Arab J. Math. Sc.
Volume 9, Number 1, June 2003
pp. 15-28

# ON THE GEOMETRY OF 3-CONTACT CR-SUBMANIFOLDS OF MANIFOLDS WITH GENERALISED 3-SASAKIAN STRUCTURE

Falleh R. Al-Solamy

ABSTRACT. In a recent paper [4] Bejancu and Farran introduced a new structure on manifolds called a generalised 3-Sasakian structure which is in fact a generalisation of a 3-Sasakian structure [10] on manifolds. The authors studied this structure locally. However, in this paper we consider this structure globally and we investigate the geometry of 3-contact CR-submanifolds of manifolds with a generalised 3-Sasakian structure.

## 1. INTRODUCTION

The differential geometry of CR-submanifolds has shown an increasing development since Bejancu defined and studied CR-submanifolds of a Kaehler manifold [2] as a natural generalisation of both invariant submanifolds and anti-invariant submanifolds. Since then many papers on CR-submanifolds in Kaehlerian, Sasakian, trans-Sasakian, nearly trans Sasakian, and Sasakian 3-structures have been published [6, 8, 9, 11, 12, etc.].

Recently in [4] Bejancu and Farran have introduced a new structure defined locally on manifolds called a generalised 3-Sasakian structure, which is a generalisation of 3-Sasakian structures [10]. The aim of this paper is to study what we call 3-contact CR-submanifolds of manifolds with generalised 3-Sasakian structure defined globally.

*Key words and pharases*: 3-contact CR-submanifolds, Sasakian 3-structures, Generalised 3-Sasakian structure, D-totally geodesic submanifold. MSC 2000: 53C40

In section 2, we review basic formulas and definitions which we shall use later. In section 3, we study integrability conditions of distributions and canonical parallel structure. In section 4, we obtain results on leaves of distributions in 3-contact CR-submanifold. We also study totally umbilical 3-contact CR-submanifold.

## 2. PRELIMINARIES

Let $\overline{M} = \overline{M}^{4m+3}$ be a manifold with almost contact metric 3-structures $(\phi_i, \xi_i, \eta_i, g)$ $(i = 1, 2, 3)$. On this manifold we have (cf.[10])

$$\eta_i(\xi_j) = \eta_j(\xi_i) = 0, \tag{2.1}$$

$$\phi_i \xi_j = -\phi_j \xi_i = \xi_k, \tag{2.2}$$

$$\eta_i \circ \phi_j = -\eta_j \circ \phi_i = \eta_k, \tag{2.3}$$

$$\phi_i \phi_j - \xi_i \otimes \eta_j = -\phi_j \phi_i + \xi_j \otimes \eta_i = \phi_k, \tag{2.4}$$

$$g(\phi_i X, \phi_i Y) = g(X, Y) - \eta_i(X)\eta_i(Y), \tag{2.5}$$

$$g(X, \xi_i) = \eta_i(X), \tag{2.6}$$

for any $X, Y$ tangent to $\overline{M}$ and cyclic permutation $(i, j, k)$ of $(1, 2, 3)$. $\overline{M}$ is called a manifold with generalised 3-Sasakian manifold if

$$(\overline{\nabla}_X \phi_i)(Y) = g(X, Y)\xi_i - \eta_i(Y)X + \alpha_{ij}(X)\phi_j Y + \alpha_{ik}(X)\phi_k Y, \tag{2.7}$$

$$\overline{\nabla}_X \xi_i = -\phi_i X + \alpha_{ij}(X)\xi_j + \alpha_{ik}(X)\xi_k, \tag{2.8}$$

where $\overline{\nabla}$ is the operator of covariant differentiation with respect to the metric $g$ on $\overline{M}$, $\alpha_{ij}$ are 1-forms on $\overline{M}$ and $\alpha_{ij} + \alpha_{ji} = 0$. We give an example of this structure in the end of this section.

Let $M$ be an $m$-dimensional isometrically immersed submanifold of a manifold $\overline{M}$ with gerenralised 3-Sasakian structure. Let us denote by the same $g$ the Riemannian metric tensor field induced on $M$ from that of $\overline{M}$. Through out this paper, we assume that $M$ is tangent to $\langle \xi_1, \xi_2, \xi_3 \rangle$.

**Definition 2.1.** A submanifold $M$ of a manifold $\overline{M}$ with a generalised 3-Sasakian structure is called a 3-contact CR-submanifold if there exists a differentiable distribution $D : x \in M \longrightarrow D_x \subset T_xM$ such that
**(1)** the distribution $D$ is invariant under $\phi_i$, i.e $\phi_i D \subset D$, $(i = 1, 2, 3)$.
**(2)** the complementary orthogonal distribution $D^\perp : x \in M \longrightarrow D_x^\perp \subset T_xM$ of $D$ is anti-invariant under $\phi_i$, i.e $\phi_i D_x^\perp \subset T_x^\perp M$, $(i = 1, 2, 3)$, where $T_xM$ and $T_x^\perp M$ are the tangent space and the normal space of $M$ at $x$ respectively.

For any vector field $X \in TM$, we put

$$\phi_i X = T_i X + F_i X, \tag{2.9}$$

where $T_iX$ (resp. $F_iX$) is the tangential (resp. normal) component of $\phi_i X$.

For any vector field $N \in T^\perp M$, we put

$$\phi_i N = t_i N + f_i N, \tag{2.10}$$

where $t_iN$ (resp. $f_iN$) denotes the tangential (resp. normal) component of $\phi_i N$.

Let $\overline{\nabla}$ (resp. $\nabla$) be the covariant differenitiation with respect to the Levi-Civita connection on $\overline{M}$ (resp. $M$). The Gauss and Weingarten formulas for $M$ are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \overline{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{2.11}$$

for $X, Y \in TM$ and $N \in T^\perp M$, where $h$ (resp. $A$) is the second fundamental form (resp. tensor) of $M$ in $\overline{M}$, and $\nabla^\perp$ denotes the normal connection. Moreover, we have

$$g(h(X, Y), N) = g(A_N X, Y). \tag{2.12}$$

**Definition 2.2.** A CR-submanifold $M$ is said to be D-totally geodesic if

$$h(X, Y) = 0, \qquad \text{for each } X, Y \in D.$$

Now, if we denote by $l$ and $m$ the projection operators corresponding to $D$ and $D^\perp$ respectively, then we have [9]

$$lm = ml = 0, \qquad l^2 = l, \qquad m^2 = m, \tag{2.13}$$

$$T_i m = 0, \qquad F_i l = 0. \tag{2.14}$$

We state the following Lemmas [9].

**Lemma 2.1.** *If $M$ is a 3-contact CR-submanifold. Then $\xi_i (i = 1, 2, 3)$ are in $D$.*

**Lemma 2.2.** *Let $M$ be a Submanifold of a manifold with almost contact 3-structure tangent to $\langle \xi_1, \xi_2, \xi_3 \rangle$. Then $M$ is a 3-contact CR-submanifold if and only if*

$$f_i F_i = 0 \quad \text{and} \quad f_i F_j = F_k, \tag{2.15}$$

*for any cyclic permutation $(i, j, k)$ of $(1, 2, 3)$.*

**Example:** Consider the Euclidean space $R^{4(n+1)}$ which has infinitely many quaternion structures coming from each copy of $R^4$ and their constant linear combinations, with coefficients being coordinates of points on $S^2$. In particular, $R^{4(n+1)}$ has a triplet of complex structures $(J_1, J_2, J_3)$ satisfying

$$J_i^2 = -I, \quad J_i \circ J_j = -J_j \circ J_i = J_k, \quad \langle J_i X, J_j Y \rangle = \langle X, Y \rangle$$
$$\left( \overline{\nabla}_X J_i \right)(Y) = a_{ij}(X) J_j Y + a_{ik}(X) J_k Y$$

for a cyclic permutation of $(i, j, k) = (1, 2, 3)$, where $\langle , \rangle$ is the Euclidean metric on $R^{4(n+1)}$, $\overline{\nabla}$ is the covariant derivative operator, $X, Y$ are smooth vector fields on the Euclidean space $R^{4(n+1)}$ and $a_{ij}$ are 1-forms satisfying $a_{ij} + a_{ji} = 0$.

Let $\overline{M}$ be an orientable hypersurface of $R^{4(n+1)}$ with unit normal vector field $N$. Then the Gauss and Wiengarten formulas for the hypersurface are

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \overline{\nabla}_X N = -AX$$

where $\nabla$ is the induced Riemanninan connection with respect to the Riemannian metric $g$ and $A$ is the Wiengarten map of the hypersurface $\overline{M}$. We define the structure $(\phi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$ on $\overline{M}$ by setting

$$J_i X = \phi_i X + \eta_i(X) N, \quad \eta_i(X) = g(X, \xi)$$

where $\phi_i X$ and $\eta_i(X)$ are tangential and normal components of $J_i(X)$ and $\xi_i$ is the unit vector field on $\overline{M}$ satisfying $J_i \xi_i = N$. It is easy to verify using the properties of $J_1, J_2, J_3$ that the structure $(\phi_i, \xi_i, \eta_i, g)$ satisfies the conditions (2.1)-(2.6). We shall show that on the unit sphere $\overline{M} = S^{4n+3}$, for which the Wiengarten map is given by $A = -I$ , this structure satisfies all the conditions (2.1)-(2.8), showing that $S^{4n+3}$ has generalized Sasakian 3-structure. Now taking smooth vector fields $X, Y$ on $\overline{M}$ and using $\left(\overline{\nabla}_X J_i\right)(Y) = a_{ij}(X)J_jY + a_{ik}(X)J_kY$ together with Gauss and Weingarten formulas after equating tangential and normal components, we arrive at

$$(\nabla_X\phi_i)(Y) = \eta_i(Y)AX - g(AX,Y) + a_{ij}(X)\phi_jY + a_{ik}(X)\phi_kY$$
$$g(\nabla_X\xi_i, Y) = g(\phi_iY, X) + a_{ij}(X)g(Y,\xi_j) + a_{ik}(X)g(Y,\xi_k)$$

which on taking $A = -I$ gives conditions (2.7) and (2.8).

## 3. INTEGRABILITY CONDITIONS OF DISTRIBUTIONS AND PARALLEL CANNONICAL STRUCTURES

First we prove the following lemma for later use.

**Lemma 3.1.** *Let $M$ be a 3-contact CR-submanifold of a manifold $\overline{M}$ with generalised 3-Sasakian structure. Then*

$$\nabla_X T_iY - A_{F_iY}X - T_i\nabla_X Y - t_ih(X,Y) = g(X,Y)\xi_i - \eta_i(Y)X + \alpha_{ij}(X)T_jY + \alpha_{ik}(X)T_kY, \tag{3.1}$$

$$h(X,T_iY) + \nabla^{\perp}_X F_iY - F_i\nabla_X Y - f_ih(X,Y) = \alpha_{ij}(X)F_jY + \alpha_{ik}(X)F_kY, \tag{3.2}$$

*for any $X, Y \in TM$.*

*Proof.* From (2.7), we have

$$\overline{\nabla}_X\phi_iY - \phi_i\overline{\nabla}_XY = g(X,Y)\xi_i - \eta_i(Y)X + \alpha_{ij}(X)\phi_jY + \alpha_{ik}(X)\phi_kY. \tag{3.3}$$

Now using the equations (2.9), (2.10) and (2.11), the above equation gives

$$\begin{aligned}\nabla_XT_iY + h(X,T_iY) - A_{F_iY}X + \nabla^{\perp}_XF_iY - T_i\nabla_X Y \\ -F_i\nabla_X Y - t_ih(X,Y) - f_ih(X,Y) = g(X,Y)\xi_i - \eta_i(Y)X + \alpha_{ij}(X)T_jY \\ + \alpha_{ij}(X)F_jY + \alpha_{ik}(X)T_kY + \alpha_{ik}(X)F_kY,\end{aligned} \tag{3.4}$$

for $X, Y \in TM$.

Comparing the tangential and normal parts in the above equation we get the result. □

Next we prove

**Lemma 3.2.** *Let $M$ be a 3-contact CR-submanifold of a manifold $\overline{M}$ with generalised 3-Sasakian structure. Then*

$$T_i[Y, Z] = A_{F_iY}Z - A_{F_iZ}Y, \tag{3.5}$$

*for any $Y, Z \in D^{\perp}$.*

*Proof.* From equation (3.1), we have

$$T_i(\nabla_Y Z) = -A_{F_iZ}Y - t_i h(Y, Z) - g(Y, Z)\xi_i, \tag{3.6}$$

for $Y, Z \in D^{\perp}$. Interchanging the vector field $Y$ and $Z$ in equation (3.6), we get

$$T_i(\nabla_Z Y) = -A_{F_iY}Z - t_i h(Y, Z) - g(Y, Z)\xi_i. \tag{3.7}$$

Subtracting (3.6) and (3.7), we get

$$T_i[Y, Z] = A_{F_iY}Z - A_{F_iZ}Y$$

for all $Y, Z \in D^{\perp}$. □

Now we prove

**Theorem 3.1.** *Let $M$ be a 3-contact CR-submanifold of a manifold $\overline{M}$ with generalised 3-Sasakian structure. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{F_iY}Z = A_{F_iZ}Y, \tag{3.8}$$

*for any $Y, Z \in D^{\perp}$.*

*Proof.* Let $D^{\perp}$ be integrable. Then for $Y, Z \in D^{\perp}$, we have $[Y, Z] \in D^{\perp}$. This implies that $T_i[Y, Z] = 0$. Then by Lemma (3.2), we have $A_{F_iY}Z = A_{F_iZ}Y$ for $Y, Z \in D^{\perp}$.

Conversely, let $A_{F_iY}Z = A_{F_iZ}Y$ for $Y, Z \in D^{\perp}$ which implies that $T_i[Y, Z] = 0$, for $Y, Z \in D^{\perp}$. This gives that $[Y, Z] \in D^{\perp}$. Therefore $D^{\perp}$ is integrable. □

Next we will study the integrability condition of the distribution $D$. For this, first we have

**Lemma 3.3.** *Let $M$ be a 3-contact CR-submanifold of a manifold $\overline{M}$ with generalised 3-Sasakian structure. Then*

$$F_i[X,Y] = h(X,T_iY) - h(Y,T_iX), \tag{3.9}$$

*for all $X,Y \in D$.*

*Proof.* From Equation (3.2), we have

$$h(X,T_iY)+\nabla_X^{\perp}F_iY - F_i\nabla_X Y - f_i h(X,Y) = \alpha_{ij}(X)F_jY+\alpha_{ik}(X)F_kY, \tag{3.10}$$

for $X,Y \in TM$. If $X,Y \in D$, then using the fact that $F_jY = 0$ for $Y \in D$, the above equation gives

$$F_i(\nabla_X Y) = h(X,T_iY) - f_i h(X,Y), \tag{3.11}$$

for $X,Y \in D$. Interchanging the vector field $X$ and $Y$ in (3.11), we get

$$F_i(\nabla_Y X) = h(Y,T_iX) - f_i h(X,Y), \tag{3.12}$$

for $X,Y \in D$. Subtracting (3.11) and (3.12), we obtain

$$F_i[X,Y] = h(X,T_iY) - h(Y,T_iX),$$

for all $X,Y \in D$, which gives the result. □

**Theorem 3.2.** *Let $M$ be a 3-contact CR-submanifold of a manifold $\overline{M}$ with generalised 3-Sasakian structure. Then the distribution $D$ is integrable if and only if*

$$h(X,T_iY) = h(Y,T_iX), \tag{3.13}$$

*for any $X,Y \in D$.*

*Proof.* Let $D$ be integrable. Then for $X,Y \in D$, we have $[X,Y] \in D$. This implies that $F_i[X,Y] = 0$. Then by Lemma (3.3), we have $h(X,T_iY) = h(Y,T_iX)$ for $X,Y \in D$.

Conversely, let $h(X,T_iY) = h(Y,T_iX)$ for $X,Y \in D$. Then by Lemma (3.3), we have $F_i[X,Y] = 0$ for $X,Y \in D$. This gives that $[X,Y] \in D$. Therefore $D$ is integrable. □

Before we close this section, we study 3-contact CR-submanifold with parallel cannonical structure. For this, let us define the covariant differentiation of $T_i, F_i, t_i$ and $f_i$ as follows :

$$(3.14) \qquad (\overline{\nabla}_X T_i)(Y) = \nabla_X T_i Y - T_i \nabla_X Y,$$

$$(3.15) \qquad (\overline{\nabla}_X F_i)(Y) = \nabla^{\perp}_X F_i Y - F_i \nabla_X Y,$$

$$(3.16) \qquad (\overline{\nabla}_X t_i)(N) = \nabla_X t_i N - t_i \nabla^{\perp}_X N,$$

$$(3.17) \qquad (\overline{\nabla}_X f_i)(N) = \nabla^{\perp}_X f_i N - f_i \nabla^{\perp}_X N,$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$.

**Definition 3.1.** The endomorphism $T_i$(resp. the endomorphism $f_i$, the 1-form $F_i$ and $t_i$) is called parallel if $\overline{\nabla} T_i = 0$ (resp. $\overline{\nabla} f_i = 0$, $\overline{\nabla} F_i = 0$ and $\overline{\nabla} t_i = 0$)

Now using (3.14) and (3.15) in Lemma 3.1 we have

$$(3.18) \qquad (\overline{\nabla}_X T_i)(Y) = A_{F_i Y} X + t_i h(X, Y) + g(X, Y)\xi_i - \eta_i(Y) X + \alpha_{ij}(X) T_j Y + \alpha_{ik}(X) T_k Y,$$

and

$$(3.19) \qquad (\overline{\nabla}_X F_i)(Y) = f_i h(X, Y) - h(X, T_i Y) + \alpha_{ij}(X) F_j Y + \alpha_{ik}(X) F_k Y,$$

for any $X, Y \in TM$.

Next, for $X \in TM$, $N \in T^{\perp}M$, equation (2.7) gives

$$(\overline{\nabla}_X \phi_i)(N) = \overline{\nabla}_X \phi_i N - \phi_i \overline{\nabla}_X N = \alpha_{ij}(X)\phi_j N + \alpha_{ik}(X)\phi_k N.$$

Using equations (2.9)-(2.11) in the above equation, we obtain

$$(3.20) \qquad \nabla_X t_i N + h(X, t_i N) - A_{f_i N} X + \nabla^{\perp}_X f_i N + T_i A_N X + F_i A_N X - t_i \nabla^{\perp}_X N - f_i \nabla^{\perp}_X N = \alpha_{ij}(X)(t_j N + f_j N) + \alpha_{ik}(X)(t_k N + f_k N),$$

for any $X, Y \in TM$, $N \in T^{\perp}M$.

Comparing tangential and normal components in the above equation and using (3.16), (3.17), we get

$$(3.21) \qquad (\overline{\nabla}_X t_i)(N) = A_{f_i N} X - T_i A_N X + \alpha_{ij}(X) t_j N + \alpha_{ik}(X) t_k N,$$

and

$$(3.22)\qquad (\overline{\nabla}_X f_i)(N) = -h(X, t_i N) - F_i A_N X + \alpha_{ij}(X) f_j N + \alpha_{ik}(X) f_k N,$$

for any $X, Y \in TM$, $N \in T^{\perp}M$.

Now we prove

**Proposition 3.1.** *Let $M$ be a 3-contact CR-submanifold of a manifold $\overline{M}$ with generalised 3-Sasakian structure. Then $t_i (i = 1, 2, 3)$ are parallel if and only if $F_i$ are parallel.*

*Proof.* Assume that $t_i (i = 1, 2, 3)$ are parallel, i.e $\overline{\nabla} t_i = 0$. Then from (3.21) it follows

$$g(A_{f_i N} X, Y) - g(T_i A_N X, Y) + \alpha_{ij}(X) g(t_j N, Y) + \alpha_{ik}(X) g(t_k N, Y) = 0,$$

which implies that

$$g(h(X, Y), f_i N) + g(h(X, T_i Y), N) - \alpha_{ij}(X) g(N, F_j Y) - \alpha_{ik}(X) g(N, F_k Y) = 0,$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$.

Thus we have

$$g(f_i h(X, Y), N) = g(h(X, T_i Y), N) - \alpha_{ij}(X) g(F_j Y, N) - \alpha_{ik}(X) g(F_k Y, N).$$

Transvecting $N$ on both sides, we get

$$(3.23)\qquad f_i h(X, Y) = h(X, T_i Y) - \alpha_{ij}(X) F_j Y - \alpha_{ik}(X) F_k Y.$$

From (3.19) and (3.23) it follows that $F_i (i = 1, 2, 3)$ are parallel. Converse part follows on the same line. □

**Lemma 3.4.** *Let $M$ be a 3-contact CR-submanifold of a manifold $\overline{M}$ with generalised 3-Sasakian structure Then*

$$(3.24)\qquad \nabla_X \xi_i = -T_i X + \alpha_{ij}(X) \xi_j + \alpha_{ik}(X) \xi_k,$$

$$(3.25)\qquad h(X, \xi_i) = -F_i X,$$

*for all $X \in TM$.*

*Proof.* For $X \in TM$, we have

$$\overline{\nabla}_X \xi_i = -\phi_i X + \alpha_{ij}(X) \xi_j + \alpha_{ik}(X) \xi_k,$$

which can be written as

$$\begin{aligned} \nabla_X \xi_i + h(X, \xi_i) &= -(T_i X + F_i X) + \alpha_{ij}(X)\xi_j + \alpha_{ik}(X)\xi_k \\ &= -T_i X - F_i X + \alpha_{ij}(X)\xi_j + \alpha_{ik}(X)\xi_k. \end{aligned}$$

Comparing the tangential and normal components on both sides, we get the result. □

**Proposition 3.2.** *Let $M$ be a Submanifold of a manifold with generalised 3-Sasakian structure. If $F_i (i = 1, 2, 3)$ are parallel, then $M$ is a 3-contact CR-submanifolds.*

*Proof.* From (3.19), we have

$$f_i h(X, Y) = h(X, T_i Y) - \alpha_{ij}(X) F_j Y - \alpha_{ik}(X) F_k Y. \tag{3.26}$$

Putting $Y = \xi_j$ in (3.26), we get

$$f_i h(X, \xi_j) = h(X, T_i \xi_j) - \alpha_{ij}(X) F_j \xi_j - \alpha_{ik}(X) F_k \xi_j,$$

which by virtue of equation (2.2) implies that

$$f_i h(X, \xi_j) = h(X, \xi_k) + \alpha_{ik}(X)\xi_i.$$

Now using $h(X, \xi_i) = -\phi_i X = -F_i X$, the above equation reduces to

$$-f_i F_j X + F_k X - \alpha_{ik}(X)\xi_i = 0. \tag{3.27}$$

Equating the normal component in equation (3.27), we get

$$f_i F_j = F_k.$$

Moreover putting $Y = \xi_i$ in (3.26) we get

$$f_i h(X, \xi_i) = h(X, T_i \xi_i) - \alpha_{ij}(X) F_j \xi_i - \alpha_{ik}(X) F_k \xi_i$$

or

$$f_i h(X, \xi_i) - \alpha_{ij}(X)\xi_k - \alpha_{ik}(X)\xi_j = 0$$

or

$$-f_i F_i X - \alpha_{ij}(X)\xi_k - \alpha_{ik}(X)\xi_j = 0,$$

from which we get $f_i F_i = 0$ by equating normal component. Thus we have $f_i F_j = F_k$ and $f_i F_i = 0$. Hence using Lemma (2.2), $M$ is a 3-contact CR-submanifold. □

## 4. GEOMETRY OF LEAVES ON 3-CONTACT CR-SUBMANIFOLDS

The aim of this section is to obtain results on the immersion of leaves of distributions in 3-contact CR-submanifolds $M$ of manifold with generalised 3-Sasakian structure. We shall obtain necessary and sufficient conditions in order that the leaves in $M$ to be totally geodesic.

From Equations (3.1) and (3.2) we have

$$(4.1)\qquad T_i \nabla_X Y = -A_{F_iY}X - t_ih(X,Y) - g(X,Y)\xi_i + \eta_i(Y)X - \alpha_{ij}(X)T_jY - \alpha_{ik}(X)T_kY,$$

and

$$(4.2)\qquad F_i \nabla_X Y = \nabla^{\perp}_X F_iY - f_ih(X,Y) - \alpha_{ij}(X)F_jY - \alpha_{ik}(X)F_kY,$$

for any $X \in TM$ and $Y \in D^{\perp}$.

Now we prove

**Proposition 4.1.** *Let $M$ be a 3-contact CR-submanifold of a manifold $\overline{M}$ with generalised 3-Sasakian structure. Then*

(i) *the distribution $D$ is integrable and a leaf of $D$ is totally geodesic in $M$ if and only if*

$$(4.3)\qquad g(h(X,W),\phi_iZ) = 0.$$

*for all $X, W \in D$ and $Z \in D^{\perp}$.*

(ii) *the distribution $D$ is integrable and the leaf of $D$ is totally geodesic in $\overline{M}$ if and only if $M$ is $D$-totally geodesic.*

*Proof.* (i) Suppose that the distribution $D$ is integrable and its leaf is totally geodesic in $M$. Then we have $\nabla_X W \in D$, $\nabla_X \phi_i W \in D$ for all $X, W \in D$.

Now, for $X, W \in D$ and $Y \in D^{\perp}$, equation (4.1) yields

$$\begin{aligned} 0 &= g(\nabla_X \phi_i W, Y) = -g(\phi_i W, \nabla_X Y) = g(\phi_i \nabla_X Y, W) = g(T_i \nabla_X Y, W) \\ &= -g(A_{F_i Y} X + t_i h(X,Y), W) - g(X,Y)\eta_i(W) + \eta_i(Y) g(X,W) \\ &\quad -\alpha_{ij}(X) g(T_j Y, W) - \alpha_{ik}(X) g(T_k Y, W) \\ &= -g(A_{F_i Y} X, W) = -g(h(X,W), \phi_i Y), \end{aligned}$$

from which we get (4.3).

Conversely, let equation (4.3) holds, then using (3.13), the distribution $D$ is integrable. Now, by virtue of equation (2.7), we get

$$\begin{aligned} g(h(X, \phi_i W), F_i Z) &= g(\overline{\nabla}_X \phi_i W, F_i Z) = g(\phi \overline{\nabla}_X W, \phi_i Z) \\ &= g(\nabla_X W, Z) = 0, \end{aligned} \tag{4.4}$$

for any $X, W \in D$ and $Z \in D^{\perp}$.

From (4.4), we obtain $\nabla_X W \in D$ for all $X, W \in D$, that is, each leaf of $D$ is totally geodesic in $M$.

**(ii)** Suppose that $D$ is integrable and its leaves are totally geodesic in $\overline{M}$. Then we have $\overline{\nabla}_X W \in D$ for all $X, W \in D$. Thus using the Gauss formula we obtain

$$g(h(X,W), N) = g(\overline{\nabla}_X W, N) = 0,$$

for all $X, W \in D$ and $N \in T^{\perp}M$. Hence $M$ is D-totally geodesic CR-submanifold.

Finally, we suppose that $h(X,W) = 0$ for $X, W \in D$. Then from (3.13) it follows that $D$ is integrable and $\overline{\nabla}_X W \in TM$ for all $X, W \in D$. Now using equation (2.7), we have

$$g(\overline{\nabla}_X W, Z) = g(\overline{\nabla}_X \phi_i Y, \phi_i Z) = g(h(X, \phi_i W), \phi_i Z) = 0,$$

for all $X, W \in D$ and $Z \in D^{\perp}$. Thus $\overline{\nabla}_X W \in D$, i.e each leaf of $D$ is totally geodesic in $\overline{M}$. This completes the proof. □

For the leaves of distribution $D^{\perp}$, we have:

**Proposition 4.2.** *Let $M$ be a 3-contact CR-submanifold of a manifold with generalised 3-Sasakian structure. If the distribution $D^{\perp}$ is integrable, then a* *leaf of $D^\perp$ is totally geodesic in $M$ if*

$$g(h(X,Z),F_iY)+g(X,Y)\eta_i(Z)=0. \tag{4.5}$$

*Proof.* For $X,Y\in D^\perp$ equation (4.1) gives

$$T_i\nabla_X Y=-A_{F_iY}X-t_ih(X,Y)-g(X,Y)\xi_i.$$

Taking inner product with $Z\in D$ in the above equation and using the fact that $\xi_i\in D$, we get

$$g(\nabla_XY,T_iZ)=g(h(X,Z),F_iY)+g(X,Y)\eta_i(Z),$$

for all $X,Y\in D^\perp$ and $Z\in D$. □

**Definition 4.1.** A 3-contact CR-submanifold $M$ of a manifold with generalised 3-Sasakian structure is called D-totally umbilical if $h(X,Y)=g(X,Y)H$, for $X,Y\in D$, where $H$ is the mean curvature vector.

**Proposition 4.3.** *Let $M$ be a D-totally umbilical 3-contact CR-submanifold of a manifold $\overline{M}$ with generalised 3-Sasakian structure. Then $M$ is a D-totally geodesic CR-submanifold.*

*Proof.* By definition of D-totally umbilical 3-contact CR-submanifold, we have

$$h(X,Y)=g(X,Y)H,\qquad X,Y\in D$$

Also from the Lemma 3.4, we have

$$h(X,\xi_i)=-F_iX,\qquad X\in TM.$$

For $X\in D$, we have $F_iX=0$, using this in the above equation we get

$$h(X,\xi_i)=0,$$

which means that $M$ is a D-totally geodesic. □

## REFERENCES


1. F. R. Al-Solamy, CR-submanifold of nearly trans-Sasakian manifold, *Int. J. of Math. & Math. Sci.* **31**, 3(2002), 167-175.

2. A. Bejancu, CR-submanifold of a Kaehler manifold I, *Proc. Amer. Math. Soc.* **69**, (1978), 135–142

3. A. Bejancu, *Geometry of CR-Submanifolds,* D. Reidel Publ. Co., Holland, 1986.

4. A. Bejancu and Hani Reda Farran, On totally umbilical QR-submanifolds of quaternion Kaehlerian manifolds, *Bull. Austral. Math. Soc.* **62,** (2000), 95–103.

5. D. E. Blair, *Contact Manifold in Riemannian Geometry,* Lecture Notes in Math. Vol. 509, Springer, N. York , 1978

6. B. Y. Chen, CR-submanifold of a Kaehler manifold I, & II, *J. Diff. Geometry,* **16,** (1981) 305–322, 493–509

7. B. Y. Chen, *Geometry of Submanifolds,* Marcel Dekker, New York, 1973

8. M. Kobayashi, CR-submanifold of a Sasakian manifold, *Tensor N. S.* **35**, (1981), 297–307.

9. M. Kobayashi, 3-contact CR-submanifolds of manifolds with a Sasakian 3-structure, *Tensor N. S.* **40**, (1983), 57–69.

10. Y. Y. Kuo, On almost contact 3-structures, *Tohoka Math. J.* **22**, (1970), 325–332

11. M. Hasan Shahid, CR-submanifold of a trans-Sasakian manifold I, II, *Indian J. Pure and Appl. Math.* **22,** (1991), 1007–1012, **25**, (1994), 299–307.

12. K. Yano and M. Kon, Contact CR-submanifolds, *Kodai Math. J.* **5**, (1982), 238-252.


Department of Mathematics, King AbdulAziz University, P. O. Box 80015, Jeddah 21589, Saudi Arabia.
e-mail: Falleh@hotmail.com

Date received May 04, 2003