

ON THE SPECTRUM OF THE TENSOR PRODUCTS OF JC-ALGEBRAS

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ABSTRACT. We study the relationship between the spectrum $\sigma(A)$ of a JC-algebra A and the spectrum $\sigma(C^*(A))$ of its universal enveloping C^* -algebra $C^*(A)$ (see below for definition). We also establish the Jordan analogue of some results in the context of spectra of tensor products of C^* -algebras. In particular, we extend Guichardet's results [5, Proposition 5] to JC-algebras.

1. INTRODUCTION

The first definition of a tensor product of JC-algebras was introduced by Hanche-Olsen [7]. Further a progress in the theory was made by Jamjoom [9, 10] who introduced a more general definition which provided a convenient treatment of tensor products of JC-algebras and stressed the important connection between the theory of tensor products of JC-algebras and tensor products of C^* -algebras. As the relation between JC-algebras and their universal enveloping C^* -algebras is very strong (see [2, 3, 8, 9, 11]), we rely on the theory of tensor products of C^* -algebras throughout this paper and we assume familiarity with the general theory as contained in [5, 12, 17]. We deal with C^* -algebras having an identity. If \mathfrak{A}_1 and \mathfrak{A}_2 are C^* -algebras, their minimal and maximal C^* -tensor products are denoted by $\mathfrak{A}_1 \otimes_{\min} \mathfrak{A}_2$ and $\mathfrak{A}_1 \otimes_{\max} \mathfrak{A}_2$, respectively. Given a $*$ -representation $\{\pi_i, H_i\}$ of a C^* -algebra $\mathfrak{A}_i, i = 1, 2$, let $\pi_1 \otimes_{\lambda} \pi_2$ denote the unique $*$ -representation $\pi_1 \otimes_{\lambda} \pi_2 : \mathfrak{A}_1 \otimes_{\lambda} \mathfrak{A}_2 \rightarrow B(H_1 \otimes H_2)$ of $\mathfrak{A}_1 \otimes_{\lambda} \mathfrak{A}_2$ on $H_1 \otimes H_2$ defined by $(\pi_1 \otimes_{\lambda} \pi_2)(x_1 \otimes x_2) = \pi_1(x_1) \otimes \pi_2(x_2), x_i \in \mathfrak{A}_i$, where λ is the minimum or the maximum C^* -norm [17, 4.4.7, 4.4.22].

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The state space of a C^* -algebra \mathfrak{A} will be denoted by $S(\mathfrak{A})$. It is convex and weak* compact [11, p. 257]. The extreme points of $S(\mathfrak{A})$ are called **pure states**, and the set of all pure states will be denoted by $P(\mathfrak{A})$. If $\rho \in S(\mathfrak{A})$, then π_ρ will denote the representation of \mathfrak{A} on the Hilbert space H_ρ , with cyclic vector ξ_ρ , obtained by the GNS construction (see [11, 4.5.2]).

A **JC-algebra** A is a norm (uniformly) closed Jordan subalgebra of the Jordan algebra $B(H)_{s.a}$ of all bounded self adjoint operators on a Hilbert space H . The Jordan product is given by $a \circ b = (ab + ba)/2$, where juxtaposition denotes the ordinary operator multiplication. The centre of a JC-algebra is the set $Z(A) = \{a \in A : ab = ba \ \forall b \in A\}$ [18, Proposition 1]. A subspace I of a JC-algebra A is called a **Jordan ideal** if $a \circ b \in I$ for every $a \in A$ and every $b \in I$. A **JW-algebra** $M \subseteq B(H)_{s.a}$ is a weakly closed JC-algebra. The **central support** of an element a in a JW-algebra M is the smallest central projection e in M such that $ea = a$ [18, p. 11]. A projection e of M is said to be **abelian** if eMe is associative, and it is called **minimal** if it is non-zero and contains no other non-zero projection of M , or equivalently, e is minimal if and only if $eMe = \mathbb{R}e$. A **JW-factor** is a JW-algebra with trivial centre; a **type I JW-factor** is a JW-factor which contains a minimal projection. A JW-algebra is said to be of **type** I_n if there is a family of abelian projections $(e_\alpha)_{\alpha \in J}$ such that $c_M(e_\alpha) = 1_M$, $\sum_{\alpha \in J} e_\alpha = 1_M$ and $\text{card } J = n$. It is known that (see [8, Section 5.3]) each type I JW-factor is of type I_n , for some n , probably infinite. A **spin factor** V_k is a $(k + 1)$ -dimensional JW-factor of type I_2 .

A linear map $\varphi : A \rightarrow B$ between JC-algebras A and B is called a **(Jordan) homomorphism** if it preserves the Jordan product. A Jordan homomorphism which is one to one is called a **Jordan isomorphism**. A **(concrete) representation** of a JC-algebra A is a (Jordan) homomorphism $\pi : A \rightarrow B(H)_{s.a}$, for some complex Hilbert space H . It is said to be **dense** if $\pi(A)^\ominus = B(H)_{s.a}$. A **factor representation** of a JC-algebra A is a (Jordan) homomorphism of A onto a weakly dense subalgebra of a JW-factor M . **Type I factor representations** are defined accordingly. Two factor representations $\phi_i : A \rightarrow M_i$, where M_i is a type I JW-factor, $i = 1, 2$, are said to be **Jordan equivalent** if there is an isomorphism Ψ of M_1 onto M_2 such that $\phi_2 = \Psi\phi_1$. Each type I factor representation of a JC-algebra A is Jordan equivalent to $\phi_\rho : A \rightarrow c(\rho)A^{**}(a \mapsto c(\rho)a)$, where A^{**} is the second dual of

A , ρ is a pure state of A and $c(\rho)$ the central support of ρ being a minimal central projection in A^{**} [1, Proposition 2.2], [8, 4.5.7, 4.6.2, 4.6.4].

A JC-algebra A is said to be **reversible** if $a_1 a_2 \dots a_n + a_n a_{n-1} \dots a_1 \in A$ whenever $a_1, a_2, \dots, a_n \in A$, and is said to be **universally reversible** if $\pi(A)$ is reversible for every representation π of A [7, p. 5]. A spin factor V is universally reversible when $\dim V = 3$ or 4 , non-reversible when $\dim V \neq 3, 4$ or 6 , and it can be either reversible or non-reversible if $\dim V = 6$ [1, p. 280], [8, 6.2.5]. Every JW-algebra without a direct summand of type I_2 is universally reversible [8, 5.1.5, 5.3.5, 6.2.3], and a JC-algebra is universally reversible if and only if it has no type I factor representation onto a spin factor other than V_2 and V_3 [7, Theorem 2.2].

If A is a JC-algebra, let $C^*(A)$ be the **universal enveloping C^* -algebra of A** , and let Φ_A be the **canonical involutory $*$ -antiautomorphism of $C^*(A)$** . Usually we will regard A as a generating Jordan subalgebra of $C^*(A)$ so that Φ_A fixes each point of A (see [8], Theorem 7.1.8) for the existence and properties of $C^*(A)$. It is known [7, Lemma 4.2] that a JC-algebra A is universally reversible if and only if it is reversible in $C^*(A)$.

The reader is referred to [3, 7, 8, 14, 15] for a detailed account of the theory of JC-algebras and JW-algebras. The relevant background on the theory of C^* -algebras and von Neumann algebras can be found in [4, 11, 12, 17].

Definition 1 Let A and B be a pair of JC-algebras canonically embedded in their respective universal enveloping C^* -algebras $C^*(A)$ and $C^*(B)$ and let λ be a C^* -norm on $C^*(A) \otimes C^*(B)$. Then the **JC-tensor product of A and B with respect to λ** is the completion $JC(A \otimes B)$ of the real Jordan algebra $J(A \otimes B)$ generated by $A \otimes B$ in $C^*(A) \otimes_{\lambda} C^*(B)$.

Theorem 1 (9, Corollary 2.3) *Let A and B be JC-algebras. Then*

$$C^*(JC(A \otimes B)) = C^*(A) \otimes_{\lambda} C^*(B)$$

where λ is the minimum (min) or the maximum (max) C^* -norm.

Theorem 2 (9, Proposition 4.1) *Let A_i be a JC-algebra and $\pi_i : A_i \rightarrow B(H_i)$ be a Jordan homomorphism, where H_i is a complex Hilbert space. Then*

there is a unique Jordan homomorphism of $JC(A_1 \otimes_{\min} A_2)$ into $B(H_1 \otimes H_2)$ such that $(\pi_1 \otimes_{\min} \pi_2)(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$, $a_i \in A_i$, $i = 1, 2$. Furthermore, if π_i is injective and A_i has no representations onto spin factors of the form V_{4n+1} , $n < \infty$, then $\pi_1 \otimes_{\min} \pi_2$ is injective.

Theorem 3 (9, Proposition 1.2) *Let A_i and C be JC-algebras and let $\pi_1 : A_1 \rightarrow C$, $\pi_2 : A_2 \rightarrow C$ be pointwise operator commuting Jordan homomorphisms, $i = 1, 2$. Then there exists a unique Jordan homomorphism $\pi_1 \otimes_{\max} \pi_2 : JC(A_1 \otimes_{\max} A_2) \rightarrow C$ such that $(\pi_1 \otimes_{\max} \pi_2)(a_1 \otimes a_2) = \pi_1(a_1)\pi_2(a_2) = \pi_2(a_2)\pi_1(a_1) = \pi_1(a_1) \circ \pi_2(a_2)$, $a_i \in A_i$.*

The reader is referred to [9] for the properties of the JC-tensor product of JC-algebras.

2. THE SPECTRUM

Recall that two representations $\{\pi_1, H_1\}$ and $\{\pi_2, H_2\}$ of a C^* -algebra \mathfrak{A} are said to be **spatially equivalent** (or **unitarily equivalent**) if there is an isometry u of H_1 onto H_2 such that $u\pi_1(x)u^* = \pi_2(x)$, $x \in \mathfrak{A}$. They are **equivalent** if there is an isomorphism ψ of $\pi_1(\mathfrak{A})^-$ onto $\pi_2(\mathfrak{A})^-$ such that $\psi(\pi_1(x)) = \pi_2(x)$, $x \in \mathfrak{A}$, where $\pi_i(\mathfrak{A})^-$ is the strong (= weak) closure of $\pi_i(\mathfrak{A})$ in $B(H_i)$, $i = 1, 2$. It is clear that spatially equivalent representations are equivalent [13, 3.3.6]). A representation $\{\pi, H\}$ of a C^* -algebra \mathfrak{A} is called **irreducible** if the only subspaces of H invariant under π are H and $\{0\}$, or equivalently, $\pi(\mathfrak{A})^- = B(H)$ [13, 3.13.2]. Thus, every irreducible representation of a C^* -algebra \mathfrak{A} is a type I factor representation. Two irreducible representations of a C^* -algebra \mathfrak{A} are either disjoint or spatially equivalent [13, 3.13.3]. The cyclic representation $\{\pi_\omega, H_\omega, \xi_\omega\}$ associated with a pure state ω of a C^* -algebra \mathfrak{A} by the GNS construction is irreducible, and every irreducible representation is spatially equivalent to a cyclic representation associated with some pure state of \mathfrak{A} [13, 3.13.2].

By (Størmer [14, Theorem 5.1] and [15, Theorem 4.1]), a JW-algebra is a type I JW-factor if and only if it can be faithfully represented to act irreducibly on a complex Hilbert space. It is important to note that a concrete irreducible representation of a JC-algebra A is not necessarily dense. Indeed,

let V be an infinite dimensional spin factor representation. We may suppose that V acts faithfully and irreducibly on a complex Hilbert space H . But then, $V \neq B(H)_{s.a}$, for otherwise, V would be universally reversible [8, 7.4.6]. Also, it is easy to see that a JC-algebra A acts irreducibly on a Hilbert space H if and only if its universal enveloping C^* -algebra $C^*(A)$ acts irreducibly on H [10, Lemma 1].

An ideal (respectively, a Jordan ideal) I of a C^* -algebra (respectively, a JC-algebra) \mathfrak{A} is called **primitive** if it is the kernel of some irreducible representation (respectively, a type I factor representation) of \mathfrak{A} . The **primitive space**, $\text{Prim}(\mathfrak{A})$, of \mathfrak{A} is the space of all primitive ideals of \mathfrak{A} . Given subsets $S \subseteq \mathfrak{A}$, $F \subseteq \text{Prim}(\mathfrak{A})$ we write $\text{hull}(S) = \{P \in \text{Prim}(\mathfrak{A}) : S \subseteq P\}$, $\ker(F) = \bigcap \{P : P \in F\}$. The class $\{\text{hull}(S) : S \subseteq \mathfrak{A}\}$ form the closed sets for a topology on $\text{Prim}(\mathfrak{A})$ called the **Jacobson or hull-kernel topology** (see [13, 4.1.1, 4.1.2] and [3, p. 2]). Since equivalent (respectively, Jordan equivalent) representations have the same kernels, $\text{Prim}(\mathfrak{A}) = \{\ker \pi_\rho : \rho \in P(A)\}$ [13, 3.13.7], [1, Proposition 2.2]. The set of equivalence classes of all irreducible representations of a C^* -algebra (respectively, type I factor representations of a JC-algebra) \mathfrak{A} is denoted by $\sigma(\mathfrak{A})$, and is called the **primitive spectrum** of \mathfrak{A} . It is endowed with the inverse hull-kernel topology from $\text{Prim}(\mathfrak{A})$ induced by the natural surjection $g : \sigma(\mathfrak{A}) \rightarrow \text{Prim}(\mathfrak{A})$, given by $g([\pi]) = \ker \pi$. We define $\hat{\text{hull}}(J)$ in $\sigma(\mathfrak{A})$ as the preimage $g^{-1}(\text{hull}(J))$ of $\text{hull}(J)$ in $\text{Prim}(\mathfrak{A})$.

Lemma 1 *Let A be a JC-algebra, and ρ be a pure state of A . If I_i is a closed Jordan ideal of A such that $I_1 \cap I_2 \subseteq \ker \phi_\rho$, $i = 1, 2$, then either $I_1 \subseteq \ker \phi_\rho$ or $I_2 \subseteq \ker \phi_\rho$.*

Proof. Recall that $\phi_\rho : A \rightarrow c(\rho)A^{**}(a \mapsto c(\rho)a)$ is the type I factor representation of A and let $\bar{\phi}_\rho : A^{**} \rightarrow c(\rho)A^{**}(x \mapsto c(\rho)x)$ be its normal extension to A^{**} [8, 4.5.7]. Identifying the weak closure I_i^- of I_i in A^{**} with $e_i A^{**}$, for some central projection e_i of A^{**} , as in [8, 4.5.8], we have $I_1 \cap I_2 = e_1 e_2 A^{**} \cap A$ since $I_i = e_i A^{**} \cap A$. It follows that $I_1^- \cap I_2^- = e_1 e_2 A^{**} = (I_1 \cap I_2)^- \subseteq \ker \bar{\phi}_\rho$. If neither I_1 nor I_2 are contained in $\ker \phi_\rho$, then there is an element $a_i \in I_i$, such that $c(\rho)a_i = \phi_\rho(a_i) \neq 0$, which implies that $c(\rho)c(a_i) \neq 0$, where $c(a_i)$ is the central support of a_i in A^{**} . Since $c(\rho)c(a_i) \leq c(\rho)$ and $c(\rho)$ is a minimal central projection in A^{**} , $c(\rho)c(a_i) = c(\rho)$. It follows that $c(\rho)(c(a_1)c(a_2)) = c(\rho) \neq 0$,

contradicting the fact that $c(a_1)c(a_2) \in Z(A^{**}) \subseteq Z(I_1^-) \cap Z(I_2^-) \subseteq \ker \bar{\phi}_\rho$. Hence, either $I_1 \subseteq \ker \phi_\rho$ or $I_2 \subseteq \ker \phi_\rho$.

It is shown in [3, Proposition 2.1] that if A is a universally reversible JC-algebra, then the map $\psi : \text{Prim}(C^*(A)) \rightarrow \text{Prim}(A)$ given by $I \mapsto I \cap A$ is an open, closed and continuous surjection. Our next Theorem gives a similar relationship between the spectrum $\sigma(C^*(A))$ of $C^*(A)$ and the spectrum $\sigma(A)$ of A . But first recall that if A is a universally reversible JC-algebra and if \mathfrak{J} is a Φ_A -invariant ideal of $C^*(A)$, then \mathfrak{J} is generated by $\mathfrak{J} \cap A$, that is $\mathfrak{J} = [J \cap A]$ [7, Lemma 4.3].

Theorem 4 *Let A be a universally reversible JC-algebra. Then the map*

$$\Psi : \sigma(C^*(A)) \rightarrow \sigma(A), \quad ([\pi] \mapsto [\pi|_A])$$

is an open, closed and continuous surjection.

Proof. First note that if $\pi_i : C^*(A) \rightarrow B(H_i)$ is an irreducible representation of $C^*(A)$, $i = 1, 2$, such that π_1 and π_2 are equivalent, then $\pi_1|_A$ and $\pi_2|_A$ are Jordan equivalent irreducible representations of A , hence Ψ is well defined and clearly is a surjection, by [14, Theorem 5.1], [15, Theorem 4.1] and [10, Lemma 1].

Suppose that I is a closed Jordan ideal of A , and let $[I]$ be the closed ideal generated by I in $C^*(A)$. Recall the open continuous surjection maps $\tilde{g} : \sigma(C^*(A)) \rightarrow \text{Prim}(C^*(A))$, $g : \sigma(A) \rightarrow \text{Prim}(A)$ ($[\pi] \mapsto \ker \pi$). If $[\pi] \in \sigma(C^*(A)) \setminus \hat{hull}([I])$, that is $[\pi] \notin \hat{hull}([I]) = \tilde{g}^{-1}(\hat{hull}([I]))$, then $\ker \pi = \tilde{g}([\pi]) \notin \hat{hull}([I])$, which implies that $\ker \pi \not\supseteq [I]$, and so, $\ker \pi \cap A \not\supseteq [I] \cap A = I$. Hence $\ker \pi|_A = \ker \pi \cap A \notin \hat{hull}(I)$, which implies that $[\pi|_A] \notin \hat{hull}(I)$, that is, $[\pi|_A] \in \sigma(A) \setminus \hat{hull}(I)$. On the other hand, if $[\pi|_A] \in \sigma(A) \setminus \hat{hull}(I)$, for some irreducible representation π of $C^*(A)$, then $\ker \pi \cap A = \ker \pi|_A \notin \hat{hull}(I)$, which implies that $\ker \pi \not\supseteq [I]$, that is $\ker \pi \notin \hat{hull}([I])$, and so $[\pi] \notin \hat{hull}([I])$. Therefore, $\Psi^{-1}(\sigma(A) \setminus \hat{hull}(I)) = \sigma(C^*(A)) \setminus \hat{hull}([I])$ and Ψ is continuous.

Next, note that if \mathfrak{J} is a closed ideal of $C^*(A)$ and $[\pi|_A] \in \Psi(\hat{hull}(\mathfrak{J}))$, for some $[\pi] \in \hat{hull}(\mathfrak{J}) = \tilde{g}^{-1}(\hat{hull}(\mathfrak{J}))$, then $\ker \pi \supseteq \mathfrak{J}$. It follows that, $g([\pi|_A$

$\}) = \ker \pi|_A = \ker \pi \cap A \supseteq \mathfrak{J} \cap A$ and so, $[\pi|_A] \in \widehat{\text{hull}}(\mathfrak{J} \cap A)$. That is, $\Psi(\widehat{\text{hull}}(\mathfrak{J})) \subseteq \widehat{\text{hull}}(J \cap A)$. Conversely, let $[\pi|_A] \in \widehat{\text{hull}}(\mathfrak{J} \cap A) = g^{-1}(\widehat{\text{hull}}(J \cap A))$, then $g([\pi|_A]) = \ker \pi|_A = \ker \pi \cap A \supseteq \mathfrak{J} \cap A \in \widehat{\text{hull}}(J \cap A)$. Since $\mathfrak{J} \cap A = \Phi_A(J) \cap A$, and $\mathfrak{J} \cap \Phi_A(J)$ is Φ_A -invariant, $[\mathfrak{J} \cap A] = J \cap \Phi_A(J)$, by [7, Lemma 4.3]. Therefore, $\ker \pi \supseteq [\ker \pi|_A] \supseteq [\mathfrak{J} \cap A] = J \cap \Phi_A(J)$. But then $\ker \pi \supseteq \mathfrak{J}$ or $\ker \pi \supseteq \Phi_A(\mathfrak{J})$, because $\ker \pi$ is primitive [13, 3.13.9], so, $\mathfrak{J} \subseteq \ker \pi$ or $\mathfrak{J} \subseteq \Phi_A(\ker \pi)$, which implies that both $\ker \pi$ and $\Phi_A(\ker \pi)$ are contained in $\widehat{\text{hull}}(\mathfrak{J})$. Since $\Phi_A(\ker \pi) = \ker(\pi \circ \Phi_A)$ and π and $\pi \circ \Phi_A$ are obviously equivalent irreducible representations of $C^*(A)$, we have $[\pi] \in \widetilde{g}^{-1}(\widehat{\text{hull}}(\mathfrak{J})) = \widehat{\text{hull}}(J)$, and so, $[\pi|_A] = \Psi([\pi]) \in \Psi(\widehat{\text{hull}}(\mathfrak{J}))$. That is, $\widehat{\text{hull}}(\mathfrak{J} \cap A) \subseteq \Psi(\widehat{\text{hull}}(J))$. Therefore, $\Psi(\widehat{\text{hull}}(\mathfrak{J})) = \widehat{\text{hull}}(J \cap A)$, and Ψ is a closed map.

It remains to prove that Ψ is open, so let \mathfrak{J} be a closed ideal of $C^*(A)$. We claim that $\Psi\left(\sigma(C^*(A)) \setminus \widehat{\text{hull}}(\mathfrak{J})\right) = \sigma(A) \setminus \widehat{\text{hull}}(\mathfrak{J} \cap A)$. Indeed, if $[\pi|_A] = \Psi([\pi]) \in \Psi\left(\sigma(C^*(A)) \setminus \widehat{\text{hull}}(\mathfrak{J})\right)$, then $\ker \pi \not\supseteq \mathfrak{J}$, which implies that $\Phi_A(\ker \pi) \not\supseteq \Phi_A(\mathfrak{J})$. But π and $\pi \circ \Phi_A$ are equivalent irreducible representations of $C^*(A)$ and so $\ker \pi = \ker(\pi \circ \Phi_A) = \Phi_A(\ker \pi)$. It follows that $\ker \pi$ does not contain the Φ_A -invariant closed ideal $\mathfrak{J} + \Phi_A(J)$ [12, 10.1.9] of $C^*(A)$, and hence $\ker \pi \not\supseteq \mathfrak{J} + \Phi_A(J) = [(J + \Phi_A(J)) \cap A] = [J \cap A]$. Therefore, $\ker \pi|_A \not\supseteq \mathfrak{J} \cap A$ which implies that $[\pi|_A] \notin \widehat{\text{hull}}(\mathfrak{J} \cap A)$. That is, $[\pi|_A] \in \sigma(A) \setminus \widehat{\text{hull}}(\mathfrak{J} \cap A)$. On the other hand, if $[\pi|_A] \notin \widehat{\text{hull}}(\mathfrak{J} \cap A)$, then $\ker \pi \cap A = \ker \pi|_A \not\supseteq \mathfrak{J} \cap A$ which implies that $\ker \pi \not\supseteq \mathfrak{J}$. Therefore, $\widetilde{g}([\pi]) = \ker \pi \notin \widehat{\text{hull}}(\mathfrak{J})$, and so, $[\pi] \notin \widetilde{g}^{-1}(\widehat{\text{hull}}(\mathfrak{J})) = \widehat{\text{hull}}(J)$. It follows that $[\pi|_A] = \Psi([\pi]) \in \Psi\left(\sigma(C^*(A)) \setminus \widehat{\text{hull}}(\mathfrak{J})\right)$. Hence,

$$\Psi\left(\sigma(C^*(A)) \setminus \widehat{\text{hull}}(\mathfrak{J})\right) = \sigma(A) \setminus \widehat{\text{hull}}(\mathfrak{J} \cap A), \text{ proving that } \Psi \text{ is open.}$$

3. TENSOR PRODUCT OF STATES AND REPRESENTATIONS OF JC-ALGEBRAS

In this section we give the Jordan analogue of (Guichardet [5, Proposition 5]) which asserts that if $\mathfrak{A}_i, i = 1, 2$ is a C^* -algebra, then the mappings $(\rho_1, \rho_2) \mapsto \rho_1 \otimes_{\max} \rho_2$ of $S(\mathfrak{A}_1) \times S(\mathfrak{A}_2)$ into $S(\mathfrak{A}_1 \otimes_{\max} \mathfrak{A}_2)$ and $(\pi_1, \pi_2) \mapsto \pi_1 \otimes_{\max} \pi_2$ of $\sigma(\mathfrak{A}_1) \times \sigma(\mathfrak{A}_2)$ into $\sigma(\mathfrak{A}_1 \otimes_{\max} \mathfrak{A}_2)$ are continuous, where $\rho_1 \otimes_{\max} \rho_2$ is the product state of ρ_1 and ρ_2 (see [12, 11.3.8], [10, Theorem 6]). Then we establish the Jordan analogue of the following commutative diagram

$$\begin{array}{ccc} P(\mathfrak{A}_1) \times P(\mathfrak{A}_2) & \xrightarrow{\theta} & P(\mathfrak{A}_1 \otimes_{\max} \mathfrak{A}_2) \\ T' \downarrow & & \downarrow T \\ \sigma(\mathfrak{A}_1) \times \sigma(\mathfrak{A}_2) & \xrightarrow{\Phi} & \sigma(\mathfrak{A}_1 \otimes_{\max} \mathfrak{A}_2) \end{array}$$

[5, p. 14], where T is the canonical map $(\rho \mapsto \pi_\rho) : P(\mathfrak{A}) \rightarrow \sigma(\mathfrak{A})$, which is known to be open, continuous and surjective [4, 3.4.11], and π_ρ is the irreducible representation associated with ρ by the GNS construction [11, 4.5.2], [17, I.9.14].

For clarity and convenience we prove the following Theorem which is used in [5, P. 13] without proof.

Theorem 5 *If $\{\pi_i, H_i\}$ is a representation of a C^* -algebra $\mathfrak{A}_i, i = 1, 2$, then $\pi_1 \otimes_{\max} \pi_2$ is irreducible.*

Proof. Since the identity map on $\mathfrak{A}_1 \otimes_{\max} \mathfrak{A}_2$ extends to a surjective *-homomorphism $\psi : \mathfrak{A}_1 \otimes_{\max} \mathfrak{A}_2 \rightarrow \mathfrak{A}_1 \otimes_{\min} \mathfrak{A}_2$, the following diagram commutes

$$\begin{array}{ccc} \mathfrak{A}_1 \otimes_{\max} \mathfrak{A}_2 & \xrightarrow{\psi} & \mathfrak{A}_1 \otimes_{\min} \mathfrak{A}_2 \\ \pi_1 \otimes_{\max} \pi_2 \searrow & & \swarrow \pi_1 \otimes_{\min} \pi_2 \\ & & B(H_1 \otimes H_2) \end{array}$$

Hence

$$\begin{aligned} (\pi_1 \otimes_{\max} \pi_2)(\mathfrak{A}_1 \otimes_{\max} \mathfrak{A}_2) &= ((\pi_1 \otimes_{\min} \pi_2) \circ \psi)(\mathfrak{A}_1 \otimes_{\max} \mathfrak{A}_2) \\ &= (\pi_1 \otimes_{\min} \pi_2)(\mathfrak{A}_1 \otimes_{\min} \mathfrak{A}_2), \end{aligned}$$

so it follows from the commutation theorem [12, 11.2.6] that

$$((\pi_1 \otimes_{\max} \pi_2)(\mathfrak{A}_1 \otimes_{\max} A_2))' = ((\pi_1 \otimes_{\min} \pi_2)(A_1 \otimes_{\min} A_2))' = (\pi_1(A_1))' \bar{\otimes} (\pi_2(A_2))'.$$

Therefore, $((\pi_1 \otimes_{\max} \pi_2)(\mathfrak{A}_1 \otimes_{\max} A_2))'$ consists of scalars if and only if $(\pi_i(\mathfrak{A}_i))'$ consists of scalars. Hence, $\pi_1 \otimes_{\max} \pi_2$ is irreducible by [13, 3.13.2].

Theorem 6 *Let \mathfrak{A}_i be a C^* -algebra, H_i and K_i be Hilbert spaces, $i = 1, 2$. If $\pi_i : \mathfrak{A}_i \rightarrow B(H_i)$ and $\phi_i : \mathfrak{A}_i \rightarrow B(K_i)$ are representations of \mathfrak{A}_i such that π_i is spatially equivalent to ϕ_i , then $\pi_1 \otimes_{\max} \pi_2$ is spatially equivalent to $\phi_1 \otimes_{\max} \phi_2$.*

Proof. Let $u_i : H_i \rightarrow K_i, i = 1, 2$, be the isometries that effectuate the equivalence, i.e. $u_i \pi_i(x_i) u_i^* = \phi_i(x_i)$ for all $x_i \in \mathfrak{A}_i$. Then $u_1 \otimes u_2 : H_1 \otimes H_2 \rightarrow K_1 \otimes K_2$ given by $(u_1 \otimes u_2)(\xi_1 \otimes \xi_2) = u_1(\xi_1) \otimes u_2(\xi_2)$ is an isometry [11, 2.6.12]. Thus,

$$\begin{aligned} (u_1 \otimes u_2)(\pi_1 \otimes \pi_2)(x_1 \otimes x_2)(u_1^* \otimes u_2^*) &= u_1 \pi_1(x_1) u_1^* \otimes u_2 \pi_2(x_2) u_2^* \\ &= \phi_1(x_1) \otimes \phi_2(x_2), \quad \forall x_i \in \mathfrak{A}_i. \end{aligned}$$

The proof is completed by linearity and continuity of $\pi_1 \otimes_{\max} \pi_2$ and $\phi_1 \otimes_{\max} \phi_2$.

Remark 1 Note that if ρ_i is a pure state of a C^* -algebra $A_i, i = 1, 2$, then $\rho_1 \otimes_{\max} \rho_2$ is pure [5, p.13]. Indeed, since the cyclic representation $\{\pi_{\rho_i}, H_{\rho_i}, \xi_{\rho_i}\}$ associated with ρ_i by the GNS construction is irreducible [17, I.9.22], $\pi_{\rho_1} \otimes_{\max} \pi_{\rho_2}$ is irreducible by Theorem 5. If $x_i \in A_i, i = 1, 2$, then

$$\begin{aligned} (\rho_1 \otimes_{\max} \rho_2)(x_1 \otimes x_2) &= \rho_1(x_1) \rho_2(x_2) \\ &= \langle \pi_{\rho_1}(x_1) \xi_{\rho_1}, \xi_{\rho_1} \rangle \langle \pi_{\rho_2}(x_2) \xi_{\rho_2}, \xi_{\rho_2} \rangle \\ &= \langle (\pi_{\rho_1} \otimes \pi_{\rho_2})(x_1 \otimes x_2)(\xi_{\rho_1} \otimes \xi_{\rho_2}), \xi_{\rho_1} \otimes \xi_{\rho_2} \rangle. \end{aligned}$$

By linearity and continuity of $\pi_{\rho_1} \otimes_{\max} \pi_{\rho_2}$, and continuity of inner product, we have

$$(\rho_1 \otimes_{\max} \rho_2)(z) = \langle (\pi_{\rho_1} \otimes_{\max} \pi_{\rho_2})(z)(\xi_{\rho_1} \otimes \xi_{\rho_2}), \xi_{\rho_1} \otimes \xi_{\rho_2} \rangle, \quad \forall z \in \mathfrak{A}_1 \otimes_{\max} A_2.$$

Therefore, $\pi_{\rho_1} \otimes_{\max} \pi_{\rho_2}$ is spatially equivalent to the representation $\pi_{\rho_1 \otimes_{\max} \rho_2}$ [11, 4.5.3], which implies that $\rho_1 \otimes_{\max} \rho_2$ is pure, by [17, I.9.22].

Definition 2 A JC-algebra A is said to be of **complex type** if all its type I factor representations are into a type I JW-factor isomorphic to $B(H)_{s,a}$ for some complex Hilbert space H . That is, if $\phi : A \rightarrow M$ is a type I factor representation, where M is a type I JW-factor, then $\phi(A)^- = M \cong B(H)_{s,a}$.

Remark 2 (i) Let $A = \mathfrak{A}_{s,a}$, where \mathfrak{A} is a C^* -algebra, and let $\phi : A \rightarrow M$ be a type I factor representation of A . Then ϕ extends to a normal surjection $\bar{\phi} : A^{**} \rightarrow M$, by [8, 4.5.7]. Thus, there exists a central projection e in A^{**} such that $eA^{**} \cong M$. Since A^{**} is the self-adjoint part of a von Neumann algebra, M is isomorphic to the self-adjoint part of a type I von Neumann factor. Hence $M \cong B(H)_{s,a}$, for some Hilbert space H , by [8, 7.5.2].

(ii) If A is a JC-algebra, then A is of complex type if and only if all its concrete irreducible representations are dense [1, Proposition 3.5]. Indeed, if A is of complex type, and $\pi : A \rightarrow B(H)$ is an irreducible representation of A , then $\pi(A)^-$ is an irreducible JW-algebra, and so it is a type I JW-factor [15, Theorem 4.1]. Since A is of complex type, $\pi(A)^-$ is the self adjoint part of a von Neumann algebra, and so $\pi(A)^- = B(H)_{s,a}$. The converse is obvious by [14, Theorem 5.1].

Lemma 2 Let A_i be a JC-algebra of complex type, H_i and K_i be Hilbert spaces, $i = 1, 2$. If $\pi_i : A_i \rightarrow B(H_i)_{s,a}$ and $\phi_i : A_i \rightarrow B(K_i)_{s,a}$ are irreducible representations of A_i such that π_i is Jordan equivalent to ϕ_i , then $\pi_1 \otimes_{\max} \pi_2$ is Jordan equivalent to $\phi_1 \otimes_{\max} \phi_2$.

Proof. Since A_i is of complex type and π_i is Jordan equivalent to ϕ_i , $B(H_i)_{s,a} \cong B(K_i)_{s,a}$ which implies that $B(H_i) \cong B(K_i)$. Therefore, $\hat{\pi}_i : C^*(A_i) \rightarrow B(H_i)$ and $\hat{\phi}_i : C^*(A_i) \rightarrow B(K_i)$ are equivalent irreducible representations of $C^*(A_i)$ and hence they are spatially equivalent, by [12, 10.3.7]. It follows that $\hat{\pi}_1 \otimes_{\max} \hat{\pi}_2$ is spatially equivalent to $\hat{\phi}_1 \otimes_{\max} \hat{\phi}_2$, which implies that $\hat{\pi}_1 \otimes_{\max} \hat{\pi}_2$ is equivalent to $\hat{\phi}_1 \otimes_{\max} \hat{\phi}_2$ and so, $\pi_1 \otimes_{\max} \pi_2 = (\hat{\pi}_1 \otimes_{\max} \hat{\pi}_2) |_{JC(A_1 \otimes_{\max} A_2)}$ is Jordan equivalent to $\phi_1 \otimes_{\max} \phi_2 = (\hat{\phi}_1 \otimes_{\max} \hat{\phi}_2) |_{JC(A_1 \otimes_{\max} A_2)}$.

Lemma 3 Every JC-algebra of complex type is universally reversible.

Proof. Let A be a JC-algebra of complex type, and let $\phi : A \rightarrow V$ be a spin factor representation of A . Since A is of complex type $\phi(A)^- = V \cong B(H)_{s,a}$, for some complex Hilber space H , which implies that V is universally reversible, by [8, 7.4.6], and so, $V = V_2$ or $V = V_3$, by [1, p.280]. Therefore, all the spin factor representations of A are onto V_2 or V_3 . That is, A is universally reversible by [7, Proposition 2.2].

Definition 3 Let A be a JC-algebra of complex type and let ρ be a pure state of A . An irreducible concrete representation $\pi : A \rightarrow B(H)_{s,a}$ is said to be **associated with** ρ if there is a unit vector ξ such that $\rho(a) = \langle \pi(a)\xi, \xi \rangle$ for all $a \in A$.

Note that the unit vector in the above definition is uniquely determined up to a scalar multiple of modulus one by virtue of the density of $\pi(A)$ in $B(H)_{s,a}$.

Lemma 4 Let A be a JC-algebra and let $\phi : A \rightarrow M \subseteq B(H)_{s,a}$ be a type I factor of A . Then there is a unit vector $\xi \in H$ such that $\omega_\xi \circ \phi$ is a pure state of A .

Proof. Let $\bar{\phi} : A^{**} \rightarrow M$, the normal extension of ϕ to A^{**} . Then there exists a central projection e in A^{**} such that $eA^{**} \cong \bar{\phi}(A^{**}) = M$, hence the restriction ψ of $\bar{\phi}$ to eA^{**} is an isomorphism of eA^{**} onto M . Since M is a JW-factor of type I, there is a unit vector $\xi \in H$ such that the vector state ω_ξ on M is pure [15, Theorem 4.3], and so, $\omega_\xi \circ \psi$ is a pure state of eA^{**} . Since $(1-e)A^{**} \subseteq \ker \omega_\xi \circ \bar{\phi}$ and $A^{**} = eA^{**} \oplus (1-e)A^{**}$, it is now easy to see that $\omega_\xi \circ \bar{\phi}$ is a pure state of A^{**} which is normal. Since a normal state ρ on A^{**} is pure if and only if $\rho|_A$ is pure on A , we see that $\omega_\xi \circ \phi = (\omega_\xi \circ \bar{\phi})|_A$ is a pure state of A .

Remark 3 Let A be a JC-algebra of complex type.

(i) If ρ is a pure state of A and $\{\pi, H, \xi\}$ is the cyclic representation associated with $\hat{\rho}$ obtained by the GNS construction, where $\hat{\rho}$ is a pure state

extension of ρ to $C^*(A)$, then $\{\pi|_A, H, \xi\}$ is an irreducible concrete representation associated with ρ .

(ii) If $\pi : A \rightarrow B(H)$ is an irreducible representation associated with a pure state ρ of A , then π is a type I factor representation and hence π is Jordan equivalent to ϕ_v for some $v \in P(A)$ and so $\pi^* : B(H)^* \rightarrow A^*$ maps the set S of normal states of $B(H)$ injectively onto the smallest split face F_v of $S(A)$ containing v [1, Proposition 2.2]. Since π is associated with ρ

$$\rho(a) = \langle \pi(a)\xi, \xi \rangle = (\pi^*(\omega_\xi))(a), \quad a \in A,$$

for some unit vector $\xi \in H$. Thus $\rho = \pi^*(\omega_\xi) \in \pi^*(S) = F_v$, so $F_\rho \subseteq F_v$. Since F_v is a minimal split face, $F_\rho = F_v$ which implies that ϕ_v (and hence π) is Jordan equivalent to ϕ_ρ [1, p. 273].

(iii) If $\rho_1, \rho_2 \in P(C^*(A))$ such that $\rho = \rho_1|_A = \rho_2|_A$ and $\{\pi_1, H_1, \xi_1\}, \{\pi_2, H_2, \xi_2\}$ are the cyclic representations associated with ρ_1, ρ_2 , respectively, then for each $a \in A$

$$\langle \pi_1(a)\xi_1, \xi_1 \rangle = \rho_1(a) = \rho(a) = \rho_2(a) = \langle \pi_2(a)\xi_2, \xi_2 \rangle.$$

Hence $\pi_1|_A$ and $\pi_2|_A$ are both associated with ρ which implies that $\pi_1|_A$ is Jordan equivalent to $\pi_2|_A$, by (ii).

The following Theorem is the Jordan analogue of [4, 3.4.11].

Theorem 7 *Let A be a JC -algebra of complex type and let $T_o : P(A) \rightarrow \sigma(A)$ be the map given by $\rho \mapsto [\phi_\rho]$. Then T_o is an open continuous surjection.*

Proof. It is clear from Remark 3(iii) that T_o is well defined, and is obviously surjective. To see that T_o is an open map, note that if $\rho \in P(C^*(A))$ then $\rho|_A \in P(A)$, since A is of complex type [1, Proposition 5.3]. So, let r be the restriction map $(\psi_A)^*|_{P(C^*(A))}$ from $P(C^*(A))$ onto $P(A)$, where $(\psi_A)^* : C^*(A)^* \rightarrow A^*$ is the dual of the embedding $\psi_A : A \rightarrow C^*(A)$, and consider the following commutative diagram.

$$\begin{array}{ccc} P(C^*(A)) & \xrightarrow{T_o} & \sigma(C^*(A)) \\ r \downarrow & & \downarrow \Psi \\ P(A) & \xrightarrow{T_o} & \sigma(A) \end{array}$$

The fact that T_\circ is open follows immediately, since T , Ψ and r are all surjective maps, T and Ψ are open and r is continuous. It remains to prove that T_\circ is continuous, so let $\eta : P(A) \rightarrow \text{Pr im}(A)$ ($\rho \mapsto \ker \phi_\rho$). By [6, Theorem 4.1], η is a continuous, open and surjective map. Since $g : \sigma(A) \rightarrow \text{Pr im}(A)$ is open and continuous surjection (see [13, 4.1.12]), it is easy now to conclude from the following commutative diagram

$$\begin{array}{ccc} P(A) & \xrightarrow{T_\circ} & \sigma(A) \\ \eta \searrow & & \swarrow g \\ & & \text{Pr im}(A) \end{array}$$

that T_\circ is also continuous. This completes the proof.

Let \mathfrak{A}_i be a C^* -algebra, $i = 1, 2$. Then the map

$$([\pi_1], [\pi_2]) \mapsto [\pi_1 \otimes_{\max} \pi_2] : \sigma(\mathfrak{A}_1) \times \sigma(A_2) \rightarrow \sigma(A_1 \otimes_{\max} A_2)$$

is continuous. The following Theorem gives the Jordan analogue of this result.

Theorem 8 *If A_i , $i = 1, 2$, is a JC-algebra of complex type, then the map $\Phi_\circ : \sigma(A_1) \times \sigma(A_2) \rightarrow \sigma(JC(A_1 \otimes_{\max} A_2))$ given by $([\pi_1], [\pi_2]) \mapsto [\pi_1 \otimes_{\max} \pi_2]$, where $[\pi_i] \in \sigma(A_i)$, is continuous.*

Proof. By Lemma 2, if $\pi_i : A_i \rightarrow B(H_i)_{s.a}$ and $\phi_i : A_i \rightarrow B(K_i)_{s.a}$ are irreducible representations of A_i such that π_i is Jordan equivalent to ϕ_i , then $\pi_1 \otimes_{\max} \pi_2$ is Jordan equivalent to $\phi_1 \otimes_{\max} \phi_2$, hence Φ_\circ is well defined. To show that Φ_\circ is continuous, let $\Phi : \sigma(C^*(A_1)) \times \sigma(C^*(A_2)) \rightarrow \sigma(C^*(A_1) \otimes_{\max} C^*(A_2))$ be the continuous map given in [5, Proposition 5], then consider the following commutative diagram

$$\begin{array}{ccc} \sigma(C^*(A_1)) \times \sigma(C^*(A_2)) & \xrightarrow{\Phi} & \sigma(C^*(A_1) \otimes_{\max} C^*(A_2)) \\ \Psi' \downarrow & & \downarrow \Psi \\ \sigma(A_1) \times \sigma(A_2) & \xrightarrow{\Phi_\circ} & \sigma(JC(A_1 \otimes_{\max} A_2)) \end{array}$$

from which it is easy to conclude that Φ_\circ is continuous, since Ψ and Ψ' are surjective, continuous and open maps (cf. Theorem 4), and Φ is continuous.

Theorem 9 *Let A_i , $i = 1, 2$, be a JC-algebra of complex type, then the map $\theta_o : P(A_1) \times P(A_2) \rightarrow P(JC(A_1 \otimes_{\max} A_2))$ given by $(\rho_1, \rho_2) \mapsto \rho_1 \otimes_{\max} \rho_2$ is continuous.*

Proof. Let $\rho_i \in P(A_i)$, we shall first show that $\rho_1 \otimes_{\max} \rho_2$ is a pure state of $JC(A_1 \otimes_{\max} A_2)$. So let $\hat{\rho}_i$ be an extension of ρ_i to $C^*(A_i)$, then $\hat{\rho}_i$ is pure and hence the cyclic representation $(\hat{\pi}_i, H_i, \xi_i)$ associated with $\hat{\rho}_i$ is irreducible, by [17, I.9.22]. If π_i is the restriction of $\hat{\pi}_i$ to A_i , then π_i is irreducible [10, Lemma 1]. Therefore $\pi_1 \otimes_{\max} \pi_2 = (\hat{\pi}_1 \otimes_{\max} \hat{\pi}_2) |_{JC(A_1 \otimes_{\max} A_2)}$ is irreducible, by Theorem 5 and [10, Lemma 1] and so the weak closure $\left((\pi_1 \otimes_{\max} \pi_2)(JC(A_1 \otimes_{\max} A_2)) \right)^-$ of $(\pi_1 \otimes_{\max} \pi_2)(JC(A_1 \otimes_{\max} A_2))$ in $B(H_1 \otimes H_2)$ is a type I JW-factor. Note that $\hat{\rho}_i = \omega_{\xi_i} \circ \hat{\pi}_i$, where ω_{ξ_i} is the vector state $x \mapsto \langle x\xi_i, \xi_i \rangle$ which is a pure state of $B(H_i)$ (see [12, 10.2.5]). Since A_i is of complex type, $\pi_i(A_i)^- = B(H_i)_{s.a}$ and hence the restriction of ω_{ξ_i} to $\pi_i(A_i)^-$ is a pure state (cf. Remark 2(ii)). Therefore, the support e_i of ω_{ξ_i} in $\pi_i(A_i)^-$ is a minimal (and hence abelian) projection [15, Corollary 4.4]. Since $\pi_1(A_1) \otimes 1 \hookrightarrow (\pi_1 \otimes_{\max} \pi_2)(JC(A_1 \otimes_{\max} A_2))$ and $1 \otimes \pi_2(A_2) \hookrightarrow (\pi_1 \otimes_{\max} \pi_2)(JC(A_1 \otimes_{\max} A_2))$, we have $\pi_1(A_1)^- \otimes 1 \hookrightarrow \left((\pi_1 \otimes_{\max} \pi_2)(JC(A_1 \otimes_{\max} A_2)) \right)^-$ and $1 \otimes \pi_2(A_2)^- \hookrightarrow \left((\pi_1 \otimes_{\max} \pi_2)(JC(A_1 \otimes_{\max} A_2)) \right)^-$. Hence $\left((\pi_1 \otimes_{\max} \pi_2)(JC(A_1 \otimes_{\max} A_2)) \right)^-$ contains the JW-algebra generated by the JW-algebra $\pi_1(A_1)^- \otimes 1$ and the JW-algebra $1 \otimes \pi_2(A_2)^-$, which implies that $e_1 \otimes e_2$ belongs to $\left((\pi_1 \otimes_{\max} \pi_2)(JC(A_1 \otimes_{\max} A_2)) \right)^-$. Since $\pi_1(A_1) \otimes \pi_2(A_2) = (\pi_1 \otimes_{\max} \pi_2)(A_1 \otimes A_2) \hookrightarrow (\pi_1 \otimes_{\max} \pi_2)(JC(A_1 \otimes_{\max} A_2))$, $\left((\pi_1 \otimes_{\max} \pi_2)(JC(A_1 \otimes_{\max} A_2)) \right)^-$ is the JW-algebra generated by $\pi_1(A_1)^- \otimes \pi_2(A_2)^-$. By [18, III.5.12], $e_1 \otimes e_2$ is the support of $\omega_{\xi_1} \otimes \omega_{\xi_2} = \omega_{\xi_1 \otimes \xi_2}$, and is obviously abelian since $(e_1 \otimes e_2)B(H_1 \otimes H_2)(e_1 \otimes e_2)$ is unitarily equivalent to $e_1 B(H_1) e_1 \otimes e_2 B(H_2) e_2$. Therefore, $\omega_{\xi_1 \otimes \xi_2}$ is a pure state of $\left((\pi_1 \otimes_{\max} \pi_2)(JC(A_1 \otimes_{\max} A_2)) \right)^-$, by [15, Corollary 4.4]. Note that for all

$a_i \in A_i$

$$\begin{aligned}
(\rho_1 \otimes_{\max} \rho_2)(a_1 \otimes a_2) &= \rho_1(a_1)\rho_2(a_2) \\
&= \langle \pi_1(a_1)\xi_1, \xi_1 \rangle \langle \pi_2(a_2)\xi_2, \xi_2 \rangle \\
&= \omega_{\xi_1 \otimes \xi_2} \circ (\pi_1 \otimes_{\max} \pi_2)(a_1 \otimes a_2),
\end{aligned}$$

and hence by the uniqueness of $\rho_1 \otimes_{\max} \rho_2$, we must have $\rho_1 \otimes_{\max} \rho_2 = \omega_{\xi_1 \otimes \xi_2} \circ (\pi_1 \otimes_{\max} \pi_2)$, which implies that $\rho_1 \otimes_{\max} \rho_2$ is a pure state of $JC(A_1 \otimes_{\max} A_2)$, and that θ_o is well defined.

The continuity of θ_o is immediate, since θ_o is the restriction of the map $S(A_1) \times S(A_2) \rightarrow S(JC(A_1 \otimes_{\max} A_2))$ $((\rho_1, \rho_2) \mapsto (\rho_1 \otimes_{\max} \rho_2))$ to $P(A_1) \times P(A_2)$ and the continuity of this map follows easily from the continuity of $\theta : S(C^*(A_1)) \times S(C^*(A_2)) \mapsto S(C^*(A_1) \otimes_{\max} C^*(A_2))$ $((f_1, f_2) \mapsto (f_1 \otimes_{\max} f_2))$ [5, Proposition 5] by Theorem 1 and since each state of A_i extends to a state of $C^*(A_i)$ and each state of $C^*(A_i)$ restricts to a state of A_i .

Collecting the results of Theorem 7 and Theorem 9 we have

Theorem 10 *Let A_i , $i = 1, 2$, be JC -algebras of complex type. Then the following diagram of continuous maps is commutative.*

$$\begin{array}{ccc}
P(A_1) \times P(A_2) & \xrightarrow{\theta_o} & P(JC(A_1 \otimes_{\max} A_2)) \\
T'_o \downarrow & & \downarrow T_o \\
\sigma(A_1) \times \sigma(A_2) & \xrightarrow{\Phi_o} & \sigma(JC(A_1 \otimes_{\max} A_2))
\end{array}$$

Corollary 11 *Let A_i , $i = 1, 2$, be JC -algebras of complex type. Then the following diagram is commutative.*

$$\begin{array}{ccccc}
P(C^*(A_1)) \times P(C^*(A_2)) & & \xrightarrow{\theta} & & P(C^*(A_1) \otimes_{\max} C^*(A_2)) \\
r' \searrow & & & & \nearrow r \\
T' \downarrow & P(A_1) \times P(A_2) & \xrightarrow{\theta_o} & P(JC(A_1 \otimes_{\max} A_2)) & \downarrow T \\
& T'_o \downarrow & & \downarrow T_o & \\
& \sigma(A_1) \times \sigma(A_2) & \xrightarrow{\Phi_o} & \sigma(JC(A_1 \otimes_{\max} A_2)) & \\
& \Psi' \nearrow & & \nwarrow \Psi & \\
\sigma(C^*(A_1)) \times \sigma(C^*(A_2)) & & \xrightarrow{\Phi} & & \sigma(C^*(A_1) \otimes_{\max} C^*(A_2))
\end{array}$$

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