Arab J. Math. Sc.
Volume 3, Number 1, June 1997
pp. 21-36

DERIVATIVES OF MATRIX ORDER

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ABSTRACT. In this paper we provide a generalization of fractional calculus and define the derivative of order D^M for M, a square matrix. Some examples are given to clarify the algebra. As an application we generalize Legendre's equation.

1. INTRODUCTION

Fractional calculus is now a well-developed discipline and has many applications in various fields such as quantitative biology, electro-chemistry, scattering theory, diffusion and elasticity, etc. For a detailed account of the development of fractional calculus and its applications, we refer to [7, 8, 9]. To mention briefly, fractional calculus has its origin in the question of the extension of meaning. A well-known example is the extension of meaning of real numbers to complex numbers. In generalized integration and differentiation, the question of extension of meaning is: can the meaning of derivatives of integral order $\frac{d^n y}{dx^n}$ be extended to have meaning where n is any number ... irrational or fraction? The first textual mention of a derivative of fractional order appeared in 1819 in a text by a French mathematician S.F. Lacroix. In 1823, Abel applied the fractional calculus in the solution of an integral equation which arises in the formulation of the tautochrone problem. In 1832, Liouville made the first major attempt to give a logical definition of a fractional derivative and applied it to problems in potential theory. Later Riemann developed the theory of fractional integration and as a result Riemann-Liouville's definition of fractional calculus emerged.

1991 Mathematics Subject Classification. Primary 26A33; Secondary 33C45; 15A24.

We mention Riemann-Liouville definition in some specific cases for the reader's interest and for more details we refer to [7, 8, 9]:

$$\text{(1.1)} \qquad \frac{d^v}{dx^v}(x^a) = D^v(x^a) = \frac{\Gamma(a+1)}{\Gamma(a-v+1)}x^{a-v}$$

$$\text{(1.2)} \qquad \frac{d^v}{dx^v}(x^{-a}) = D^v(x^{-a}) = \frac{(-1)^v\Gamma(a+v)}{\Gamma(a)}x^{-a-v}$$

where a is any positive real number and v is any real number.

For a fractional derivative of the logarithmic function, we have

$$\text{(1.3)} \qquad D^v(\ln ax) = -e^{-\pi iv}x^{-v}\Gamma(v) = -(-1)^v x^{-v}\Gamma(v).$$

For example,

$$\text{(1.4)} \qquad D^{1/2}(\ln x) = -(-1)^{1/2}x^{-1/2}\Gamma(1/2) = -i\sqrt{\pi}x^{-1/2}.$$

Then using (1.2),

$$\begin{aligned} D^{1/2}(x^{-1/2}) &= \frac{(-1)^{1/2}\Gamma(1)}{\Gamma(1/2)}x^{-1/2-1/2} = \frac{i}{\sqrt{\pi}}x^{-1}, \\ D^{1/2}(D^{1/2}\ln x) &= D^{1/2}(-i\sqrt{\pi}x^{-1/2}) = -i\sqrt{\pi}\frac{i}{\sqrt{\pi}}x^{-1} = x^{-1} \\ &= D(\ln x) \end{aligned}$$

as must be expected for consistency.

Another generalization of the (scalar) differential calculus is the matrix calculus which plays an important role in fields such as multivariate analysis where the scalar calculus has its limitations. For a full account of the matrix calculus and its applications we refer to the book of Graham [4] where further references are given. The book contains natural concepts in the context of matrix calculus such as derivative of scalar functions of a matrix with respect to a matrix, the derivatives of the powers of a matrix and the derivative of a matrix with respect to a matrix (see also Vetter [11]).

The main purpose of this note is to develop an analog of the fractional calculus in the framework of the matrix calculus. In section 2 we define

A^B where A or B are square matrices and outline some of its algebraic consequences. The definition makes use of the eigenvalues and the associated projection matrices. Section 3 deals with derivatives. The notation $D^n = \frac{d^n}{dx^n}$ is given an interpretation in the case D^M where M is a square matrix. In section 4, we use D^M to generalize Legendre functions. In case M is not semisimple, this generalization reveals some new structure worthy of further Investigation.

2. MATRICES AS EXPONENTS AND BASES

In complex algebra we can define $a^b = e^{b \ln a} = e^{(\ln a)b}$. If a is a square matrix, then this definition is restricted to matrices with nonzero eigenvalues. If both a and b are square matrices of the same size then this definition is ambiguous because matrices do not necessarily commute. Our objective is to define A^B for a given pair of matrices A and B. Certainly, A^B is a function of A and B. The theory of functions of matrices deals with ordinary functions with real or complex arguments and values and defines the corresponding function of a square matrix exactly when the eigenvalues of the matrix are in the domain of the function (see e.g. [3]).

One way that any admissible function of a matrices can be determined is by their decomposition into eigenvalues and projectors as follows (see e.g. [3]). To begin with, for a semisimple matrix A, we have

$$A = \sum_j \lambda_j P_j$$

where λ_j are distinct eigenvalues, P_j are projections such that $P_j P_i = 0$ for $i \neq j$ and $\sum_j P_j = I$, where I is the identity. In fact, P_j projects onto the null space of $(A - \lambda_j I)^{m_j}$, an irreducible factor of the minimal polynomial in A. Thus for any function $f(z), z \in C$, we have $f(A) = \sum_j f(\lambda_j) P_j$ where P_j are determined as follows: Since $(zI - A)^{-1} = \sum_j \frac{1}{(z-\lambda_j)} P_j$, we have $P_j = \frac{1}{2\pi i} \oint \frac{dz}{(zI - A)}$, where the only eigenvalue inside the contour is λ_j. Calculation of $(zI - A)^{-1}$ can be done by Fadeev's algorithm as outlined below (see for instance [3, 5]):

For an $n \times n$ matrix A,

$$\begin{array}{lll} A = A_1 & q_1 = trA_1, & B_1 = A_1 - q_1 I, \\ AB_1 = A_2, & q_2 = \frac{1}{2} trA_2, & B_2 = A_2 - q_2 I, \\ AB_{n-1} = A_n, & q_n = \frac{1}{n} trA_n, & B_n = 0 \end{array}$$

Then $(zI - A)^{-1} = \dfrac{1}{\varphi(A)}(z^{n-1}I + z^{n-2}B_1 + \cdots + B_{n-1})$, where $\varphi(A) = z^n - \sum_{j=1}^{n} q_j z^{n-j}$ is the characteristic equation of A.

Definition 2.1. We define the matrix A^B as follows:

$$A^B = \sum_i A^{\lambda_i} S_i$$

where λ_i are the eigenvalues of B and S_i are the corresponding projection matrices. Of course, we must remember the domain of this "function". For example, if A is singular and B has a negative eigenvalue, the function is not defined.

The following example illustrates the definition of A^B. The example also gives a decomposition of matrices A and B.

Example 1. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$. Then,

$$A = (-1)R_1 + (4)R_2 = (1)\begin{pmatrix} 0.6 & -0.4 \\ -0.6 & 0.4 \end{pmatrix} + (4)\begin{pmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{pmatrix}$$

and

$$B = (1)S_1 + (2)S_2 = (1)\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} + (2)\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$$

where R_1, R_2 and S_1 and S_2 are the projectors. Now we can define A^B in the same way as we define the matrix functions, using the eigenvalues: $A^B = AS_1 + A^2 S_2$.

We have used the eigenvalues to determine the powers of A. The notation A^B indicates that A remains on the left while projectors of B

remain on the right. Similarly, we can define ${}^{B}A$ keeping the projectors of B on the left while A is on the right. Thus, we get

$$A^B = \begin{pmatrix} -13 & 16 \\ -19 & 24 \end{pmatrix} \quad \text{and} \quad {}^{B}A = \begin{pmatrix} 1 & 6 \\ 3 & 10 \end{pmatrix}.$$

We have used eigenvalues and the projectors of B when defining A^B and ${}^{B}A$. We now use the eigenvalues and projectors of A to define A^B as follows:

$$\begin{aligned} A &= R_1(-1) + R_2(4) \\ A^B &= R_1(-1)^B + R_2(4)^B \\ &= R_1((-1)^1 S_1 + (-1)^2 S_2) + R_2((4)^1 S_1 + (4)^2 S_2) \\ &= R_1(-S_1 + S_2) + R_2(4S_1 + 16S_2) \\ &= \begin{pmatrix} -13 & 16 \\ -19 & 24 \end{pmatrix}. \end{aligned}$$

Thus the two definitions of A^B coincide. Similar computations for ${}^{B}A$ yield that the two procedures lead to the same value. In fact, the two definitions are equivalent for matrices of higher order as well.

We observe that if A and B are any two semisimple matrices of the same size, it can be shown that A^B and ${}^{B}A$ have the same characteristic equation. Furthermore, if we denote the transpose of A as $\tilde{A}$ then we have $(A^B)^{\sim} = {}^{\tilde{B}}\tilde{A}$. In general, the additive law for exponents does not hold, but we do have $A^k A^B = A^{B+kI}$ for any complex number k. If A and B have a common eigenvector v with $Av = \lambda v$ and $Bv = \mu v$, then $A^B v = \lambda^\mu v$.

The interpretation of A^B in the semisimple case is the following: If the space is decomposed by the projectors of B, then A^{λ_j} acts on these subspaces where λ_j is the eigenvalue associated with the projector P_j. If the space is decomposed by the projectors of A (acting on the right with the space written as row vectors) then these subspaces are acted on by $(\mu_j)^B$ where μ_j is an eigenvalue of A.

Example 2 (Hyperbolic Numbers). One of the simplest examples of a Clifford algebra has been called the "hyperbolic numbers" by Sobczyk

[10] and "perplex numbers" by Fjelstad [2]. Hyperbolic numbers extend the real number system in much the same way as complex numbers do. Any hyperbolic number z can be written as $z = a + ub$ with a, b as real and u, a new quantity such that $u^2 = 1$. The hyperbolic number u is called the unipotent. Hyperbolic numbers serve as coordinates in the Lorentzian plane in much the same way as complex numbers serve as coordinates in the Euclidean plane. Sobczyk has observed that a hyperbolic number $a + ub$ has the matrix representation $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. We write $a + bu \cong \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. The decomposition of this matrix is:

$$\begin{aligned} \begin{pmatrix} a & b \\ b & a \end{pmatrix} &= (a+b)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + (a-b)\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= (a+b)E_1 + (a-b)E_2. \end{aligned}$$

Clearly, $u \cong (1)E_1 + (-1)E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Whether we regard u as a hyperbolic number or a matrix will be clear from the context. For example, we may write $E_1 = \frac{1}{2}I + \frac{1}{2}u$ and $E_2 = \frac{1}{2}I - \frac{1}{2}u$. Here we regard u as a matrix.

Now using the decomposition of matrices as above and the definition A^B, we obtain some additional properties of hyperbolic numbers as follows:

$$\begin{aligned} u^u &= u^{(1)}E_1 + u^{(-1)}E_2 \\ &= u\left(\frac{1}{2}I + \frac{1}{2}u\right) + u\left(\frac{1}{2}I - \frac{1}{2}u\right) \\ &= u. \end{aligned}$$

Since $1 - u = (0)E_1 + (2)E_2$, we get

$$(1-u)! = (0!)E_1 + (2!)E_2 = \frac{3}{2} - \frac{u}{2}.$$

Similar calculations give

$$i^u = iu, \; \ln u = \frac{i\pi}{2}(1-u), \quad u^i = \frac{1}{2}(1 + e^{-\pi} + (1 - e^{-\pi})u).$$

Just as quaternions become "hypercomplex" numbers when the field is taken to be complex numbers, rather than the reals, we have complexified the algebra of course, iu is no longer a hyperbolic number.

For a hyperbolic number $z = a + bu$, we have $z\bar{z} = (a+bu)(a-bu) = a^2 - b^2 = |z|^2$. So if $|z| = 0$, z does not have an inverse. In general,

$$(a+bu)^{-1} = \frac{1}{a+b}E_1 + \frac{1}{a-b}E_2.$$

We also note how factorials in the denominator behave:

$$\frac{1}{u!} = \frac{1}{1!}E_1 + \frac{1}{(-1)!}E_2 = E_1 = \frac{1}{2}(1+u).$$

Several other identities about hyperbolic numbers can be found by similar calculations

Example 3 (Quaternions). Quaternions may be interpreted as the algebra of certain two-by-two complex matrices. If we write

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for the unit quaternions, we find that

$$\begin{aligned} I^J &= \left(\cosh\frac{\pi}{2}\right) + \left(\sinh\frac{\pi}{2}\right)K, \\ {}^J I &= \left(\cosh\frac{\pi}{2}\right) - \left(\sinh\frac{\pi}{2}\right)K, \end{aligned}$$

the kind of behavior we might expect from quaternions.

We have seen how the hyperbolic numbers are analogous to the complex numbers. In the same way, the Klein group algebra can be regarded as analogous to the quaternions. Our methods and definition can be applied to the elements of the Klein group algebra as illustrated in the following:

Example 4 (Klein Group). The Klein group consists of unity and three commuting elements whose square is unity: $A^2 = B^2 = C^2 = 1$

and the product of any two matrices gives the third, e.g. $AB = C$. We can represent A, B and C as follows:

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ B &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ C &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

where $\otimes$ indicates the standard Kronecker product $P \otimes Q = [P_{ij}Q]$ (see e.g. [4]). For the sake of completeness we define the Kronecker product of matrices A and B as follows (see [4]):

Consider a matrix $A = [a_{ij}]$ of order $(m \times n)$ and a matrix B of order $(r \times s)$. The Kronecker product of the two matrices, denoted by $A \otimes B$ is defined as the partitioned matrix.

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

$A \otimes B$ is seen to be a matrix of order $(mr \times ns)$. For example, let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

then

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

Now define a group algebra by writing a "Klein number" as

$$\begin{aligned} K &= w + xA + yB + zC = (w - x - y + z)P + (w - x + y - z)Q \\ &\quad +(w + x - y - z)R + (w + x + y + z)S \end{aligned}$$

where P, Q, R and S are projections given by

$$\begin{aligned} P &= \frac{1}{4}(I - A - B + C), \quad Q = \frac{1}{4}(I - A + B - C) \\ R &= \frac{1}{4}(I + A - B - C), \quad S = \frac{1}{4}(I + A + B + C) \end{aligned}$$

That the elements P, Q, R and S are to be understood as elements of the group or matrices should be clear from the context.

We can now use our definition and method to find results like

$$B^A = B, \quad i^A = (1 - 2R) \quad \text{etc.}$$

So far we considered only semisimple matrices to define A^B and ${}^B A$. We now consider the case where matrices are not semisimple. It is well-known that any square matrix M decomposes into a sum: $M = S + N$, where S is semisimple, N is nilpotent, and $NS = SN$ [5]. A further decomposition of M implies that

$$M = \sum_j (\lambda_j P_j + N_j),$$

where $P_j N_j = N_j = N_j P_j$ and N_j are nilpotent. The projectors P_j can be determined as above and then $(M - \lambda_j I)P_j = N_j$ because the projectors P_j are mutually annihilating. The Taylor series expansion of an appropriate function f in the several-variable case implies

$$f(M) = \sum_j f(\lambda_j P_j + N_j) = \sum_j \left(f(\lambda_j)P_j + f'(\lambda_j)N_j + \frac{f''(\lambda_j)}{2!}N_j^2 + \cdots \right).$$

The sum $(f(\lambda_j)P_j + f'(\lambda_j)N_j + \cdots)$ terminates for each value of j because N_j are nilpotent. A more concise formula is

$$f(M) = \sum_j P_j e^{N_j D_\lambda} f(\lambda)|_{\lambda=\lambda_j}.$$

A simple illustration follows:

Example 5. $M = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Let $f(z) = z^3$. Then $f(2) = 8, f'(2) = 12$ and

$$f(M) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (8) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (12) \right] = \begin{pmatrix} 8 & 12 \\ 0 & 8 \end{pmatrix}.$$

Our definition of exponentiation goes through by using $g(z) = A^z$ and $g'(z) = (\ln A)A^z$, etc.

Example 6. With B as specified in Example 1 and $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have

$$\ln B = (0)S_1 + (\ln 2)S_2 = (\ln 2)\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}.$$

Then

$$\begin{aligned} B^E &= B^0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B^0(\ln 2)\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\ln 2)\begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\ln 2 \\ 0 & 1 - 2\ln 2 \end{pmatrix}. \end{aligned}$$

Similar calculations imply that

$${}^{E}B = \begin{pmatrix} 1 - 2\ln 2 & 2\ln 2 \\ 0 & 1 \end{pmatrix}.$$

3. DERIVATIVES OF MATRIX ORDER

In the fractional calculus, D^n is given an interpretation by letting n be a complex number, not merely an integer. We can extend this further by letting the "order" of the derivative be a square matrix. When the matrix is semisimple, this generalization is straightforward. For example, in the case of hyperbolic numbers we can use the decomposition

$$D^{a+bu} = D^{a+b}E_1 + D^{a-b}E_2$$

and find that with some obvious restrictions on a, b, c and d, the following holds:

$$D^{a+bu}x^{c+du} = \frac{(c+du)!}{(c-a+(d-b)u)!}x^{c-a+(d-b)u}.$$

This can be compared with the familiar differentiation formula $D^m x^n = \frac{n!}{(n-m)!}x^{n-m}$. But in the case when M is not semisimple, how are we to understand D^M? Suppose that $M = \sum_j \lambda_j P_j + N_j$. Then (as we saw before Example 5) that for $\lambda_j > -1$,

$$D^M = \sum_j D^{\lambda_j} P_j + D^{\lambda_j}(\ln D)N_j + \cdots = \sum_j P_j D^{\lambda_j}(e^{N_j \ln D})$$

So, it is a matter of determining $\ln D$. We begin with the familiar expression

$$\ln z = \lim_{\alpha\to 0} \frac{(z^\alpha - 1)}{\alpha}. \tag{$*$}$$

Before we define $\ln D$, it is convenient to give the following preliminary results (see Arfken [1, p. 550]),

$$\left[\begin{array}{ll} \dfrac{\partial}{\partial\alpha}(n-\alpha)! & = (n-\alpha)!\frac{\partial}{\partial\alpha}\ln((n-\alpha)!) \\ & = -(n-\alpha)!\left(-\gamma + \displaystyle\sum_{m\geq 1}\frac{(n-\alpha)}{m(m+n-\alpha)}\right) \end{array}\right] \tag{$**$}$$

where γ is the Euler constant, and

$$\sum_{m\geq 1}\frac{n}{m(m+n)} = \sum_{m\geq 1}\frac{1}{m} - \frac{1}{m+n} = \sum_{m=1}^{n}\frac{1}{m} = h_n \tag{$*\,*\,*$}$$

where h_n is the n-th harmonic number and we define $h_0 = 0$.

We now generalize $\ln z$ in (*). Using (**) and (***), we get

$$\begin{aligned} (\ln D)(x^n) &= \lim_{\alpha\to 0}\frac{D^\alpha - 1}{\alpha}x^n = \lim_{\alpha\to 0}\frac{x^n}{\alpha}\left(\frac{n!}{(n-\alpha)!}x^{-\alpha} - 1\right) \\ &= \lim_{\alpha\to 0} x^n \frac{\partial}{\partial\alpha}\left(\frac{n!}{(n-\alpha)!}x^{-\alpha}\right) \end{aligned}$$

$$\begin{aligned}
&= x^n(-\ln x) + x^n n! \lim_{\alpha\to 0} \frac{\partial}{\partial\alpha}\frac{1}{(n-\alpha)!} \\
&= x^n(-\ln x) - x^n n! \lim_{\alpha\to 0} \frac{1}{[(n-\alpha)!]^2}\frac{\partial}{\partial\alpha}(n-\alpha)! \\
&= x^n(-\ln x) + x^n n! \lim_{\alpha\to 0} \frac{1}{(n-\alpha)!}\left(-\gamma + \sum_{n\geq 1} \frac{(n-\alpha)}{m(m+n-\alpha)}\right) \\
&= x^n(-\ln x) + x^n \left(-\gamma + \sum_{m\geq 1}\frac{n}{m(m+n)}\right) \\
&= x^n(-\ln x - \gamma + h_n).
\end{aligned}$$

It immediately follows that $(\ln D)(1) = -(\ln x + \gamma)$. (I)

This definition of $\ln D$ is consistent with $D^A x^A = A!$, where A is a square matrix.

For example, let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $x^A = \begin{pmatrix} 1 & \ln x \\ 0 & 1 \end{pmatrix}$, $D^A = \begin{pmatrix} 1 & \ln D \\ 0 & 1 \end{pmatrix}$, $A! = \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & \ln D \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \ln x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}$.

In general, the factorial function $f(z) = z!$ has the derivative $f'(z) = z!\psi(z+1)$, where $\psi(z) = D_z(\ln\Gamma(z))$. For a positive integer n, we have $\psi(n+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma$ and $\psi(1) = -\gamma$. So, $\psi(n+1) = h_n - \gamma$. It is useful to note that $D^n(\ln D)(x^m) = (\ln D)(D^n x^m)$ for $n \leq m$.

4. AN APPLICATION

As an example of classical polynomials, we consider the Legendre polynomials indexed by non-negative integers $p_n(x)$. They are a solution to the differential equation

$$\mathcal{L}y = (x^2-1)D^2y + 2xDy = n(n+1)y.$$

The polynomials may be given by the Rodrigues formula

$$p_n(x) = \frac{1}{2^n n!}D^n(x^2-1)^n.$$

In the fractional calculus we may generalize n and find the functions

$$p_\nu(x) = \frac{1}{2^\nu \nu!} D^\nu (x^2 - 1)^\nu$$

which are a solution of

$$\mathcal{L}y = \nu(\nu + 1)y.$$

We can further generalize the Rodrigues formula by letting the index of the solution function be a semisimple matrix. For example, if we use the hyperbolic numbers, we can calculate $u + 1 = (2)E_1 + (0)E_2$ and

$$\begin{aligned} p_{u+1}(x) &= \frac{1}{2^{u+1}(u+1)!} D^{u+1}(x^2 - 1)^{u+1} \\ &= \left(\frac{3}{2}x^2 - \frac{1}{2}\right) E_1 + E_2 = p_2(x)E_1 + p_0(x)E_2, \end{aligned}$$

and

$$\mathcal{L}p_{u+1}(x) = (u + 1)(u + 2)p_{u+1}$$

Some reflection should make it clear that in the case of semisimple matrices it is a simple exercise. In fact, if $M = \sum_i \lambda_i Q_i$, (where Q_i are projectors), then using M in the Rodrigues formula we obtain $p_M(x) = \sum_i P_{\lambda_i}(x) Q_i$ which will be a solution of $\mathcal{L}_{y_M} = M(M + I)_{y_M}$.

Something more interesting happens in the case where M is not semisimple. Let us consider the archetypal case of two-by-two matrices

$$N = \begin{pmatrix} n & 1 \\ 0 & n \end{pmatrix}$$

with non-negative integer n. The Rodrigues formula gives

$$p_N(x) = \begin{pmatrix} p_n(x) & \square \\ 0 & p_n(x) \end{pmatrix}$$

where

$$\begin{aligned} \square &= \frac{1}{2^n n!} D^n[(x^2 - 1)^n \ln(x^2 - 1)] - (\ln 2)p_n(x) \\ &\quad - h_n p_n(x) + \gamma p_n(x) + (\ln D)(p_n(x)). \end{aligned}$$

On the other hand,

$$N(N+I)p_N(x) = \begin{pmatrix} n(n+1)p_n(x) & \square\square \\ 0 & n(n+1)p_n(x) \end{pmatrix}$$

where $\square\square = n(n+1)\square + (2n+1)p_n(x)$.

If we try these results in Legendre's equation we find that it does not quite work. The problem is the row 1, column 2 entry.

A tedious calculation gives

$$\mathcal{L}\square - \square\square = \frac{1}{x^2}(-1)^{n+1}t_n,$$

where t_n is the term of lowest degree in $p_n(x)$. As it happens, $\frac{1}{x^2}(-1)^{n+1}t_n = -D^2(\ln D)(t_n)$ (by I). This gives us a clue to a generalization of Legendre's equation. Note that $[D^2, \ln D]x^m = 0$ for $m \geq 2$, and $[D^2, \ln D]x^m = \frac{1}{x^2}(-1)^m x^m$ for $m = 0$ or 1. Using the commutator bracket notation $[X, Y] = XY - YX$, we have

$$\begin{pmatrix} \mathcal{L} & [D^2, \ln D] \\ 0 & \mathcal{L} \end{pmatrix} p_N(x) = N(N+I)p_N(x).$$

The generalization to non-integer eigenvalues is apparent (See [6] for a summary of properties of $\psi(z)$). The generalization for larger matrices is more detailed. However, at this point we want to draw attention to the fact that $[D^2, \ln D]$ seems to have little to do with Legendre's equation as such. In general, given a Rodriques formula and the corresponding differential equation, it seems likely that the generalization of the equation involves $[D^2, \ln D]$ as the nilpotent part of the matrix of differential operators. To conclude, it should be interesting to consider the cases of Laguere and Hermite functions as well as the general form of the Legendre's functions of the second kind. We could further generalize D as a matrix of partial derivatives.

ACKNOWLEDGMENT

The authors wish to acknowledge the support provided by King Fahd University of Petroleum and Minerals during this research. The authors are also grateful to the referee for suggestions leading to an improvement of the paper.

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Date received May 6, 1996.