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ASYMPTOTIC SOLUTIONS FOR SECOND-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. We investigate the asymptotic form of two linearly independent solutions of the second-order differential equation

$$(py')' + ry' - qy = 0 \quad \text{as} \quad x \to \infty \tag{1}$$

The functions p, r and q are defined on the interval $[a, \infty)$ and are not necessarily real-valued, while p is nowhere zero in this interval. We shall consider the case where r is small compared to p and q in the sense that

$$\frac{r}{(pq)^{1/2}} \to 0 \text{ as } x \to \infty. \tag{2}$$

In this situation we identify two cases:

$$\frac{(pq)'}{(pq)^{1/2}} = o\{(q/p)^{1/2}\} \ (x \to \infty), \tag{3}$$

$$\frac{(pq)'}{(pq)^{1/2}} \sim \text{const}(q/p)^{1/2} \ (x \to \infty) \tag{4}$$

Here (4) is Euler's case for (1) and under this case and case (3) we shall obtain asymptotic solutions for (1) which extend those of Eastham [5].

1. INTRODUCTION

In this paper we consider the asymptotic form of two linearly independent solutions of the second-order differential equation

$$(p(x)y'(x))' + r(x)y'(x) - q(x)y(x) = 0 \tag{1.1}$$

as $x \to \infty$, where x is the independent variable and the prime denotes $\frac{d}{dx}$. The functions p, r and q are defined on the interval $[a, \infty)$ and are not necessarily real-valued, while p is nowhere zero in this interval. We shall consider the situation where r is small compared to p and q as $x \to \infty$.

In this situation we identify two cases

$$\frac{(pq)'}{(pq)^{1/2}} = o\{(q/p)^{1/2}\} \ \ (x \to \infty) \tag{1.2}$$

$$\frac{(pq)'}{(pq)^{1/2}} \sim \text{const.}(q/p)^{1/2} \ \ (x \to \infty) \tag{1.3}$$

Here (1.3) is Euler's case for (1.1) which is given in section 5.

Eastham [5] considered (1.1) with $r = 0$ subject to the condition (1.2) as $x \to \infty$. We also use the recent asymptotic theorem of Eastham [5, section 1.6] to obtain the solutions of (1.1).

The general features of our method are given in sections 2 and 3, with the main theorem for (1.1) in section 4. Finally, we give some examples in section 6.

2. THE GENERAL METHOD

We consider (1.1) in a standard way [5] as a first-order system

$$Y'(x) = A(x)Y(x), \tag{2.1}$$

where the first component of Y is y and

$$A = \begin{pmatrix} 0 & p^{-1} \\ q & -rp^{-1} \end{pmatrix} \tag{2.2}$$

As in [5], we express A in its diagonal form

$$T^{-1}AT = \Lambda \tag{2.3}$$

and we therefore require the eigenvalues $\lambda_j(x)$ and eigenvectors $v_j(x)$ $(j = 1, 2)$ of A. The characteristic equation of A is given by

$$p\lambda^2 + r\lambda - q = 0 \tag{2.4}$$

An eigenvector $v_j(x)$ of A corresponding to $\lambda_j(x)$ $(j = 1, 2)$ is

$$v_j\,(1 \quad p\lambda j)^t \tag{2.5}$$

where the superscript t denotes the transpose. We assume at this stage that the $\lambda_j(x)$ are distinct and we define the matrix T in (2.3) by

$$T = (v_1 \quad v_2) \tag{2.6}$$

Then by (2.5) and (2.6),

$$T^{-1} = \{p(\lambda_2 - \lambda_1)\}^{-1} \begin{pmatrix} p\lambda_2 & -1 \\ -p\lambda_1 & 1 \end{pmatrix} \tag{2.7}$$

and

$$T' = \begin{pmatrix} 0 & 0 \\ (p\lambda_1)' & (p\lambda_2)' \end{pmatrix} \tag{2.8}$$

By (2.3), the transformation

$$Y = TZ \tag{2.9}$$

takes (2.1) into

$$Z' = (\Lambda + R)Z, \tag{2.10}$$

where

$$\Lambda = dg(\lambda_1, \lambda_2) \tag{2.11}$$

From (2.5)-(2.8), we obtain

$$R = -T^{-1}T' = -\{p(\lambda_2 - \lambda_1)\}^{-1} \begin{pmatrix} -(p\lambda_1)' & -(p\lambda_2)' \\ (p\lambda_1)' & (p\lambda_2)' \end{pmatrix} \tag{2.12}$$

Now we need to work out (2.11) and (2.12) in some detail in terms of p, q and r in order to determine the form of (2.10) and then to apply the general asymptotic theorem of Eastham [5, section 1.6). We then transform back to Y by means of (2.9) and (2.6)

3. THE MATRICES Λ AND R

In our analysis, the basic conditions required to express the fact that r is small compared to p and q as $x \to \infty$ are as follows:

(1) p and q are nowhere zero in some interval $[a, \infty)$ and

$$r = o\{(pq)^{1/2}\} \quad (x \to \infty) \tag{3.1}$$

and we write

$$\delta = \frac{r}{(pq)^{1/2}} = o(1) \tag{3.2}$$

Also, we require that p, q and r to be $C^{(2)}[a, \infty)$ such that the functions

$$\frac{r'}{(pq)^{1/2}}, \ \frac{q'}{q}\delta, \ \frac{p'}{p}\delta \quad \text{are all} \quad L(a, \infty) \tag{3.3}$$

Now, subject to (3.1), the characteristic equation (2.4) can be solved as in [1] to obtain

$$\lambda_j = \pm(q/p)^{1/2}\left\{\left(1 + \frac{\delta^2/4}{1 + (1 + \delta^2/4)^{1/2}}\right)^{1/2} \mp \frac{\delta}{2}\right\} \qquad (j = 1, 2) \tag{3.4}$$

i.e.

$$\lambda_j = \pm(q/p)^{1/2}\{1 + \delta_j\} \qquad (j = 1, 2) \tag{3.5}$$

where

$$\delta_j = O(\delta) \qquad (j = 1, 2) \tag{3.6}$$

By (3.5), we have

$$p\lambda_j = \pm(pq)^{1/2}(1 + \delta_j) \qquad (j = 1, 2) \tag{3.7}$$

Differentiating (3.7) to get

$$(p\lambda_j)' = \pm\frac{(pq)'}{2(pq)^{1/2}} \pm (pq)^{1/2}\left\{\delta_j' + o\left(\delta_j\frac{(pq)'}{pq}\right)\right\} \qquad (j = 1, 2) \tag{3.8}$$

Let

$$I = \delta_j' + o\left(\delta_j\frac{(pq)'}{pq}\right) \qquad (j = 1, 2) \tag{3.9}$$

On substituting (3.5) into (2.4) and then differentiating we obtain

$$\delta_j' = -\frac{1}{2}\delta'(1+\delta_j)\left\{1+\delta_j+\frac{\delta}{2}\right\}^{-1} \quad (j=1,2) \tag{3.10}$$

Hence

$$\begin{aligned} \delta_j' &= O(\delta') \\ &= O\left(\frac{r'}{(pq)^{1/2}}\right)+O\left(\frac{q'}{q}\delta\right)+O\left(\frac{p'}{p}\delta\right) \quad (j=1,2) \end{aligned} \tag{3.11}$$

Therefore, by (3.5) and (3.11),

$$\delta_j' \quad \text{is in } L(a,\infty) \tag{3.12}$$

So (3.8) becomes

$$(p\lambda_j)' = \pm\frac{(pq)'}{2(pq)^{1/2}} \pm (pq)^{1/2} I \tag{3.13}$$

Hence (2.10) can be written as

$$Z' = (\Lambda + Q + S)Z, \tag{3.14}$$

where

$$Q = -\frac{(pq)'}{4pq}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{3.15}$$

and

$$S = O\left(\frac{r'}{(pq)^{1/2}}\right)+O\left(\frac{q'}{q}\delta\right)+O\left(\frac{p'}{p}\delta\right) \quad (j=1,2) \tag{3.16}$$

By (3.12), S is in $L(a,\infty)$.

4. CASE I

Let

$$(4.1) \qquad \frac{(pq)'}{(pq)} = 0\{(q/p)^{1/2}\}$$

Also let

$$(4.2) \qquad \{p^{-1/2}q^{-3/2}(pq)'\}' \in L(a,\infty)$$

Now we can apply the asymptotic theorem of Eastham [5, section 1.6] provided that Λ and Q satisfy the conditions in [5, section 1.6]. By (3.15) and (3.5), the conditions (i) and (ii) in Theorem 1.6.1 [5, section 1.6] are satisfied by (3.3) , (4.1) and (4.2). We now state our main theorem for (1.1).

Theorem 4.1. *Let (3.1), (3.3), (4.1) and (4.2) hold. Also let*

$$(4.3) \quad re\left\{\left(\frac{(pq)'}{4pq}\right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2\right\}^{1/2} \quad \textit{have one sign in } [a,\infty)$$

Then (1.1) has solutions y_1 and y_2 such that

$$(4.4) \qquad y_1 \sim (pq)^{-1/4} \exp\left(\int_a^x \left\{\left(\frac{(pq)'}{4pq}\right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2\right\}^{1/2} dt\right)$$

$$(4.5) \qquad py_1' \sim (pq)^{1/4} \exp\left(\int_a^x \left\{\left(\frac{(pq)'}{4pq}\right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2\right\}^{1/2} dt\right)$$

with similar formulae for y_2 containing $-(\ldots.)$ in the exponential term.

Proof. Since (3.14) satisfies all conditions for the asymptotic theorem of Eastham [5, section 1.6], it follows that, as $x \to \infty$, (3.14) has two linearly independent solutions:

$$(4.6)\ Z_j(x) = \{e_j + o(1)\}\ \exp\left(\int_a^x \left\{\left(\frac{(pq)'}{4pq}\right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2\right\}^{1/2} dt\right)$$

where $e_j (j = 1,2)$ is the coordinate vector with j-th component unity and other components zero. Now by transforming back to Y by means of

(2.6) and (2.9), we obtain (4.4) and (4.5) after adjusting y_1 by a constant multiple, and similarly for y_2.

5. CASE II (EULER TYPE)

$$\text{Let } \frac{(pq)'}{pq} = k(q/p)^{1/2}(1+\phi) \tag{5.1}$$

where k is a non-zero constant with

$$k^2 \neq -16 \tag{5.2}$$

and

$$\phi = o(1) \qquad (x \to \infty) \tag{5.3}$$

and

$$\phi' \in L(a,\infty) \tag{5.4}$$

Now it is easy to see that first condition of Eastham theorem [5, section 1.6] is not true subject to (5.1). So it is necessary to carry out a second diagonalization. To do this, we first use (5.1), (3.15) and (3.5) to write the system (3.14) in the form

$$Z' = \left(\frac{q}{p}\right)^{1/2}(C + R_1 + S)Z, \tag{5.5}$$

where C is a constant matrix and $R_1 \to 0$ as $x \to \infty$.

To obtain the system (5.5), we write

$$(5.6)\Lambda + Q = \begin{pmatrix} \left(\frac{q}{p}\right)^{1/2}(1+\delta_1) - \frac{(pq)'}{4pq} & \frac{(pq)'}{4pq} \\ \frac{(pq)'}{4pq} & -\left(\frac{q}{p}\right)^{1/2}(1+\delta_2) - \frac{(pq)'}{4pq} \end{pmatrix}$$

Our aim now is to write the matrix on the right of (5.6) as $\left(\frac{q}{p}\right)^{1/2}(C+R_1)$, and this can be done as follows:

$$\Lambda + Q \;=\; \left(\frac{q}{p}\right)^{1/2} \begin{pmatrix} 1+\delta_1 - \frac{(pq)'}{4pq}\left(\frac{p}{q}\right)^{1/2} & \frac{(pq)'}{4pq}\left(\frac{p}{q}\right)^{1/2} \\ \frac{(pq)'}{4pq}\left(\frac{p}{q}\right)^{1/2} & -1-\delta_2 - \left(\frac{(pq)'}{4pq}\right)^{1/2} \end{pmatrix}$$

$$= \left(\frac{q}{p}\right)^{1/2} \begin{pmatrix} 1+\delta_1 - \frac{1}{4}k(1+\phi) & \frac{1}{4}k(1+\phi) \\ \frac{1}{4}k(1+\phi) & -1-\delta_2 - \frac{1}{4}k(1+\phi) \end{pmatrix}$$

$$= \left(\frac{q}{p}\right)^{1/2} \begin{pmatrix} 1+\delta_1 - \frac{1}{4}k - \frac{1}{4}k\phi & \frac{1}{4}k + \frac{1}{4}k\phi \\ \frac{1}{4}k + \frac{1}{4}k\phi & -1-\delta_2 - \frac{1}{4}k\phi \end{pmatrix}$$

$$= \left(\frac{q}{p}\right)^{1/2} (C+R_1),$$

where C is a constant matrix given by

$$C = \begin{pmatrix} 1-\frac{1}{4}k & \frac{1}{4}k \\ \frac{1}{4}k & -1-\frac{1}{4}k \end{pmatrix} \tag{5.7}$$

and $R_1 = o(1)$ as $x \to \infty$, where

$$R_1 = \begin{pmatrix} \delta_1 - \frac{1}{4}k\phi & \frac{1}{4}k\phi \\ \frac{1}{4}k\phi & -\delta_2 - \frac{1}{4}k\phi \end{pmatrix} \tag{5.8}$$

As in [5], since C is not a diagonal matrix, we have to express C in its diagonal form and then apply Eastham theorem. We write

$$T_1^{-1} C T_1 = \Lambda_1 \tag{5.9}$$

where Λ_1 is diagonal and

$$T_1 = \begin{pmatrix} \frac{1}{4}k & -\frac{1}{4}k \\ -1+\frac{1}{4}k(1+16k^{-2})^{1/2} & 1+\frac{1}{4}k(1+16k^{-2})^{1/2} \end{pmatrix} \tag{5.10}$$

T is non-singular when $k \neq 0$ and $k^2 \neq -16$.

The transformation

$$Z = T_1 W \tag{5.11}$$

gives, after differentiating and using (2.10),

$$W' = \left(\frac{q}{p}\right)^{1/2} (\Lambda_1 + T_1^{-1} R_1 T_1 + T_1^{-1} S T_1) W \tag{5.12}$$

Now we can apply the asymptotic theorem of Eastham [5, section 1.6] provided that Λ_1 and $T_1^{-1} R_1 T_1$ satisfy the conditions in [5, section 1.6].

By (5.10), (5.8) and (5.12), the conditions (i) and (ii) in Theorem 1.6.1 [5, section 1.6] are satisfied by (3.1), (3.3), (5.3) and (5.4). The next theorem is as follows:

Theorem 5.1. *Let p and q be nowhere zero and $C^{(2)}[a,\infty)$. Let (3.1), (3.3), (5.1) and (5.4) hold. Also let*

$$(5.13)\quad re\left\{\left(\frac{(pq)'}{4pq}\right)^2+\left(\frac{\lambda_1-\lambda_2}{2}\right)^2\right\}^{1/2} \quad \text{have one sign in } [a,\infty)$$

Then (1.1) has solutions y_1 and y_2 such that

$$(5.14)\qquad y_1\sim(pq)^{-\frac{1}{4}+\frac{1}{4}(1+16k^{-2})^{1/2}}\exp\left(\frac{1}{4}\int_a^x\left\{\left(\frac{(pq)'}{pq}\right)\Phi\right\}dt\right)$$

$$(5.15)\qquad py_1'\sim ck(pq)^{\frac{1}{4}+\frac{1}{4}(1+16k^{-2})^{1/2}}\exp\left(\frac{1}{4}\int_a^x\left\{\left(\frac{(pq)'}{pq}\right)\Phi\right\}dt\right)$$

and

$$(5.16)\qquad y_2\sim(pq)^{-\frac{1}{4}-\frac{1}{4}(1+16k^{-2})^{1/2}}\exp\left(-\frac{1}{4}\int_a^x\left\{\left(\frac{(pq)'}{pq}\right)\Phi\right\}dt\right)$$

$$(5.17)\qquad py_2'\sim(ck)^{-1}(pq)^{\frac{1}{4}+\frac{1}{4}(1+16k^{-2})^{1/2}}\exp\left(-\frac{1}{4}\int_a^x\left\{\left(\frac{(pq)'}{pq}\right)\Phi\right\}dt\right)$$

where

$$(5.18)\qquad \Phi=-16k^{-2}(1+16k^{-2})^{-1/2}\phi+O(\phi\delta)+O(\phi^2)$$

Proof. Since (5.12) satisfies all the conditions of the asymptotic theorem of Eastham [5, section 1.6], it follows that, as $x\to\infty$, (5.12) has two linearly independent solutions:

$$(5.19)\quad W_j(x)=\{e_j+o(1)\}\exp\left(\int_a^x\left\{\left(\frac{(pq)'}{4pq}\right)^2+\left(\frac{\lambda_1-\lambda_2}{2}\right)^2\right\}^{1/2}dt\right)$$

where $e_j(j=1,2)$ is as in the proof of Theorem (4.1). Now by transforming back to Y by means of (5.11), (2.9), (2.6) and (2.5), we obtain (5.14), (5.15), (5.16) and (5.17).

6. EXAMPLES

(i) We consider Theorem (4.1) applied to the coefficients

$$p(x) = Ax^{\alpha}, \; q(x) = Bx^{\beta}, \;\; r(x) = Cx^{\gamma},$$

where $\alpha, \beta, \gamma, A(\neq 0), B(\neq 0)$ and C are real constants with $AB > 0$. Then (3.1) and (3.2) hold under the condition

$$\alpha + \beta > 2\gamma \tag{6.1}$$

(ii) We consider Theorem (5.1) applied to the coefficients

$$p(x) = Ax^{\alpha}, \;\; q(x) = Bx^{\beta}, \;\; r(x) = Cx^{\gamma},$$

where $\alpha, \beta, \gamma, A(\neq 0), B(\neq 0)$ and C ar real constants such that $AB > 0$ and $\alpha + \beta \neq 0$. Then (3.1) and (3.2) hold under the condition

$$\alpha + \beta > 2\gamma \tag{6.2}$$

The condition (5.1) holds if

$$\alpha - \beta = 2 \tag{6.3}$$

The constant k which is in (5.1) is $k = (\alpha + \beta)\left(\frac{A}{B}\right)^{1/2}$. Here $\phi = 0$, so $\Phi = 0$. Therefore, the asymptotic formulae give:

$$y_j \sim (pq)^{-\frac{1}{4}-\frac{1}{4}(1+16k^{-2})^{1/2}} \qquad (j = 1, 2) \tag{6.4}$$

i.e.

$$y_j \sim \left((x)^{\alpha+\beta}\right)^{-\frac{1}{4}\pm\frac{1}{4}[1+\frac{4B}{A}(1+\beta)^{-2}]^{1/2}}$$

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