Arab J. Math. Sc.
Volume 13, December 2007
pp.15-26

NEW FUNCTION THEORETIC PROOFS OF BROWN-HALMOS THEOREMS

HOCINE GUEDIRI[1]

ABSTRACT. In this note, based on complex function theory, we present new elegant proofs to the famous Brown-Halmos theorems for Hardy space Toeplitz operators. As a byproduct, we provide an alternative characterization of Toeplitz operators on the Hardy space.

1. INTRODUCTION

The paper of Brown and Halmos [6] is fairly acknowledged by experts to be a main pillar of the modern theory of Toeplitz operators for at least two reasons: firstly, it studies extensively Hardy space Toeplitz operators and secondly, it offers insight into the study of Toeplitz operators on the Bergman space as well as on other function spaces. In that paper, they have investigated various properties of Toeplitz operators on the Hardy space, for more details we refer to [6, 9]. A natural question occurs at once, namely whether the set of all Toeplitz operators is a commutative algebra. It turns out that it is not closed neither under multiplication nor under commutation. Accordingly, Brown and Halmos established conditions on the symbols in order that the product of two Toeplitz operators is again a Toeplitz operator and for two Toeplitz operators to commute. Such results are known as Brown-Halmos theorems. Their original proofs were based on matrix methods. Stroethoff in [14, 15] presented new nice corresponding proofs based on the concept of the Berezin transform. A lot has been already done for the Bergman space versions of Brown-Halmos theorems, mainly by P. Ahern, S. Axler and Ž.

[1]Supported by College of Science-Research Center Project No. (Math/2007/27).
Mathematics Subject Classification: 47B35.
Key words: Toeplitz operator; Hardy space; Brown-Halmos; Intertwining relations.

Čučković [2, 3, 4, 7, 8] as well as many others, though the problem is not completely solved yet.

Recently, Stroethoff and Zheng [16, 17] investigated a new class of Toeplitz-type operators, which they called dual Toeplitz operators. They have proved many corresponding properties. Among their results, they have established Brown-Halmos type theorems for dual Toeplitz operators, whose wonderful proofs were based on function theory. They mentioned that their approach cannot be carried over to Toeplitz operators because there is no canonical transformation of Hankel products $H^*_{\overline{f}}H_g$ into rank-one operators such as the one used in [17]. For Bergman space Toeplitz operators, it is indeed unclear how such an approach can be adapted. However, for Hardy space Toeplitz operators, Stroethoff-Zheng's elegant technique can be successfully invoked with, of course, appropriate modifications. This probably provides insight into further techniques in the theory. In addition, we provide an alternative intertwining characterization for Toeplitz operators and we study an associated C*-algebra. Such task represents the aim of our present observation.

2. PRELIMINARIES

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and let $\partial\mathbb{D}$ be the unit circle. The Lebesgue space $L^2(\partial\mathbb{D})$ is the space of square integrable functions on $\partial\mathbb{D}$ and the Hardy space $\mathcal{H}^2(\partial\mathbb{D})$ is the closed Hilbert subspace of $L^2(\partial\mathbb{D})$ spanned by the analytic polynomials. The space of essentially bounded functions on $\partial\mathbb{D}$ is denoted by $L^\infty(\partial\mathbb{D})$. The space of bounded analytic functions, which is dense in $\mathcal{H}^2(\partial\mathbb{D})$, is denoted by $\mathcal{H}^\infty(\partial\mathbb{D})$. Note that the analyticity throughout has to be understood in the customary sense that the Poisson extension of the concerned function is analytic in $\mathbb{D}$. The orthogonal projection from $L^2(\partial\mathbb{D})$ onto $\mathcal{H}^2(\partial\mathbb{D})$ is denoted by $\mathcal{P}$. Denote the orthogonal projection from $L^2(\partial\mathbb{D})$ onto $\left(\mathcal{H}^2(\partial\mathbb{D})\right)^\perp$ by $\mathcal{Q} = I - \mathcal{P}$. Let $\mathcal{J}$ be the unitary operator on $L^2(\partial\mathbb{D})$ defined by $\mathcal{J}h(w) = \overline{w}h(\overline{w})$. $\mathcal{J}$ satisfies $\mathcal{J}^2 = I$ and $\mathcal{J}(I-\mathcal{P}) = \mathcal{P}\mathcal{J}$, see for instance [13, 18, 19]. For a symbol $f \in L^\infty(\partial\mathbb{D})$, the Toeplitz and Hankel operators on $\mathcal{H}^2(\partial\mathbb{D})$ are defined respectively by $T_f g = \mathcal{P}(fg)$ and $H_f g = \mathcal{J}(I-\mathcal{P})(fg)$, for all $g \in \mathcal{H}^2(\partial\mathbb{D})$. We shall need the following notation $f^*(z) = \overline{f(\overline{w})}$. It is well-known that f is analytic (co-analytic) if and only if

f^* is analytic (co-analytic).
The following crucial facts can be obtained by direct calculations, (see [13, 18]):

Lemma 2.1. *Let f and g be in $L^\infty(\partial\mathbb{D})$. Then, one has*

(1) $H_f^* = H_{f^*}$.
(2) $H_{\overline{f}}^* H_g = T_{\overline{f}g} - T_{\overline{f}} T_g$.
(3) $T_f^* H_g = H_g T_{f^*}$ *and* $H_g^* T_f = T_{\overline{f^*}} H_g^*$, *if* $f \in \mathcal{H}^\infty(\partial\mathbb{D})$.

For f and g in $L^2(\partial\mathbb{D})$, consider the rank one operator defined by $(f \otimes g)h =< h, g > f, \forall f \in L^2(\partial\mathbb{D})$. Note that $\|f \otimes g\| = \|f\| \, \|g\|$. For $w \in \mathbb{D}$ and $z \in \partial\mathbb{D}$, let $k_w(z)$ be the normalized reproducing kernel of $\mathcal{H}^2(\partial\mathbb{D})$ defined by

$$k_w(z) = \frac{\sqrt{1-|w|^2}}{1-\overline{w}z},$$

and let φ_w be defined by

$$\varphi_w(z) = \frac{w-z}{1-\overline{w}z}.$$

It is easily seen that φ_w is in $\mathcal{H}^\infty(\partial\mathbb{D})$ and that $(\varphi_w)^{-1} = \varphi_w$. Moreover, it is an inner function, i.e. $|\varphi_w| = 1$, and $\varphi_w^* = \varphi_{\overline{w}}$. The Toeplitz product $T_{\varphi_w} T_{\overline{\varphi_w}}$ is the orthogonal projection onto $\mathcal{H}^2(\partial\mathbb{D}) \ominus \{k_w\}$, [13, 19]. Thus $I - T_{\varphi_w} T_{\overline{\varphi_w}}$ is the rank one operator $k_w \otimes k_w$. In other words, we have

$$k_w \otimes k_w = I - T_{\varphi_w} T_{\overline{\varphi_w}}. \tag{2.1}$$

For a linear operator $\mathcal{S}$ on $\mathcal{H}^2(\partial\mathbb{D})$ and $w \in \mathbb{D}$, define the linear operator $S_w(\mathcal{S})$ by

$$S_w(\mathcal{S}) = \mathcal{S} - T_{\overline{\varphi_w^*}} \mathcal{S} T_{\varphi_w^*}. \tag{2.2}$$

From (ii) of Lemma 2.1 it follows that the commutator $[T_f, T_g]$ is given by

$$[T_f, T_g] = H_{\overline{f}}^* H_g - H_{\overline{g}}^* H_f. \tag{2.3}$$

From Equation (2.1), we see that

$$H_{\overline{f}}^*(k_w \otimes k_w) H_g = H_{\overline{f}}^* \left(I - T_{\varphi_w} T_{\overline{\varphi_w}} \right) H_g = H_{\overline{f}}^* H_g - H_{\overline{f}}^* T_{\varphi_w} T_{\overline{\varphi_w}} H_g. \tag{2.4}$$

Using part (iii) of Lemma 2.1 and Equation (2.2), we see that Equation (2.4) reduces to

$$H_{\overline{f}}^*(k_w \otimes k_w) H_g = H_{\overline{f}}^* H_g - T_{\overline{\varphi_w^*}} H_{\overline{f}}^* H_g T_{\varphi_w^*} = S_w \left(H_{\overline{f}}^* H_g \right). \tag{2.5}$$

Similarly, we obtain

$$H_{\bar{g}}^{*}(k_w \otimes k_w)H_f = S_w\left(H_{\bar{g}}^{*}H_f\right). \tag{2.6}$$

Combining Equations (2.3), (2.5) and (2.6), we obtain

$$H_{\bar{f}}^{*}(k_w \otimes k_w)H_g - H_{\bar{g}}^{*}(k_w \otimes k_w)H_f = S_w\left([T_f, T_g]\right). \tag{2.7}$$

On the other hand, for operators $\mathbf{T}$ and $\mathbf{S}$, we can easily verify that

$$\mathbf{T}^{*}(f \otimes g)\mathbf{S} = \mathbf{T}^{*}f \otimes \mathbf{S}^{*}g.$$

Finally, combining the latter with Equation (2.7), we obtain

$$H_{\bar{f}}^{*}k_w \otimes H_g^{*}k_w - H_{\bar{g}}^{*}k_w \otimes H_f^{*}k_w = S_w\left([T_f, T_g]\right). \tag{2.8}$$

3. INTERTWINING CHARACTERIZATIONS OF TOEPLITZ OPERATORS

A given matrix is said to be Toeplitz if it is constant along diagonals parallel to the main one. The famous Brown-Halmos characterization of Toeplitz operators asserts that: *an operator T on $\mathcal{H}^2(\partial\mathbb{D})$ is Toeplitz if and only if its matrix with respect to the generic basis is a Toeplitz matrix.* An alternative characterization in terms of an operator equation involving the shift operator reads as [6, 9]:

Theorem 3.1. *A bounded linear operator T on $\mathcal{H}^2(\partial\mathbb{D})$ is a Toeplitz operator if and only if*

$$\mathbf{S}^{*}T\mathbf{S} = T,$$

where $\mathbf{S}$ is the unilateral shift operator $\mathbf{S}f(z) = zf(z) = T_z f(z)$.

Note that for Bergman space Toeplitz operators there is no such characterization, (except in certain trivial cases), as pointed out by Engliš [10].

A Toeplitz operator is said to be analytic, (co-analytic), if its symbol is analytic, (co-analytic). The characterization of Theorem 3.1 restricted to analytic and co-analytic Toeplitz operators has the following reduced form:

Corollary 3.1.

(1) *A bounded linear operator T on $\mathcal{H}^2(\partial\mathbb{D})$ is an analytic Toeplitz operator if and only if it satisfies the operator equation $\mathbf{S}T = T\mathbf{S}$.*
(2) *A bounded linear operator T on $\mathcal{H}^2(\partial\mathbb{D})$ is a co-analytic Toeplitz operator if and only if it satisfies the operator equation $\mathbf{S}^*T = T\mathbf{S}^*$.*

The operator S_w introduced in the previous section reveals on a new characterization of Hardy space Toeplitz operators:

Theorem 3.2. *A bounded linear operator T on $\mathcal{H}^2(\partial\mathbb{D})$ is a Toeplitz operator if and only if*

$$S_w(T) = 0, \quad \textit{for all } w \in \mathbb{D}.$$

Proof. The only if part: fix a $w \in \mathbb{D}$ and consider a Toeplitz operator T_f on $\mathcal{H}^2(\partial\mathbb{D})$, with symbol $f \in L^\infty(\partial\mathbb{D})$. Since φ_w is bounded analytic, φ_w^* behaves similarly, whence $T_{\overline{\varphi_w^*}} T_f T_{\varphi_w^*} = T_{\overline{\varphi_w^*}} T_{f\varphi_w^*} = T_{\overline{\varphi_w^*} f \varphi_w^*} = T_{f|\varphi_w^*|^2} = T_f$, as $|\varphi_w^*| = 1$ on the circle. Therefore, we infer that $S_w(T_f) = T_f - T_{\overline{\varphi_w^*}} T_f T_{\varphi_w^*} = 0$.
The if part: Suppose that $S_w(T) = 0$, for all $w \in \mathbb{D}$. Then, in particular for $w = 0$, we obtain $T - T_{-\bar{z}} T T_{-z} = 0$, whence $T - T_z^* T T_z = 0$. By Theorem 3.1, (or directly by using the original matrix characterization), we see that T must be a Toeplitz operator. □

Notice that a result similar to that of Corollary 3.1 involving S_w is also valid.

Remark 3.1. Theorem 3.2 can be rephrased as follows:

$$T_f \text{ is a Toeplitz operator } \iff T_f \in \bigcap_{w \in \mathbb{D}} \ker S_w.$$

It seems to be interesting to see whether a less restricted characterization is valid, namely given $w \in \mathbb{D}$ (fixed): T_f is Toeplitz $\iff S_w(T_f) = 0$?
Also, it might be interesting to consider the following operator equation

$$T_{\overline{\varphi_w^*}} T T_{\varphi_w^*} = \lambda T,$$

for instance.

Denote by K the ideal of compact operators in the algebra $B\left(\mathcal{H}^2(\partial\mathbb{D})\right)$. Following the ideas of [11, 12], for fixed $w \in \mathbb{D}$ introduce the following set

$$F(T_{\varphi_w^*}) := \left\{T \in B\left(\mathcal{H}^2(\partial\mathbb{D})\right), \ S_w(T) \in K\right\}.$$

The following assertion characterizes $F(T_{\varphi_w^*})$:

Theorem 3.3.

(1) *In terms of commutators, we have*

$$F(T_{\varphi_w^*}) = \left\{T \in B\left(\mathcal{H}^2(\partial\mathbb{D})\right),\ \left[T, T_{\varphi_w^*}\right] \in K\right\}.$$

(2) $F(T_{\varphi_w^*})$ *is a proper* C^**-subalgebra of* $B\left(\mathcal{H}^2(\partial\mathbb{D})\right)$.

Proof. 1) First, observe that $I - T_{\overline{\varphi_w^*}}T_{\varphi_w^*} = T_{1-|\varphi_w^*|^2} = 0$ and $I - T_{\varphi_w^*}T_{\overline{\varphi_w^*}}$ is the rank one operator $k_{\overline{w}} \otimes k_{\overline{w}}$, whence they are both compact operators. Therefore, if $T - T_{\overline{\varphi_w^*}}TT_{\varphi_w^*} \in K$, then

$$T_{\varphi_w^*}\left(T - T_{\overline{\varphi_w^*}}TT_{\varphi_w^*}\right) = \left(T_{\varphi_w^*}T - TT_{\varphi_w^*}\right) + \left(I - T_{\varphi_w^*}T_{\overline{\varphi_w^*}}\right)TT_{\varphi_w^*} \in K.$$

Thus, we infer that $T_{\varphi_w^*}T - TT_{\varphi_w^*} = \left[T, T_{\varphi_w^*}\right] \in K$.

Conversely, if $T_{\varphi_w^*}T - TT_{\varphi_w^*} \in K$, then we see that $T_{\overline{\varphi_w^*}}\left(T_{\varphi_w^*}T - TT_{\varphi_w^*}\right) = T_{|\varphi_w^*|^2}T - T_{\overline{\varphi_w^*}}TT_{\varphi_w^*} = S_w(T) \in K$.

2) Clearly $F(T_{\varphi_w^*})$ is a linear self-adjoint subspace of $B\left(\mathcal{H}^2(\partial\mathbb{D})\right)$. Moreover, it is easy to see that the operator S_w is bounded on $B\left(\mathcal{H}^2(\partial\mathbb{D})\right)$, whence $F(T_{\varphi_w^*})$ is norm-closed. Besides, it is closed under multiplication because $\left[T_1T_2, T_{\varphi_w^*}\right] = T_1\left(T_2T_{\varphi_w^*} - T_{\varphi_w^*}T_2\right) + \left(T_1T_{\varphi_w^*} - T_{\varphi_w^*}T_1\right)T_2$. For example, for $w = 0$, as in [11], we see that $T = \mathsf{diag}(-1)^n$ is in $B\left(\mathcal{H}^2(\partial\mathbb{D})\right)$ but not in $F(T_z)$, whence the inclusion is proper. $\square$

4. BROWN-HALMOS THEOREMS

The first Brown-Halmos theorem characterizes commuting Hardy space Toeplitz operators. Its original proof [6] hinges on matrix methods. Stroethoff [14, 15] provided an alternative proof which makes use of the Berezin transform. The more involved Bergman space version is due to Axler and Čučković [3], see also [14].

Theorem 4.1. *Let* f *and* g *be in* $L^\infty(\partial\mathbb{D})$. *Then,* T_f *and* T_g *commute if and only if one of the following conditions holds:*

(1) *Both* f *and* g *are analytic.*

(2) *Both $\overline{f}$ and $\overline{g}$ are analytic.*
(3) *There exist constants α and β, not both zero, such that $\alpha f + \beta g$ is constant.*

Proof. The sufficiency of these conditions is straightforward (it immediately follows from elementary properties of Toeplitz operators). So, clearly it is sufficient to show the necessity of one of the conditions (i), (ii) and (iii) when T_f and T_g commute.
Suppose that T_f and T_g commute, then it follows from Equation (2.8) that

$$H_{\overline{f}}^* k_w \otimes H_g^* k_w = H_{\overline{g}}^* k_w \otimes H_f^* k_w, \ \forall w \in \mathbb{D}.$$

Since $k_0 = 1$, we obtain

$$\left(H_{\overline{f}}^* 1\right) \otimes \left(H_g^* 1\right) = \left(H_{\overline{g}}^* 1\right) \otimes \left(H_f^* 1\right),$$

which can be rewritten as

$$\left\langle h, H_g^* 1\right\rangle H_{\overline{f}}^* 1 = \left\langle h, H_f^* 1\right\rangle H_{\overline{g}}^* 1, \quad \forall\, h \in \mathcal{H}^2(\partial\mathbb{D}).$$

At this stage, we distinguish several cases:

- If $H_{\overline{f}}^* 1 \neq 0$ and $H_f^* 1 \neq 0$, then there exists a complex number λ, such that $H_{\overline{g}}^* 1 = \lambda H_{\overline{f}}^* 1$ and $H_g^* 1 = \overline{\lambda} H_f^* 1$. That is to say $\mathcal{Q}\left(\overline{g^*} - \lambda \overline{f^*}\right) = 0$ and $\mathcal{Q}\left(g^* - \overline{\lambda} f^*\right) = 0$; whence $\overline{g^*} - \lambda \overline{f^*}$ and $g^* - \overline{\lambda} f^*$ are analytic. Thus $g^* - \overline{\lambda} f^*$ is constant. It follows that a non-trivial linear combination of f and g, namely $g - \lambda f$, is constant, (which corresponds to condition (iii)).
- If $H_{\overline{f}}^* 1 = 0$, then $\overline{f^*}$ is analytic, i.e. f^* is co-analytic and either $H_f^* 1 = 0$ or $H_{\overline{g}}^* 1 = 0$, that is, f^* is also analytic, (in which case f^* is constant, which corresponds to condition (iii)), or g^* is also co-analytic, (which corresponds to condition (ii)).
- If $H_f^* 1 = 0$, then f^* is analytic and $H_g^* 1 = 0$ or $H_{\overline{f}}^* 1 = 0$, that is $\overline{f^*}$ is analytic, (in which case f^* is constant, which corresponds to condition (iii)), or g^* is analytic, (which corresponds to condition (i)). □

Theorem 4.1 has an important corollary on the characterization of normal Toeplitz operators. Before stating it, note that for the range of a complex-valued function to lie on a line in the complex plane, it is necessary and sufficient that there exist complex constants α, β and λ not all zero such that $\alpha f + \beta \overline{f} = \lambda$.

Corollary 4.1. *Let $f \in L^\infty(\mathbb{D})$. Then, the Toeplitz operator T_f is normal if and only if the range of f lies on a line.*

Proof. Since f and $\overline{f}$ cannot be simultaneously analytic or co-analytic unless f is constant, by Theorem 4.1 T_f and $T_f^* = T_{\overline{f}}$ commute if and only if there are constants α, β and λ not all zero such that $\alpha f + \beta \overline{f} = \lambda$. Then, we infer that T_f and T_f^* commute if and only if the range of f lies on a line. □

The second Brown-Halmos theorem answers the crucial question: when the product of two Toeplitz operators is again a Toeplitz operator? The Bergman space version was proved rather recently by Ahern and Čučković [2]. They pointed out that their approach can be extended to the Hardy space case too in the following way: since any f from $L^\infty(\partial\mathbb{D})$ can be written as $f = f_1 + \overline{f_2}$ with *BMOA* components f_1 and f_2, then invoking the method of Proposition 1 of [2], we see that if $T_f T_g = T_h$, then

$$f_1(z)g_1(z) + \overline{f_2}(z)\overline{g_2}(z) + f_1(z)\overline{g_2}(z) + \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|1-ze^{-i\theta}|^2}\overline{f}_2(\theta)g_1(\theta)d\theta = h(z).$$

Since the last term of the L.H.S. of the latter is the harmonic extension (Berezin transform) of $\overline{f_2}g_1$, it is harmonic in particular. Hence every term in the above equation is obviously harmonic except $f_1\overline{g_2}$. It follows that $f_1\overline{g_2}$ is harmonic too which implies that

$$\Delta\left(f_1\overline{g_2}\right) = \frac{\partial f_1}{\partial z}\frac{\partial \overline{g_2}}{\partial \overline{z}} = 0.$$

Since $\frac{\partial f_1}{\partial z}$ is analytic, we see that either $\frac{\partial f_1}{\partial z} = 0$ or $\frac{\partial \overline{g_2}}{\partial \overline{z}} = \overline{\frac{\partial g_2}{\partial z}} = 0$, whence f_1 is constant or g_2 is constant. In other words either f is co-analytic or g is analytic. Therefore, we arrive to the statement of the forthcoming Brown-Halmos theorem. In contrast to the Hardy space situation, on the Bergman space the Berezin transform does not always yield harmonic functions. Thus, a similar conclusion requires extra analysis, which has been impressively developed in the nice piece of art [2].

Theorem 4.2. *Let f and g be in $L^\infty(\partial\mathbb{D})$. Then, the product $T_f T_g$ is again a Toeplitz operator if and only if one of the following conditions holds:*

(1) *f is co-analytic.*
(2) *g is analytic.*

In which case $T_f T_g = T_{fg}$.

Proof. From the elementary properties of Toeplitz operators, the "if part" is obvious. While the "only if part" is less trivial. For, suppose that $T_f T_g = T_h$ for some $h \in$

$L^\infty(\partial\mathbb{D})$. From part (ii) of Lemma 2.1, we see that

$$T_h - T_f T_g = T_{h-fg} + H_{\overline{f}}^* H_g = 0.$$

So, introducing the operator S_w, we see from Theorem 3.2 that since $z \in \partial\mathbb{D}$ we have

$$S_w(T_{fg-h}) = S_w\left(H_{\overline{f}}^* H_g\right) = H_{\overline{f}}^* k_w \otimes H_g^* k_w = 0.$$

In particular for $w = 0$, we obtain $\left\|H_{\overline{f}}^* 1 \otimes H_g^* 1\right\| = \left\|H_{\overline{f}}^* 1\right\| \left\|H_g^* 1\right\| = 0$, whence $H_{\overline{f}}^* 1 = 0$ or $H_g^* 1 = 0$. Therefore, we have two possibilities

- If $H_{\overline{f}}^* 1 = 0$, then $\overline{f^*}$ is analytic, whence f is co-analytic, (which corresponds to (i)).
- If $H_g^* 1 = 0$, then g^* is analytic, whence g is analytic, (which corresponds to (ii)). □

A first corollary is about the so-called zero product problem:

Corollary 4.2. *A necessary and sufficient condition for the product $T_f T_g$ of two Toeplitz operators on the Hardy space $\mathcal{H}^2(\partial\mathbb{D})$ to be zero is that at least one factor be zero. In other words, among the class of Toeplitz operators there are no zero divisors.*

Proof. Clearly if $T_f = 0$ or $T_g = 0$, then $T_f T_g = 0$. Now, suppose $T_f T_g = 0$, then $T_f T_g$ is a Toeplitz operator with zero symbol. By Theorem 4.2 we see that either f is co-analytic or g is analytic and $T_f T_g = T_{fg} = 0$. Thus $fg = 0$ a.e. on $\mathbb{D}$. We distinguish several cases:

(1) If f or g is identically zero, then the problem is solved.

(2) If $f \not\equiv 0$ and $g \not\equiv 0$, then
  (a) If $fg = 0$ and g is analytic, then $g = 0$.
  (b) If $fg = 0$ and f is co-analytic, then $\overline{f}\overline{g} = 0$ and $\overline{f}$ is analytic. Therefore $\overline{f} = 0$.

Hence we infer that either $T_f = 0$ or $T_g = 0$. □

Yet, three more corollaries of Brown-Halmos Theorem 4.2 are listed below:

Corollary 4.3.

(1) *A Toeplitz operator T_f is an isometry if and only if f is a unimodular constant.*
(2) *A Toeplitz operator T_f is unitary if and only if f is a constant function of modulus 1.*

Proof. 1) If T_f is an isometry, then $T_f^*T_f = I$, i.e. $T_{\overline{f}}T_f = T_1$. By Theorem 4.2, this implies that f is analytic and $\overline{f}f = |f|^2 = 1$. Conversely if f is a constant function of modulus 1 then it is clear that $T_{\overline{f}}T_f = T_{\overline{f}f} = T_{|f|^2} = T_1 = I$. Therefore, T_f is an isometry.
2) If $T_f^* = T_{\overline{f}}$ and T_f are both isometries, then by (1) above f and $\overline{f}$ are analytic. Hence f is constant and $|f| = 1$. □

Remark 4.1. It turns out that even more is true, namely: there are no Toeplitz operators that are partial isometries other than the isometric ones T_f with $|f| = 1$, see [5]. Note that the Bergman space case was addressed by Čučković [7, 8]. Čučković's approach makes use of the Toeplitz algebras structure theorems, and should work for the Hardy space situation as well.

Corollary 4.4. *The only idempotent Toeplitz operators are the trivial ones, i.e. 0 and I.*

Proof. If $T_f^2 = T_f$, then $T_f^2 - T_f = T_f(T_f - I) = T_f(T_f - T_1) = T_f T_{f-1} = 0$. Corollary 4.2 implies $T_f = 0$ or $T_{f-1} = 0$. Thus $T_f = 0$ or $T_f = T_1 = I$. □

Corollary 4.5. *If a Toeplitz operator T_f is invertible, then T_f^{-1} is a Toeplitz operator if and only if f is analytic or f is co-analytic.*

Proof. If f is analytic then, by Theorem 3.2, for every $w \in \mathbb{D}$ we have $T_{\varphi_w^*}T_f = T_fT_{\varphi_w^*}$, which implies that

$$T_f^{-1}(T_{\varphi_w^*}T_f)T_f^{-1} = T_f^{-1}(T_fT_{\varphi_w^*})T_f^{-1}.$$

Hence, we obtain

$$T_f^{-1}T_{\varphi_w^*} = T_{\varphi_w^*}T_f^{-1}.$$

Now, Theorem 3.2, implies that T_f^{-1} must be a (analytic) Toeplitz operator. If f is co-analytic, then again by Theorem 3.2, we have $T^*_{\varphi_w^*}T_f = T_fT^*_{\varphi_w^*}$, which implies that

$$T_f^{-1}T^*_{\varphi_w^*} = T^*_{\varphi_w^*}T_f^{-1}.$$

From Theorem 3.2, we conclude that T_f^{-1} is a (co-analytic) Toeplitz operator. For the only if part, suppose that T_f^{-1} is a Toeplitz operator T_g say. Since $T_f^{-1}T_f = T_gT_f = I = T_1$, which is a Toeplitz operator, Theorem 4.2 implies that either f or $\overline{g}$ is analytic. On the other hand since we have $T_fT_f^{-1} = T_fT_g = I = T_1$, again by Theorem 4.2, we see that either g or $\overline{f}$ is analytic. Now, if $\overline{f}$ is analytic then the proof is complete. But if $\overline{f}$ is not analytic, then g must be analytic and non-constant (because if g is constant then $T_g = T_f^{-1} = \lambda I$ which means that $T_f = \frac{1}{\lambda}I$, i.e. $f = \frac{1}{\lambda}$ and $\overline{f} = \frac{1}{\lambda}$ which is analytic). Thus $\overline{g}$ is not analytic and hence f must be analytic, which completes the proof. □

ACKNOWLEDGEMENTS

The author would like to sincerely thank the referee for useful remarks.

REFERENCES

[1]

[2] AHERN, P. AND ČUČKOVIĆ, Ž.: *A theorem of Brown-Halmos type for Bergman space Toeplitz operators.* J. Funct. Anal., 187 (2001), 200- -210.

[3] AXLER, S. AND ČUČKOVIĆ, Ž.: *Commuting Toeplitz operators with harmonic symbols.* Integr. Equat. Oper. Th., 14 (1991), 1- -12.

[4] AXLER, S.; ČUČKOVIĆ, Ž. AND RAO, N.V.: *Commutants of analytic Toeplitz operators on the Bergman space.* Proc. Amer. Math. Soc., 128 (2000), 1951- -1953.

[5] BROWN, A. AND DOUGLAS, R.G.: *Partially isometric Toeplitz operators.* Proc. Amer. Math. Soc., 16 (4), (1965), 681- -682.

[6] BROWN, A. AND HALMOS, P.R.: *Algebraic properties of Toeplitz operators.* J. Reine Angew. Math. 213 (1963/64), 89- -102.

[7] Čučković, Ž.: *Commutants of Toeplitz operators on the Bergman space.* Pacific J. Math., 162 (1994), 277- -285.
[8] Čučković, Ž.: *Commutants of Toeplitz operators on the Bergman space.* (Ph.D. Thesis), Michigan State University, 1991.
[9] Douglas, R.G.: *Banach algebra techniques in operator theory.* Academic Press, New York, 1972.
[10] Engliš, M.: *A note on Toeplitz operators on Bergman spaces.* Comm. Math. Univ. Carolinae, 29 (1988), 217- -219.
[11] Engliš, M.: *Toeplitz operators on Bergman-type spaces.* Kandidatska disertacni prace, (Ph.D. thesis), Mu CSAV, Praha (Čezch Republic), 1991.
[12] Engliš, M.: *Density of algebras generated by Toeplitz operators on Bergman spaces.* Ark. Mat. 30 (2), (1992), 227- -243.
[13] Hamada, M.: *Remark on application of distribution function inequality for Toeplitz and Hankel operators.* Hokkaido Math. J. 32 (2003), 193- -208.
[14] Stroethoff, K.: *The Berezin transform and operators on spaces of analytic functions.* In "Linear Operators" (J. Zem ánek, Ed.), Banach Center Publications, Vol. 38, pp. 361- -380, Polish Academy of Sciences, Warsaw, 1997.
[15] Stroethoff, K.: *Algebraic properties of Toeplitz operators on the Hardy space via the Berezin transform.* Contemporary Math. 232 (1999), 313- -319.
[16] Stroethoff, K. and Zheng, D.: *Products of Hankel and Toeplitz operators on the Bergman space.* J. Funct. Anal. 169, (1999), 289- -313.
[17] Stroethoff, K. and Zheng, D.: *Algebraic and spectral properties of dual Toeplitz operators.* Trans. Amer. Math. Soc. 354 (6), (2002), 2495- -2520.
[18] Yoshino, T.: *The conditions that the product of Hankel operators is also a Hankel operator.* Arch. Math. 73 (1999), 146- -153.
[19] Zheng, D.: *The distribution function inequality and products of Toeplitz operators and Hankel operators.* J. Funct. Anal. 138 (1996), 477- -501.

Department of Mathematics

College of Science

King Saud University

P.O. Box 2455, Riyadh 11451

Saudi Arabia

Email: hguediri@ksu.edu.sa

Date received April 4, 2007