside a compact set K. Let D be an open disc containing K. Then, by the above theorem, $f = f_1 + f_2$ on $D \setminus K$, where f_1 is analytic on D and f_2 is analytic on K^c tending to 0 at the point at infinity.

Define

$$g = \begin{cases} f - f_2 & \text{if } z \notin K \\ f_1 & \text{if } z \in D. \end{cases}$$

Then, g is a well-defined entire function such that $g - f$ tends to 0 at the point at infinity. □

However, an attempt to generalize the above result in the following form turns futile: Given a (real -valued) harmonic function u outside a compact set in the complex plane, is it possible to find a harmonic function v on the whole plane such that $| u(z) - v(z) | \to 0$ when $| z | \to \infty$? For example, consider $u(z) = \log | z |$ which is harmonic outside the origin. Assume that there is some harmonic function v on the whole plane such that $| u(z) - v(z) | \to 0$ when $| z | \to \infty$. Then, this assumption leads to a contradiction. For, if such a function v were to exist, then v should be lower bounded on the plane and hence by the maximum principle it should be a constant α. This means that $\alpha - \log | z | \to 0$ when $| z | \to \infty$. This is not possible.

Let ω be a bounded open set in $\mathbb{R}^2$ and Ω be an open set containing ω. Let f be a C^2 (continuously twice differentiable) function on Ω. If $\frac{\partial g}{\partial n^+}(s)$ denotes the outer normal derivative at a point s on $\partial\omega$, then $\int\limits_{\partial\omega} \frac{\partial g}{\partial n^+}(s)\, ds$ is defined as the outward flux of g on ω. As a particular case of the Green's Formula, we see that $\iint\limits_{\omega} \Delta g(x) = \int\limits_{\partial\omega} \frac{\partial g}{\partial n^+}(s)\, ds$.

Suppose $h(z)$ is a harmonic function defined on $| z |> R$. Let a, b be two positive numbers larger than R. Since $\Delta h(z) = 0$ when $| z |> R$, we obtain from the Green's Formula on the annulus $\omega = \{z : a <| z |< b\}$, $\int\limits_{|s|=a} \frac{\partial h}{\partial n^-}(s)\, ds + \int\limits_{|s|=b} \frac{\partial h}{\partial n^+}(s)\, ds = 0$. This implies that $\int\limits_{|s|=a} \frac{\partial h}{\partial n^+}(s)\, ds = \int\limits_{|s|=b} \frac{\partial h}{\partial n^+}(s)\, ds$. Since a and b are arbitrary, the constant $\lambda = \int\limits_{|s|=r} \frac{\partial h}{\partial n^+}(s)\, ds$ is independent of $r\,(> R)$. We define λ as the flux at infinity of h.

Suppose H is a harmonic function on $\mathbb{R}^2$. Then $\int\limits_{|s|=r} \frac{\partial H}{\partial n^+}(s)\, ds = 0$ for any $r > 0$. Hence, the flux at infinity of H is 0. Suppose b is a bounded harmonic function defined outside a compact set in $\mathbb{R}^2$. Then, $b(z)$ tends to a finite limit when $\mid z \mid\rightarrow\infty$. Hence, using the inversion which preserves the harmonicity, $b(z)$ can be considered as a function harmonic at the point at infinity also (Brelot $[4, p.195]$) so that the flux at infinity of b is 0.

Let $u(z)$ be a harmonic function outside a compact set in the plane, then we can find a harmonic function $f(z)$ on the whole plane, a unique real number λ and a bounded harmonic function $b(z)$ outside a compact set such that $u(z) = \lambda \log \mid z \mid + f(z) + b(z)$ outside a compact set. The constant λ is the flux at infinity of the function u. (Brelot $[4, p.194]$ has proved this result, by using a power series expansion of harmonic functions on the plane; in Anandam [1], this result is deduced from a general result in a locally compact space. See also, Axler $[2, p.173]$)

Actually we can prove the following: Let $u(z)$ be a harmonic function defined outside a compact set in the plane. Then there exists a harmonic function $v(z)$ in the plane such that $\mid u(z) - v(z) \mid\rightarrow 0$ when $\mid z \mid\rightarrow\infty$ if and only if the flux at infinity of u is 0.

For, if u is harmonic off some compact set and α is its flux, then u is of the form $u(z) = \lambda \log \mid z \mid + f(z) + b(z)$. Since $b(z)$ is a bounded harmonic function outside a compact set, $b(z) \rightarrow \beta$, a finite value, when $\mid z \mid\rightarrow\infty$. Consequently, if the flux $\alpha = 0$, then the harmonic function $v(z) = f(z) + \beta$ defined on the whole plane is such that $\mid u(z) - v(z) \mid\rightarrow 0$ when $\mid z \mid\rightarrow\infty$. Clearly $v(z)$ is unique, from the minimum principle.

Conversely, suppose there exists a harmonic function $v(z)$ on the complex plane such that $\mid u(z) - v(z) \mid\rightarrow 0$ when $\mid z \mid\rightarrow\infty$. Then, by using the representation of $u(z)$, we see that $\mid \lambda \log \mid z \mid + [f(z) - v(z) + \beta] + [b(z) - \beta] \mid\rightarrow 0$ when $\mid z \mid\rightarrow\infty$. If $\lambda \neq 0$, this will mean that $\mid \log \mid z \mid + \frac{1}{\lambda}[f(z) - v(z) + \beta] \mid\rightarrow 0$ when $\mid z \mid\rightarrow\infty$, which we have just seen is not possible. Hence, the flux $\lambda = 0$.

Recall that when g is an entire function, the quantity $M(r, g)$ denotes $\max\limits_{|z|=r} \mid g(z) \mid$, and $ordg$ equals $\limsup\limits_{r\rightarrow\infty} \frac{\log\log M(r,g)}{\log r}$ (see, Hille $[6, p.182]$ or Copson $[5, p.165]$).

The above mentioned consequence permits us to define the order of an analytic function even when it is defined only outside a compact set, thus bringing out the fact that the notion of order of an entire function is introduced essentially to study the behavior of entire functions near the point at infinity. For, if $f(z)$ is an analytic function defined outside a compact set, then $f(z) = g(z) + b(z)$ in a neighborhood of the point at infinity, where $b(z)$ is bounded analytic outside a compact set tending to 0 at the point at infinity and $g(z)$ is a uniquely determined entire function. Hence, without ambiguity, we can define the order of $f(z)$ as $ordf = ordg$. When $ordf$ is finite, we can use the Hadamard's factorization (Copson $[5, p.174]$) for $g(z)$ to obtain a representation for $f(z)$ outside a compact set up to an additive bounded analytic function. As a simple example, we prove the following result.

Proposition 0.1. *Let $f(z)$ be an analytic function defined outside a compact set and $k \in \mathbb{N}$. Then there exists a polynomial $p(z)$ of degree $n \leq k$ such that $\mid f(z) - p(z) \mid \rightarrow 0$ when $\mid z \mid \rightarrow \infty$ if and only if $\mid f(z) \mid \leq A \mid z \mid^k$ for $\mid z \mid \geq R, R$ large.*

Proof. Write as before $f(z) = g(z) + b(z)$ outside a compact set. If $\mid f(z) \mid \leq A \mid z \mid^k$, then $\mid g(z) \mid \leq B \mid z \mid^k$. Then by Cauchy inequalities, $g(z)$ is a polynomial of degree $n \leq k$.

Conversely, if $p(z)$ is a polynomial of degree $n \leq k$, such that $\mid f(z) - p(z) \mid \rightarrow 0$ when $\mid z \mid \rightarrow \infty$, then $\mid f(z) \mid \leq \mid p(z) \mid +$a constant outside a compact set. Hence $\mid f(z) \mid \leq A \mid z \mid^n \leq A \mid z \mid^k$ for $\mid z \mid \geq R$, for large R. ☐

It is known (see, for example Titchmarsh $[7, p.284a]$) that if an entire function $g(z)$ does not take the value a, then a is an asymptotic value of g. (That is, there is a continuous curve from a given point to the point at infinity along which $g(z) \rightarrow a$ when $\mid z \mid \rightarrow \infty$). Consequently, the following proposition is easy to establish.

Proposition 0.2. *Let $f(z)$ be an analytic function defined outside a compact set. Then, either $\lim\limits_{z \rightarrow \infty} f(z)$ is finite or $f(z)$ has the asymptotic value ∞.*

Proof. Write $f(z) = g(z) + b(z)$ as before. If $f(z)$ is bounded, then the entire function $g(z)$ is bounded and hence a constant α. Since $\lim\limits_{z \rightarrow \infty} b(z) = 0$, we conclude $\lim\limits_{z \rightarrow \infty} f(z) = \alpha$. On the other hand, if $f(z)$ is not bounded , then $g(z)$ is a non-constant entire function and hence $\lim_{z\to\infty} g(z) = \infty$ along a continuous curve going to infinity. Consequently, $f(z)$ has the asymptotic value ∞. □

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